PERIODIC ORBITS

BY

F. R. MOULTON

IN COLLABORATION WITH

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INTRODUCTION.

The problem of three bodies received a great impetus in 1878, when Hill published his celebrated researches upon the lunar theory. His investigations were carried out with practical objects in mind, and comparatively little attention was given to the underlying logic of the processes which he invented. For example, the legitimacy of the use of infinite determinants was assumed, the validity of the solution of infinite systems of non-linear equations was not questioned, and the conditions for the convergence of the infinite series which he used were stated to be quite unknown. These deficiencies in the logic of his work do not detract from the brilliancy and value of his ideas, and his skill in carrying them out excites only the highest admiration.

The work of Hill was followed in the early nineties by the epoch-making researches of Poincaré, which were published in detail in his *Les Méthodes Nouvelles de la Mécanique Céleste.* Poincaré brought to bear on the problem all the resources of modern analysis. The new methods of treating the difficult problem of three bodies which he invented were so numerous and powerful as to be positively bewildering. They opened so many new fields that a generation will be required for their complete exploration. On the one hand, the results were in the direction of purely theoretical considerations, in which Birkoff has recently made noteworthy extensions; on the other hand, they foreshadowed somewhat dimly methods which will doubtless be of great importance in practical applications in celestial mechanics. The researches of Poincaré are scarcely less revolutionary in character than were those of Newton when he discovered the law of gravitation and laid the foundations of celestial mechanics.

In 1896 Sir George Darwin published an extensive paper on the problem of three bodies in *Acta Mathematica.* In mathematical spirit it was similar to the work of Hill; indeed, the methods used were essentially those of Hill, but the problem treated was considerably different. For a ratio of the finite masses of ten to one, Darwin undertook to discover by numerical processes all the periodic orbits of certain types and to follow their changes with varying values of the Jacobian constant of integration. This program was excellently carried out at the cost of a great amount of labor. It gave specific numerical results for many orbits in a particular example.

The investigations contained in this volume were begun in 1900 and, with the exception of the last chapter, they were completed by 1912. Those not made by myself were carried out by students who made their doctorates under my direction.

The following chapters have been heretofore published in substance:

I. Sections III and IV American Journal of Mathematics, vol. XXXIII (1911).
II. Astronomical Journal, vol. XXV (1907).
III. Rendiconti Matematico di Palermo, vol. XXXII (1911).
IV. Transactions of the American Mathematical Society, vol. XI (1910).
VII. Mathematische Annalen, vol. LXXIII (1913).
VIII. Annals of Mathematics, 2d Series, vol. 12 (1910)
XI. Transactions of the American Mathematical Society, vol. VII (1906).
XII. Transactions of the American Mathematical Society, vol. XIII (1912).
XIII. Transactions of the American Mathematical Society, vol. VIII (1907).
XIV. Transactions of the American Mathematical Society, vol. IX (1908).
XV. Proceedings of the London Mathematical Society, Series II, vol. 2 (1912).

The investigations and computations contained in the last chapter were completed in 1917.

It was originally intended to publish only the first fifteen chapters, and if that program had been carried out they would have appeared in 1912. But as the work of printing progressed the ideas contained in the last chapter were being developed and the computations were begun. It was thought that an even more nearly complete and certain idea of the evolution of periodic orbits with changing parameters could be obtained in a year than were obtained in five years. The difficulties and enormous amount of labor involved were not foreseen. No one can now read with better appreciation than I the following words from Darwin's introduction to his paper:

"As far as I can see, the search resolves itself into the discussion of particular cases by numerical processes, and such a search necessarily involves a prodigious amount of work. It is not for me to say whether the enormous amount of labor I have undertaken was justifiable in the first instance; but I may remark that I have been led on by the interest of my results, step by step, to investigate more, and again more, cases."

The results which now appear had all been obtained when service in the army made it necessary to lay them aside before the final chapter could be put into form for publication. After they had been laid aside for about two years it was not easy to gather up the details again and to arrange them in a systematic order. This explains the long delay in the appearance of this volume. It is clear that it was in no wise due to the Carnegie Institution of Washington. Indeed, the patience of President Woodward with long and expensive delays has been far beyond what could reasonably have been expected.

In the greater part of this work complete mathematical rigor has been insisted upon. On the other hand, the developments have been in a form applicable to practical problems in celestial mechanics. For example, sections III and IV of Chapter I treat non-homogeneous equations of the types which arise in practical problems; Chapter II is devoted to questions which have long been classic in celestial mechanics; Chapter III contains, among other things, a new and rigorous treatment of Hill's differential equation with periodic coefficients; Chapter IV treats a problem that arises,

at least approximately, in the solar system; Chapter V is developed in a form suitable for numerical applications; Chapter VI is an alternative treatment of the same problem, and in Chapter VII an extension of the problem involving entirely new types of difficulties is found and more powerful methods of treatment are required; Chapter XI covers the same ground as Hill's work on the moon's variational orbit and Brown's work on the parallactic terms; Chapter XII contains the corresponding discussion for superior planets; and Chapter XIV treats a problem similar to that presented by the satellite systems of Jupiter and Saturn or by the planetary system. Chapter XV contains a discussion of limiting cases of periodic orbits, namely, closed orbits of ejection. It forms a basis for part of the work of the last chapter, and it may be found to have practical applications in the escape of atmospheres. The last chapter is an attempt to trace out the evolution of periodic orbits as the parameters on which they depend are varied. In spite of the fact that infinitely many families of periodic orbits were found, whereas only a few such families were previously known, the discussion remains in certain respects incomplete. It should be stated that many results were found which have not been included because they did not contribute to the solution of the particular question under consideration. For example, a series of orbits asymptotic to the collinear equilibrium points was computed. The amount of labor the last chapter cost can scarcely be overestimated.

I should not be true to my own feelings if I did not express the appreciation of the assistance of my collaborators. The association with them has been a deep source of satisfaction and inspiration. They are to be held accountable only for those chapters which appear under their names. Most of the computations on which many of the results of the last chapter are based were made by Dr. W. L. Hart and Dr. I. A. Barnett. Without assistance of such a high order the laborious computations could not have been carried out.

F. R. MOULTON.

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CHAPTER I.

CERTAIN THEOREMS ON IMPLICIT FUNCTIONS AND DIFFERENTIAL EQUATIONS.

BY F. R. MOULTON AND W. D. MACMILLAN.

I. SOLUTION OF IMPLICIT FUNCTIONS.

1. Formal Solution of Simultaneous Equations when the Functional Determinant is Distinct from Zero at the Origin.—In applying the conditions for periodicity of the solutions of differential equations after the method of Poincaré,* there will be frequent occasion to consider the solution of

$$P_i(a_1, \ldots, a_n; \mu) = 0 \qquad (i=1, \ldots, n), \tag{1}$$

for $a_1, \ldots, a_n$ in terms of μ, where the P_i are power series in the a_j and μ, vanishing with $a_j=0$, $\mu=0$, but not with $a_j=0$, $\mu\neq 0$, and converging for $|a_j|<r_j>0$, $|\mu|<\rho>0$. We are interested in only those solutions which vanish with μ; that is, if we regard $a_1, \ldots, a_n, \mu$ as coördinates in $(n+1)$-space, in those curves satisfying (1) which pass through the origin.

Equations (1) can be satisfied formally by the series

$$a_i = \sum_{j=1}^{\infty} a_i^{(j)}\mu^j, \tag{2}$$

where the $a_i^{(j)}$ are functions of the coefficients of the P_i which are to be determined. Upon substituting (2) in (1), expanding and arranging as power series in μ, it is found that

$$\left.\begin{aligned}
0 = &\left[\sum_{j=1}^{n}\frac{\partial P_i}{\partial a_j}a_j^{(1)} + \frac{\partial P_i}{\partial \mu}\right]\mu \\
&+\left[\sum_{j=1}^{n}\frac{\partial P_i}{\partial a_j}a_j^{(2)} + \tfrac{1}{2}\sum_{j=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 P_i}{\partial a_j\partial a_k}a_j^{(1)}a_k^{(1)} + \sum_{j=1}^{n}\frac{\partial^2 P_i}{\partial a_j\partial\mu}a_j^{(1)} + \tfrac{1}{2}\frac{\partial^2 P_i}{\partial\mu^2}\right]\mu^2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+\left[\sum_{j=1}^{n}\frac{\partial P_i}{\partial a_j}a_j^{(k)} + F_i^{(k)}(a_j^{(1)}, \ldots, a_j^{(k-1)})\right]\mu^k \qquad (i=1, \ldots, n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3}$$

where the $F_i^{(k)}$ are polynomials in $a_j^{(1)}, \ldots, a_j^{(k-1)}$.

Upon assuming for the moment that the series (2) converge and satisfy (1), it follows that (3) are identities in μ. Hence the coefficients of each power of μ separately are zero.

**Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, chap. 3.

From the coefficients of the first power of μ we get

$$\sum_{j=1}^{n} \frac{\partial P_i}{\partial \alpha_j} \alpha_j^{(1)} = - \frac{\partial P_i}{\partial \mu} \qquad (i=1, \ldots, n). \tag{4}$$

Since by hypothesis the functional determinant

$$\Delta = \begin{vmatrix} \frac{\partial P_1}{\partial \alpha_1}, & \ldots, & \frac{\partial P_1}{\partial \alpha_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial P_n}{\partial \alpha_1}, & \ldots, & \frac{\partial P_n}{\partial \alpha_n} \end{vmatrix}$$

is distinct from zero for $\alpha_1 = \cdots = \alpha_n = \mu = 0$, equations (4) can be solved uniquely for the $\alpha_j^{(1)}$. If not all of the $\partial P_i / \partial \mu$ are zero, then not all of the $\alpha_j^{(1)}$ are zero; but if all of the $\partial P_i / \partial \mu$ are zero, then all of the $\alpha_j^{(1)}$ are zero.

Equating the coefficients of the second power of μ in (3) to zero, we get

$$\sum_{j=1}^{n} \frac{\partial P_i}{\partial \alpha_j} \alpha_j^{(2)} = - F_i^{(2)} (\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}) \qquad (i=1, \ldots, n),$$

the right members of which are completely known. The determinant of the coefficients of the $\alpha_j^{(2)}$ is Δ, and the $\alpha_j^{(2)}$ are therefore uniquely determined, being all zero or not all zero according as the $F_i^{(2)}$ are all zero or not all zero. And it is seen from (3) that the treatment of the general term is entirely similar and depends upon the same determinant Δ. Hence, under the hypothesis as to Δ, a formal solution is possible, and it is unique.

2. Proof of Convergence of the Solutions.—In order to prove the convergence of the series (2), consider the solution of the comparison equations

$$Q_i (\beta_1, \ldots, \beta_n; \mu) = 0 \qquad (i=1, \ldots, n) \tag{1'}$$

for $\beta_1, \ldots, \beta_n$ in terms of μ, where the Q_i are power series in the β_j and μ, vanishing for $\beta_j = 0$, $\mu = 0$, and convergent for $|\beta_j| < r > 0$, $|\mu| < \rho' > 0$; and where also the coefficients of all terms beyond the linear in each Q_i are real, positive, independent of i, and greater than the moduli of the corresponding coefficients* in the expansions of any of the P_i. The character of the coefficients of the linear terms will be specified when they are used.

Suppose the solutions of (1′) have the form

$$\beta_i = \sum_{j=1}^{\infty} \beta_i^{(j)} \mu^j \qquad (i=1, \ldots, n). \tag{2'}$$

*For proof of the possibility of satisfying these conditions see Picard's *Traité d'Analyse*, edition of 1905, vol. II, pp. 255–260.

On substituting (2′) in (1′) and arranging in powers of μ, there results a system of equations similar to (3). The $\beta_j^{(1)}$ are determined by

$$\sum_{j=1}^{n} \frac{\partial Q_i}{\partial \beta_j} \beta_j^{(1)} = - \frac{\partial Q_i}{\partial \mu} \qquad (i=1, \ldots, n). \qquad (4')$$

It is necessary now to specify the properties of the linear terms of the Q_i. It will be supposed first that the $\partial Q_i/\partial \mu$ are real and positive, and that $\partial Q_1/\partial \mu = \cdots = \partial Q_n/\partial \mu = \partial Q/\partial \mu$. It follows that for fixed values of the $\partial Q_i/\partial \beta_j$ the values of the $\beta_j^{(1)}$ satisfying (4′) are proportional to $\partial Q/\partial \mu$. The $\partial Q_i/\partial \beta_j$ will now be so determined that when $\partial Q_i/\partial \mu$ is replaced by the greatest $|\partial P_i/\partial \mu|$ the $\beta_j^{(1)}$ determined by (4′) shall be equal, positive, and at least as great as the greatest $|\alpha_j^{(1)}|$ for all values of $\partial P_i/\partial \mu$ such that $|\partial P_i/\partial \mu| \leqq \partial Q/\partial \mu$. This must be done in such a way that the determinant

$$\Delta' = \begin{vmatrix} \frac{\partial Q_1}{\partial \beta_1}, & \ldots, & \frac{\partial Q_1}{\partial \beta_n} \\ \vdots & & \vdots \\ \frac{\partial Q_n}{\partial \beta_1}, & \ldots, & \frac{\partial Q_n}{\partial \beta_n} \end{vmatrix}$$

shall be distinct from zero for $\beta_j = \mu = 0$. These conditions can be satisfied in infinitely many ways. A simple way is to choose $\partial Q_i/\partial \beta_j = -1$ if $j \neq i$ and $\partial Q_i/\partial \beta_i = -(1+c)$. Then the determinant of (4′) is $\Delta' = (-1)^n c^{n-1} (c+n)$, which can vanish only if $c=0$ or $c=-n$, and the solutions of (4′) are

$$\beta_1^{(1)} = \cdots = \beta_n^{(1)} = \frac{\frac{\partial Q}{\partial \mu}}{(c+n)}. \qquad (5)$$

For any n we can give c such a value that the $\beta_i^{(1)}$ shall be positive and at least as great as the greatest $|\alpha_j^{(1)}|$.

The $\beta_j^{(2)}$ are determined by equations of the form

$$\sum_{j=1}^{n} \frac{\partial Q_i}{\partial \beta_j} \beta_j^{(2)} = - G_i^{(2)}(\beta_1^{(1)}, \ldots, \beta_n^{(1)}) \qquad (i=1, \ldots, n). \qquad (6)$$

It follows from the properties of the Q_i, together with the explicit structure of the $G_i^{(2)}$ and the values of the $\beta_i^{(1)}$, that

$$G_1^{(2)} = \cdots = G_n^{(2)} \geqq |F_i^{(2)}| \qquad (i=1, \ldots, n),$$

and therefore that

$$\beta_1^{(2)} = \cdots = \beta_n^{(2)} \geqq |\alpha_i^{(2)}| \qquad (i=1, \ldots, n).$$

It is very easily shown by induction that for every value of the index k

$$\left.\begin{aligned} G_1^{(k)} &= \cdots = G_n^{(k)} \geqq |F_i^{(k)}| \qquad (i=1, \ldots, n), \\ \beta_1^{(k)} &= \cdots = \beta_n^{(k)} \geqq |\alpha_i^{(k)}| \qquad (i=1, \ldots, n). \end{aligned}\right\} \quad (7)$$

Therefore if (2′) converge for $|\mu| \leqq \rho'$, then also (2) converge for $|\mu| \leqq \rho'$.

All the conditions imposed upon the Q_i can be satisfied by functions of the form*

$$\left.\begin{aligned} Q_i = -c\beta_i - (1+M)(\beta_1 + \cdots + \beta_n) + \frac{M(\beta_1 + \cdots + \beta_n + \mu)}{\left(1-\frac{\mu}{\rho}\right)\left(1-\frac{\beta_1 + \cdots + \beta_n}{r}\right)} = 0 \\ (i=1, \ldots, n), \end{aligned}\right\} \quad (8)$$

where M is a real positive constant. Adding these n equations and solving for $\beta_1 + \cdots + \beta_n$, it is found that

$$\beta_1 + \cdots + \beta_n = \frac{1+\frac{c}{n} - \frac{M\mu}{\rho-\mu} \pm \sqrt{\left(1+\frac{c}{n} - \frac{M\mu}{\rho-\mu}\right)^2 - \frac{4M\left(1+\frac{c}{n}+M\right)\frac{\rho\mu}{r}}{\rho-\mu}}}{\frac{2}{r}\left(1+\frac{c}{n}+M\right)}. \quad (9)$$

Since each β_j, and therefore the sum $\beta_1 + \cdots + \beta_n$, is zero for $\mu=0$, the negative sign must be taken before the radical in (9). The right member of (9) can be expanded as a converging power series in μ if $|\mu|$ is taken so small that $|\mu| < \rho$ and $|f(\mu)| < |\varphi(\mu)|$, where

$$f(\mu) = \frac{4M\left(1+\frac{c}{n}+M\right)}{\rho-\mu}\frac{\rho\mu}{r}, \qquad \varphi(\mu) = \left(1+\frac{c}{n} - \frac{M\mu}{\rho-\mu}\right)^2,$$

conditions which can always be satisfied, whatever may be the values of n, r, ρ, c, and M. Moreover, the coefficients of all powers of μ in the expansion of (9) are real and positive. Hence it follows that

$$\beta_1 + \cdots + \beta_n = \mu R(\mu), \quad (10)$$

where $R(\mu)$ is a power series in μ whose coefficients are all positive. It follows from (8) that all the β_i are equal. Hence

$$\beta_1 = \cdots = \beta_n = \frac{\mu R(\mu)}{n}. \quad (11)$$

For $|\mu|$ sufficiently small the right members of these equations are converging power series in μ; moreover, they identically satisfy (8). It follows from this result and the second set of (7) that $\rho'' > 0$ exists such that the series (2) converge for $|\mu| < \rho''$.

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*Picard's *Traité d'Analyse*, loc. cit.

3. Generalization to Many Parameters.—Suppose the equations to be solved are

$$P_i(a_1, \ldots, a_n; \mu_1, \ldots, \mu_k) = 0 \qquad (i=1, \ldots, n), \quad (12)$$

and that the functional determinant with respect to the a_j is distinct from zero for $a_1 = \cdots = a_n = \mu_1 = \cdots = \mu_k = 0$. Then the problem can be reduced to that discussed in §§1 and 2 by letting $\mu_j = c_j \mu$. After the solutions have been obtained the $c_j \mu$ everywhere can be replaced by μ_j, for the $c_j \mu$ will occur only in integral powers.

4. The Functional Determinant Zero, but not All of its First Minors Zero at the Origin.*—Consider the equations

$$P_i(a_1, \ldots, a_n; \mu) = 0 \qquad (i=1, \ldots, n), \quad (13)$$

where the P_i have the properties imposed upon the P_i of §1. Suppose that the determinant of the linear terms in the a_j is zero for $a_1 = \cdots = a_n = \mu = 0$, but that not all of its first minors are zero. It may be supposed without any loss of generality that the determinant of the terms remaining after deleting the last row and column of the linear terms is distinct from zero. Hence, as a consequence of the theorems proved in §§ 1, 2, 3, the first $n-1$ equations can be solved uniquely for $a_1, \ldots, a_{n-1}$ as power series in a_n and μ, vanishing for $a_n = \mu = 0$.

Suppose the solutions of the first $n-1$ equations for $a_1, \ldots, a_{n-1}$ are substituted in the last equation. It will become a function of a_n and μ which may be written, omitting the now useless subscript on the a_n,

$$P(a, \mu) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{kj}\, a^k \mu^j = 0 \qquad (k+j>0). \quad (14)$$

Since the determinant of the linear terms of (13) is zero, this equation carries no linear term in a. Suppose the term of lowest degree in a alone is $c_{k0} a^k$. Then, for each value of μ whose modulus is sufficiently small there are k values of a satisfying (14) and, moreover, the modulus of μ can be taken so small that the moduli of the solutions for a shall be as small as one pleases.† Also, for each set of values of $a = a_n$ and μ whose moduli are sufficiently small there is one set of values $a_1, \ldots, a_{n-1}$ satisfying the first $n-1$ equations of (13). Consequently, for each μ whose modulus is sufficiently small there are precisely k sets of values of $a_1, \ldots, a_n$ satisfying (13). A special discussion is required to determine the character of these solutions and the method of finding them. These questions are taken up in the immediately following articles.

*The more difficult case, in which all the first minors of the functional determinant vanish, does not arise in this work. It has only recently (in 1911) been completely solved by MacMillan, in a paper which will appear in *Mathematische Annalen*.

†Weierstrass, *Abhandlungen aus der Functionenlehre*, p. 107. Picard, *Traité d'Analyse*, vol. II, chap. 9, §7, and chap. 13. Harkness and Morley, *Treatise on the Theory of Functions*, chap. 4.

5. Case where $P(\alpha,\mu)=\alpha^k P_1(\alpha,\mu)-\mu^\lambda P_2(\alpha,\mu)$.—Suppose the $P(\alpha,\mu)$ of (14) has the form

$$\alpha^k P_1(\alpha,\mu) - \mu^\lambda P_2(\alpha,\mu) = 0,$$

where P_1 and P_2 are power series in α and μ which do not vanish for $\alpha=\mu=0$ and which converge for $|\alpha| < r > 0$, $|\mu| < \rho > 0$. Upon extracting the k^{th} root, this equation gives

$$\alpha Q_1(\alpha,\mu) - \eta \mu^{\frac{\lambda}{k}} Q_2(\alpha,\mu) = 0,$$

where Q_1 and Q_2 are power series in α and μ which do not vanish for $\alpha=\mu=0$ and which converge for $|\alpha|<r'>0$, $|\mu|<\rho'>0$, and where η is a k^{th} root of unity. If we let $\mu=\nu^k$, this equation takes the form of those treated in §§1 and 2 and can be solved uniquely for α in terms of ν for each η. The k solutions are obtained by taking for η the k roots of unity.

6. A Second Simple Case.—Suppose $P(\alpha,\mu)$ has the form

$$P(\alpha,\mu) = \sum_{i=0}^{k} c_i \alpha^{k-i}\mu^i + Q(\alpha,\mu) = 0, \tag{15}$$

where $c_0 \neq 0$ and Q contains no term of degree less than $k+1$ in α and μ. It can be supposed without loss of generality that $c_0=1$. Then

$$\sum_{i=0}^{k} c_i \alpha^{k-i}\mu^i = P_0(\alpha,\mu) = (\alpha-b_1\mu)(\alpha-b_2\mu)\cdots(\alpha-b_k\mu). \tag{16}$$

Suppose $(\alpha-b_j\mu)$ is a simple factor of the homogeneous polynomial $P_0(\alpha,\mu)$, and exclude the trivial case in which it is also a factor of $Q(\alpha,\mu)$. Then $\partial P_0/\partial\alpha \neq 0$ for $\alpha=b_j\mu$. Now let

$$\alpha = b_j\mu + \beta\mu. \tag{17}$$

After this transformation both P_0 and Q are divisible by μ^k. After μ^k is divided out, P_0 carries a term in β to the first degree whose coefficient is not zero, and Q carries no term independent of μ, but has at least one term in μ alone, for otherwise $P(\alpha,\mu)$ would be divisible by $(\alpha-b_j\mu)$. Consequently by §§ 1 and 2 the equation in β and μ can be solved for β as a converging power series in μ, vanishing for $\mu=0$. Therefore α can be expanded as a converging power series in μ, vanishing with μ, for each of the simple roots of the polynomial $P_0(\alpha,\mu)=0$. If $b_1,\ldots,b_k$ are all distinct, the expansions for the k branches of the function $P(\alpha,\mu)$ which pass through the origin can be found by this process. The actual determination of the coefficients is by the method of §1 in the simple case $n=1$.

7. General Case of Power Series in two Variables.*—The method of treatment consists in reducing the equation, by suitable transformations, to forms of a standard type from which the solutions can be found. In certain special cases successive transformations are required. The analysis in

*This problem has been treated by Puiseux, Nöther, etc. For references and discussion see Harkness and Morley's *Treatise on the Theory of Functions*, chap. 4, and Crystal's *Algebra*, part 2, chap. 30.

general, as well as the particular transformations required in any special case to reduce the equation to the standard forms, is indicated most simply by Newton's *Parallelogram*.

In constructing Newton's parallelogram it is sufficient to consider only those terms $c_{ij}\alpha^i\mu^j$ of $P(\alpha, \mu)$ for which $i \gtreqless k$, $j \gtreqless \lambda$, $c_{k0}\alpha^k$ being the term of lowest degree in α alone and $c_{0\lambda}\mu^\lambda$ the term of lowest degree in μ alone. Take a set of rectangular axes and for each of the terms $c_{ij}\alpha^i\mu^j$, $c_{ij} \neq 0$, lay down a *degree point* whose coördinates are i and j. Then the line passing through the origin and the point $(k, 0)$ is rotated around $(k, 0)$ as a pole so that it moves along the j-axis in the positive direction until it strikes at least one other degree point (it may, of course, strike several simultaneously). Let the one of those which it first strikes having the greatest j be (i_1, j_1). Then the line is rotated around (i_1, j_1) in the same direction until it strikes at least one other degree point. Letting the one of these having the greatest j be (i_2, j_2), the line is rotated around (i_2, j_2) until another is encountered. This process is continued until the point $(0, \lambda)$ is reached. The number of steps in the process evidently can not be greater than k or λ. The part of Newton's parallelogram needed in discussing the character of the function near the origin is made up of the segments $(k, 0)$ to (i_1, j_1), (i_1, j_1) to (i_2, j_2), . . . , (i_ν, j_ν) to $(0, \lambda)$. For the terms of $P(\alpha, \mu)$ corresponding to each one of these segments there is, as will be shown, a transformation which throws $P(\alpha, \mu)$ into a standard form.

In order to illustrate Newton's parallelogram, consider the example

$$P(\alpha, \mu) = c_{50}\alpha^5 + c_{31}\alpha^3\mu + c_{22}\alpha^2\mu^2 + c_{13}\alpha\mu^3 + c_{14}\alpha\mu^4 + c_{06}\mu^6 + Q(\alpha, \mu),$$

where $Q(\alpha, \mu)$ contains only terms of the seventh and higher degrees in α and μ. The coördinates of the points in Newton's Parallelogram are $(5, 0)$, $(3, 1)$, $(2, 2)$, $(1, 3)$, and $(0, 6)$, and it consists of three segments which are shown in Fig. 1.

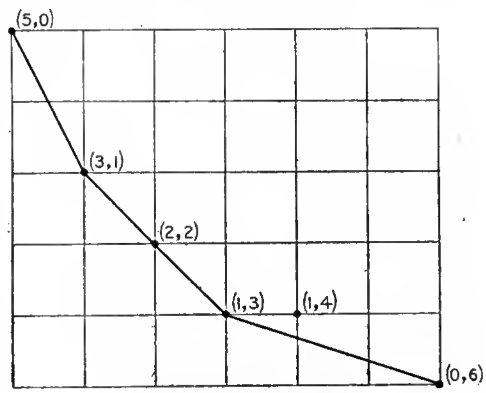

FIG. 1.

Consider the segment (i_1, j_1) to (i_2, j_2) and make the transformation

$$\alpha = \beta\mu^\sigma, \qquad \sigma = \frac{j_2 - j_1}{i_1 - i_2} = \frac{m}{n}, \quad (18)$$

where m and n are relatively prime integers. The terms $c_{i_1j_1}\alpha^{i_1}\mu^{j_1}$ and $c_{i_2j_2}\alpha^{i_2}\mu^{j_2}$ become respectively $c_{i_1j_1}\beta^{i_1}\mu^{\sigma'}$ and $c_{i_2j_2}\beta^{i_2}\mu^{\sigma'}$, where

$$\sigma' = i_1\sigma + j_1 = \frac{i_1 j_2 - i_2 j_1}{i_1 - i_2}. \quad (19)$$

If there is another degree point (i', j') on this segment its coördinates satisfy the equation

$$i'(j_2 - j_1) + j'(i_1 - i_2) + i_2 j_1 - i_1 j_2 = 0.$$

Hence after the transformation (18) the term $c_{i'j'}\alpha^{i'}\mu^{j'}$ becomes $c_{i'j'}\beta^{i'}\mu^{\sigma'}$.

It follows from the position of the segment (i_1, j_1) to (i_2, j_2) with reference to every other degree point, that in the case of any term $c_{ij}\, \alpha^i\, \mu^j$ of $P\,(\alpha, \mu)$ whose degree point is not on this segment the exponents i and j satisfy the inequality

$$i\,(j_2 - j_1) + j\,(i_1 - i_2) + i_2 j_1 - i_1 j_2 > 0.$$

Consequently, after the transformation (18) the term $c_{ij}\alpha^i\mu^j$ becomes $c_{ij}\beta^i\mu^{\sigma''}$, where

$$\sigma'' = \frac{i\,(j_2-j_1)+j\,(i_1-i_2)}{i_1-i_2} > \sigma'.$$

This discussion proves that after the transformation (18) the terms belonging to the segment (i_1, j_1) to (i_2, j_2) contain $\mu^{\sigma'}$ as a factor, and that every other term contains μ to a higher power than σ'. Since σ' is not necessarily an integer the series will not be, in general, a series in integral powers of μ, but it will be in integral powers of $\mu' = \mu^{\frac{1}{n}}$. Hence, dividing out $\mu^{\sigma'}$ the series becomes

$$0 = c_{i_1 j_1}\beta^{i_1} + \cdot\cdot\cdot + c_{i_2 j_2}\beta^{i_2} + \mu'\, P_1\,(\beta, \mu'). \tag{20}$$

For $\mu' = 0$ equation (20) becomes

$$c_{i_1 j_1}\beta^{i_2}\left[\beta^{i_1-i_2} + \cdot\cdot\cdot + \frac{c_{i_2 j_2}}{c_{i_1 j_1}}\right] = c_{i_1 j_1}\beta^{i_2}\,(\beta - c_1)\cdot\cdot\cdot(\beta - c_{i_1-i_2}) = 0. \tag{21}$$

The solution $\beta^{i_2} = 0$ is not to be considered for this transformation because it belongs to the solutions obtained from the segments having smaller values of i. Suppose $\beta - c_\nu$ is a simple factor of (21) and let

$$\beta = c_\nu + \gamma_\nu\,. \tag{22}$$

Then the right member of (20) becomes a power series in γ_ν and μ', vanishing with $\gamma_\nu = \mu' = 0$, and the coefficient of γ_ν to the first power is distinct from zero. Therefore, by §1, the equation can be solved for γ_ν uniquely as a converging power series in μ', vanishing for $\mu' = 0$. Then, on substituting back in (22) and (18), α is expressed in integral powers of μ'. This is an integral series in μ only if σ is an integer. If $c_1, \ldots, c_{i_1-i_2}$ are all distinct we obtain at this step $i_1 - i_2$ solutions, and if σ is an integer the number of them is precisely $i_1 - i_2$. Since when σ is not an integer μ' has more than one determination, and since the series obtained by the transformation (18) after removing the factor $\mu^{\sigma'}$ is not in integral powers of μ, it would seem that the number of solutions for the segment is greater than $i_1 - i_2$. But it will now be shown that the number of distinct solutions is $i_1 - i_2$, whether σ is an integer or not.

Now $\sigma = (j_2 - j_1)/(i_1 - i_2) = m/n$, where m and n are relatively prime integers. It is clear that $i_1 - i_2$ equals n, or is greater than n, according as $j_2 - j_1$ and $i_1 - i_2$ do not have, or have, a common integral divisor greater than unity. Consider first the case where $i_1 - i_2 = n$. There can be no degree point

(i', j') on the segment between (i_1, j_1) and (i_2, j_2), for its coördinates would have to satisfy the relation $(j_2 - j')/(i' - i_2) = m/n$, which is impossible when $i' - i_2 < n$. Therefore equation (21) becomes

$$\beta^{i_1 - i_2} + A = 0,$$

whose solution gives $i_1 - i_2$ values of β, differing only by the $i_1 - i_2$ roots of unity. Hence the same final results are obtained by using the principal value of μ^σ in (18) and all of these $i_1 - i_2$ values of β as are obtained if any other determination of μ^σ is used.

Suppose now $i_1 - i_2 = qn$, where q is an integer. There can not be more than $q - 1$ degree points on the segment (i_1, j_1) to (i_2, j_2), i. e. satisfying the relation $(j_2 - j')/(i' - i_2) = m/n$. Therefore in this case (21) is a polynomial in β^n of degree q, and for each of the solutions for β^n there are n values of β obtained one from the other by multiplying by the n^{th} roots of unity. Hence in this case also the final results are the same as are obtained by using any other of the n determinations of μ^σ.

It follows that in every case all the distinct solutions are obtained by taking the principal value of μ', and that the number of them for the segment (i_1, j_1) to (i_2, j_2) is precisely $i_1 - i_2$. In a similar manner the solutions associated with each of the other segments can be obtained. The whole number of solutions found in this way is

$$N = (k - i_1) + (i_1 - i_2) + \cdots + (i_\nu - 0) = k, \tag{23}$$

which is the number of solutions of $(P\, \alpha, \mu) = 0$ for α which vanish with $\mu = 0$. In this case the problem is completely solved.

Suppose, however, that in treating the terms belonging to the segment (i_1, j_1) to (i_2, j_2) it is found that $c_1 = \cdots = c_p$. The analysis above fails to give the solutions for these roots. In this case the transformation

$$\beta = c_1 + \gamma \tag{24}$$

is made, after which the right member of (20) is a power series in γ and μ'; and for $\mu' = 0$, $\gamma^p = 0$ is a solution. That is, the equation is of the same form as $P(\alpha, \mu) = 0$, only in place of having k zero roots for $\mu = 0$ there are now only p such roots. This number p is always less than k except when $i_1 = k$, $j_1 = 0$, $i_2 = 0$, $j_2 = \lambda$, and $c_1 = c_2 = \cdots = c_{i_1 - i_2}$. But whatever the value of p a new Newton's parallelogram for the $\gamma\mu'$-equation is to be constructed. It will depend upon terms of higher degree in the original $\alpha\mu$-equation because the terms which gave rise to the p equal roots, $c_1, \ldots, c_p$, have been concentrated, so to speak, by the transformations into the single one γ^p, and the parallelogram depends upon the term in μ' alone of lowest degree. By this step, or some succeeding one, the solutions will all become distinct unless, indeed, the original $P(\alpha, \mu) = 0$ has two or more solutions for α which are identical in μ.

II. SOLUTIONS OF DIFFERENTIAL EQUATIONS AS POWER SERIES IN PARAMETERS.

8. The Types of Equations Treated.—In the course of this work certain types of differential equations will arise and they will be solved by processes adapted to attaining their solutions in convenient forms. It will tend to clearness and brevity of exposition of the actual dynamical problems to set down in advance those methods of solving differential equations which will be used, and to state the conditions under which the results obtained by them are valid. Consequently, this section will be devoted to these questions without making here any applications to physical problems.

The equations which will be treated are characterized chiefly by being analytic in the independent and dependent variables and in certain parameters upon which they depend; and the solutions are considered only for those values of the variables and parameters for which the equations are all regular. In the case where the differential equations are linear, their coefficients are either constants or periodic functions of the independent variable.

9. Formal Solution of Differential Equations of Type I.*—The differential equations

$$\frac{dx_i}{dt} = \mu f_i(x_1, \ldots, x_n, \mu; t) \qquad (i=1, \ldots, n) \tag{25}$$

will be said to be of the Type I when the right members have μ as a factor and when all the f_i are analytic in $x_1, \ldots, x_n$, μ and t, and are regular at the point $x_i = a_i$, $\mu = 0$, for all $t_0 \gtreqless t \leqq T$. Then the $f_i(x_1, \ldots, x_n, \mu; t)$ can be expanded as power series in $(x_i - a_i)$ and μ which will converge if $|x_i - a_i| < r_i > 0$ and $|\mu| < \rho > 0$ for $t_0 \gtreqless t \leqq T$.

Suppose $x_i = a_i$ at $t = t_0$, whatever be the value of μ. That is, suppose

$$x_i(t_0) \underset{\mu}{\equiv} a_i, \tag{26}$$

in which the letter under the identity sign $\equiv$ indicates the parameter in which the identity is defined.

Equations (25) can be solved formally as power series in μ which have the form

$$x_i = \sum_{j=0}^{\infty} x_i^{(j)} \mu^j, \tag{27}$$

where the $x_i^{(j)}$ are functions of t. On substituting (27) in (25) and arranging in powers of μ, it is found that

$$\left.\begin{aligned} &\sum_{j=0}^{\infty} \frac{dx_i^{(j)}}{dt} \mu^j = f_i(x_k^{(0)}, 0; t)\,\mu + \left[\sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k} x_k^{(1)} + \frac{\partial f_i}{\partial \mu}\right]\mu^2 \\ &+\left[\sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k} x_k^{(2)} + \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^2 f_i}{\partial x_k \partial x_l} x_k^{(1)} x_l^{(1)} + \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial \mu \partial x_k} x_k^{(1)} + \frac{1}{2}\frac{\partial^2 f_i}{\partial \mu^2}\right]\mu^3 + \cdots \end{aligned}\right\} \tag{28}$$

*See Moulton's *Introduction to Celestial Mechanics*, pp. 264–272.

If these series are convergent the coefficients of corresponding powers of μ in the right and left members are equal. On assuming for the moment that they are convergent, the identity relations become

$$\frac{dx_i^{(0)}}{dt} = 0 \qquad (i=1, \ldots, n), \tag{29}$$

$$\frac{dx_i^{(1)}}{dt} = f_i\,(x_j^{(0)}, 0; t) \qquad (j=1, \ldots, n), \tag{30}$$

$$\frac{dx_i^{(2)}}{dt} = \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}\, x_k^{(1)} + \frac{\partial f_i}{\partial \mu}, \tag{31}$$

$$\frac{dx_i^{(3)}}{dt} = \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}\, x_k^{(2)} + \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f_i}{\partial x_k \partial x_l}\, x_k^{(1)} x_l^{(1)} + \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial \mu \partial x_k}\, x_k^{(1)} + \frac{1}{2} \frac{\partial^2 f_i}{\partial \mu^2}, \tag{32}$$

. . . .

These sets of equations can be integrated sequentially. From (29) we get

$$x_i^{(0)} = a_i^{(0)}, \tag{33}$$

where the $a_i^{(0)}$ are constants of integration. The right members of (30) are now known functions of t, and their solutions can be written

$$x_i^{(1)} = \int^t f_i\,(a_j^{(0)}, 0; t)\, dt + a_i^{(1)} = F_i^{(1)}(t) + a_i^{(1)}, \tag{34}$$

where $F_i^{(1)}(t)$ is the primitive of $f_i\,(a_j^{(0)}, 0; t)$. Then (31), (32), $\cdots$ give in order, similarly,

$$x_i^{(2)} = F_i^{(2)}(t) + a_i^{(2)}, \qquad x_i^{(3)} = F_i^{(3)}(t) + a_i^{(3)}, \qquad \cdots\cdots\cdots \tag{35}$$

In this manner the process can be continued as far as may be desired.

10. Determination of the Constants of Integration in Type I.—At each step n additive constants of integration are obtained, and they must be determined in terms of the initial values of the x_i. From (26), (27), (33), (34), (35), . . . , it follows that

$$a_i^{(0)} + \sum_{j=1}^{\infty} [F_i^{(j)}(t_0) + a_i^{(j)}]\, \mu^j \underset{\mu}{\equiv} a_i\,.$$

Therefore

$$a_i^{(0)} = a_i\,, \qquad a_i^{(j)} = -\,F_i^{(j)}(t_0) \qquad (j=1, \ldots \infty). \tag{36}$$

By these equations all of the constants of integration are uniquely determined in terms of the constants of the differential equations and of the initial values of the dependent variables.

11. Proof of the Convergence of the Solutions of Type I.—The method of integrating differential equations as power series in parameters has been in use in more or less explicit form since almost the beginnings of celestial mechanics. For example, in the year 1772 Euler published his

second Lunar Theory, in which he used a process quite analogous to this;* and the method of computing the absolute perturbations of the elements of the planetary orbits is virtually that of developing the solutions as power series in the masses. But the actual determination of the conditions for the validity of the process was not made until Cauchy published his celebrated memoirs on differential equations in 1842.† The results of Cauchy were extended by Poincaré in his prize memoir on the Problem of Three Bodies,‡ and were proved again, following Cauchy's *Calcul des Limites*, in *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, pp. 58–63. The theorem will be needed in this work in the form given by Poincaré, viz.:

If the $f_i(x_1, \ldots, x_n, \mu; t)$ of equations (25) *are analytic* || *in $x_1, \ldots, x_n, \mu$, and t, and regular at $x_i = a_i$, $\mu = 0$, for all $t_0 \gtreqless t \leqq T$, then $\rho > 0$ can be taken so small that the series* (27) *will converge for all $t_0 \gtreqless t \leqq T$ provided $|\mu| < \rho$.*

To prove this theorem consider a comparison set of differential equations

$$\frac{dy_i}{dt} = \mu\,\varphi_i(y_1, \ldots, y_n, \mu; t) \qquad (i=1, \ldots, n), \quad (25')$$

where the φ_i are analytic in $y_1, \ldots, y_n, \mu, t$, and regular at $y_i = |a_i| = b_i$, $\mu = 0$ for all $t_0 \gtreqless t \leqq T$; and where, further, the coefficients of all powers of $y_i - b_i$ and μ in the expansions of all the φ_i are real, positive, and greater than the moduli of the corresponding coefficients in the expansions of the f_i for all the values of t under consideration. Then positive constants $M, \rho, r_1, \ldots, r_n$ exist such that equations (25′) can be written in the form§

$$\frac{dy_i}{dt} = \frac{M\mu}{\left(1-\frac{\mu}{\rho}\right)\left(1-\frac{y_1-b_1}{r_1}\right) \cdots \left(1-\frac{y_n-b_n}{r_n}\right)}.$$

The conditions are evidently satisfied also by

$$\frac{dy_i}{dt} = \frac{M\mu}{\left(1-\frac{\mu}{\rho}\right)\left[1-\frac{(y_1-b_1)+(y_2-b_2)+\cdots+(y_n-b_n)}{r}\right]}, \qquad (25'')$$

where r is the smallest of $r_1, \ldots, r_n$.

Suppose now the solutions of equations (25″) are developed as power series in μ of the form

$$y_i = \sum_{j=0}^{\infty} y_i^{(j)} \mu^j. \qquad (27')$$

There will be quadratures corresponding to (29), (30), ... Moreover, by virtue of the hypotheses on the φ_i,

$$\frac{dy_i^{(1)}}{dt} \geqq \left|\frac{dx_i^{(1)}}{dt}\right|$$

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* Tisserand's *Mécanique Céleste*, vol. III, chapter 6.

† See Cauchy's *Collected Works*, 1st series, vol. VII.

‡ *Acta Mathematica*, vol. XIII, pp. 5–266.

|| The assumption that the f_i are analytic in t is not necessary for the demonstration.

§ Picard's *Traité d'Analyse*, edition of 1905, vol. II, pp. 255–260.

for all $t_0 \gtreqless t \leqq T$. Therefore it follows that $y_i^{(1)} \gtreqless |x_i^{(1)}|$ for all $t_0 \gtreqless t \leqq T$. Then it is seen from the form of (31) that

$$\frac{dy_i^{(2)}}{dt} \gtreqless \left|\frac{dx_i^{(2)}}{dt}\right|$$

for $t_0 \gtreqless t \leqq T$. From this it follows similarly that $y_i^{(2)} \gtreqless |x_i^{(2)}|$. This process can be continued indefinitely, giving by induction for the general term

$$y_i^{(j)} \gtreqless |x_i^{(j)}| \tag{37}$$

for $t_0 \gtreqless t \leqq T$. Consequently, if the right members of (27') are convergent series when $|\mu| < \rho > 0$ for $t_0 \gtreqless t \leqq T$, then likewise are the right members of (27) convergent when $|\mu| < \rho > 0$ for the same range in t.

It is a simple matter to find the explicit expression for (27') by a direct integration of (25''). Since the right members are the same for all i, we have

$$y_1 - c_1 = y_2 - c_2 = \cdot \cdot \cdot = y_n - c_n,$$

where $c_1, \ldots, c_n$ are constants of integration. By the initial conditions it follows that $c_i - c_j = b_i - b_j$ and $y_i - b_i = y_j - b_j$. Let this common value of $y_i - b_i$ be $y - b$. Then each equation of (25'') becomes

$$\frac{dy}{dt} = \frac{M\mu}{\left(1-\frac{\mu}{\rho}\right)\left(1-\frac{n(y-b)}{r}\right)}. \tag{25'''}$$

On integrating this equation and determining the constant of integration by the condition that $(y-b)=0$ at $t=t_0$, it is found that

$$\frac{n}{2r}(y-b)^2 - (y-b) + \frac{M\mu}{\left(1-\frac{\mu}{\rho}\right)}(t-t_0) = 0.$$

The solution of this equation for $(y-b)$ is

$$(y-b) = \frac{r}{n} \pm \frac{r}{n}\sqrt{1 - \frac{2nM\mu}{r\left(1-\frac{\mu}{\rho}\right)}(t-t_0)}. \tag{38}$$

Since $(y-b)=0$ at $t=t_0$, the negative sign must be taken before the radical.

It follows directly from equation (38) that whatever finite values n, M, r, ρ, and $T-t_0$ may have, $(y-b)$ can be expanded as a convergent series in μ for $t_0 \gtreqless t \leqq T$ provided the condition

$$\frac{2nM|\mu|}{r\left(1-\frac{\mu}{\rho}\right)}(T-t_0) < 1$$

is satisfied. This condition imposes the explicit limitation

$$|\mu| < \frac{1}{\frac{2nM(T-t_0)}{r} + \frac{1}{\rho}} \tag{39}$$

upon μ, which can always be satisfied by $|\mu|<\mu_0>0$ for $r>0$, $\rho>0$, and for M and $T-t_0$ finite. If these conditions are satisfied, the resulting expression for $(y-b)$ substituted in (25‴) leads to convergent series. Moreover, the series for $(y-b)$ satisfies (25″) and is identical with (27′) since (27′) is unique. Consequently (27′) converges, and therefore also (27) if $|\mu|<\mu_0$, where μ_0 is the limiting value of μ satisfying the inequality (39), for all t in the range $t_0 \gtreqless t \leqq T$. The theorem is thus established.

12. Generalization to Many Parameters.—The differential equations may involve many parameters, $\mu_1, \ldots, \mu_k$, instead of a single parameter μ. The f_i are supposed to be regular for $\mu_1=\mu_2=\cdots=\mu_k=0$ for $t_0 \gtreqless t \leqq T$. The discussion can be thrown upon the preceding case by letting

$$\mu_i=\beta_i\,\mu \qquad (i=1, \ldots, k).$$

After the solutions have been found $\beta_i\,\mu$ can be everywhere replaced by μ_i. This groups the terms of the same degree in $\mu_1, \ldots, \mu_k$ together.

The equations can also be integrated as multiple series in the parameters $\mu_1, \ldots, \mu_k$ without the use of this artifice, and then the constants of integration can be determined and the convergence proved. But the method is not essentially distinct from the other, and the details may be omitted.

13. Generalization of the Parameter.—Suppose the differential equations depend upon a single parameter μ. It may happen that this parameter enters in two distinct ways. For example, it may enter in one way so that, so far as this way alone is concerned, the f_i can be expanded very simply as power series in μ. It may enter in another way so that, so far as this way alone is concerned, the expansions of the f_i as power series in μ are very complex, or even impossible without throwing the equations into an undesirable form.

Under the circumstances thus described it is sometimes of the highest importance to generalize the parameter. Where it enters in the first way it is left simply as the parameter μ. Where it enters in the second way it is replaced by m to preserve the distinction. In forming the solutions μ is regarded as a variable parameter in terms of which identity arguments are made, while m is regarded simply as a fixed number. The solutions obtained are valid mathematically for any value of μ whose modulus is sufficiently small, but they belong to the original (physical) problem for only one particular value of μ, viz., for $\mu=m$. But it will be observed that when the differential equations are regular for a continuous range of values of m this restriction is of no importance, provided the solutions converge for $\mu=m$, if the literal value of m has been retained in the solutions.*

*For a practical application of this artifice see Moulton's *Introduction to Celestial Mechanics*, pp. 264–5.

14. Formal Solution of Differential Equations of Type II.—The differential equations

$$\frac{dx_i}{dt} = g_i(x_1, \ldots, x_n; t) + \mu f_i(x_1, \ldots, x_n, \mu; t) \qquad (i=1, \ldots, n), \quad (40)$$

will be said to be of Type II if

(a) the $g_i(x_1, \ldots, x_n; t)$ are independent of μ and not identically zero;

(b) the $g_i(x_1, \ldots, x_n; t)$ and $f_i(x_1, \ldots, x_n, \mu; t)$ are analytic* in $x_1, \ldots, x_n, \mu$, and t;

(c) the $g_i(x_1, \ldots, x_n; t)$ and $f_i(x_1, \ldots x_n, \mu; t)$ are regular at $x_i = x_i^{(0)}(t)$, $\mu=0$, for $t_0 \gtreqless t \leqq T$, where the $x_i^{(0)}$ are the solutions of equations (40) for $\mu=0$, and $x_i^{(0)} = a_i$ at $t=t_0$.

It follows from these conditions that the g_i and f_i can be expanded as power series in $(x_i - x_i^{(0)})$ and μ, which converge if $|x_i - x_i^{(0)}| < r_i > 0$ and $|\mu| < \rho > 0$ for all t in the range $t_0 \gtreqless t \leqq T$.

Equations (40) can be solved formally as power series in μ having the form

$$x_i = \sum_{j=0}^{\infty} x_i^{(j)} \mu^j, \quad (41)$$

where the $x_i^{(j)}$ are functions of t, and where

$$x_i(t_0) \underset{\mu}{\equiv} a_i. \quad (42)$$

Upon substituting (41) in (40) and equating coefficients of corresponding powers of μ, it is found that

$$\frac{dx_i^{(0)}}{dt} = g_i(x_1^{(0)}, \ldots, x_n^{(0)}; t) \qquad (i=1, \ldots, n), \quad (43)$$

$$\frac{dx_i^{(1)}}{dt} - \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_j^{(1)} = f_i(x_1^{(0)}, \ldots, x_n^{(0)}, 0; t), \quad (44)$$

$$\frac{dx_i^{(2)}}{dt} - \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_j^{(2)} = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 g_i}{\partial x_j \partial x_k} x_j^{(1)} x_k^{(1)} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} x_j^{(1)} + \frac{\partial f_i}{\partial \mu}, \quad (45)$$

. . . .

$$\frac{dx_i^{(k)}}{dt} - \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_j^{(k)} = \Phi_i^{(k)}(x_j^{(0)}, \ldots, x_j^{(k-1)}; t), \quad (46)$$

. . . .

where the $\Phi_i^{(k)}$ are linear in the coefficients of the expansions of g_i and f_i and polynomials in $x_j^{(0)}, \ldots, x_j^{(k-1)}$. In all the partial derivatives the x_j are replaced by $x_j^{(0)}$.

The solutions of equations (43) are

$$x_i^{(0)} = F_i^{(0)}(c_1, \ldots, c_n; t), \quad (47)$$

*The assumption that the g_i and f_i are analytic in t is not necessary, but is made for simplicity because the condition is always fulfilled in the applications which follow.

where $c_1, \ldots, c_n$ are the constants of integration which can be determined in terms of the a_i. Substituting these $x_i^{(0)}$ in (44) and integrating, we obtain

$$x_i^{(1)} = \sum_{j=1}^{n} A_j^{(1)} \varphi_{ij}(t) + F_i^{(1)}(t), \tag{48}$$

where the $A_j^{(1)}$ are the constants of integration. After these solutions are found, equations (45) can be integrated, and this process can be continued to the k^{th} step, which gives

$$x_i^{(k)} = \sum_{j=1}^{n} A_j^{(k)} \varphi_{ij}(t) + F_i^{(k)}(t). \tag{49}$$

The $\varphi_{ij}(t)$ belonging to the complementary function are the same for each step, but the $F_i^{(k)}(t)$, which depend upon the right members of the differential equation, are in general all different. The problem of finding the $F_i^{(0)}$, the φ_{ij}, and the $F_i^{(k)}$ depends upon the explicit form of the differential equations, and can not be given a general treatment.

15. Determination of the Constants of Integration in Type II.—At each step there are n constants of integration introduced which can be determined in terms of the initial values of the x_i. It follows from equations (41), (42), (47), (48), . . . , that

$$F_i^{(0)}(c_1, \ldots, c_n; t_0) + \sum_{k=1}^{\infty} \left[\sum_{j=1}^{n} A_j^{(k)} \varphi_{ij}(t_0) + F_i^{(k)}(t_0) \right] \mu^k \underset{\mu}{\equiv} a_i. \tag{50}$$

Hence

$$\left. \begin{aligned} &F_i^{(0)}(c_1, \ldots, c_n; t_0) = a_i \qquad && (i=1, \ldots, n), \\ &\sum_{j=1}^{n} A_j^{(k)} \varphi_{ij}(t_0) = -F_i^{(k)}(t_0) && (k=1, \ldots \infty), \end{aligned} \right\} \tag{51}$$

Suppose the constants $c_1, \ldots, c_n$ are uniquely determined in terms of a_i by the first set of equations of (51). Then the $F_i^{(1)}$ become completely defined, and from the second set of (51) the $A_j^{(1)}$ are uniquely determined since the determinant $\Delta = |\varphi_{ij}(t_0)|$ is the determinant of a fundamental set of solutions at a regular point of the differential equations and is therefore not zero (§18). Then the $F_j^{(2)}$ become entirely known and the $A_j^{(2)}$ are determined by a similar set of linear equations whose determinant is the same Δ. The whole process is unique and can be continued indefinitely.

16. Proof of the Convergence of the Solutions of Type II.—Consider the comparison set of differential equations

$$\frac{dy_i}{dt} = \varphi_i(y_1, \ldots, y_n; t) + \mu \psi_i(y_1, \ldots, y_n, \mu; t), \tag{40'}$$

where the conditions corresponding to (a), (b), (c), and (42) of §14 are satisfied, and where, in addition, the coefficients of the expansions of the φ_i and the ψ_i as power series in $(y_i - y_i^{(0)})$ and μ are real, positive, and greater than the moduli of the corresponding coefficients in the expansions of the g_i and the f_i for all t in the interval $t_0 \gtreqless t \leqq T$. Suppose $y_i = |a_i| = b_i$ at $t = t_0$.

Equations (40′) will be solved in the form

$$y_i = \sum_{j=0}^{\infty} y_i^{(j)} \mu^j, \tag{42'}$$

where the $y_i^{(j)}$ are functions of t to be determined. It will be shown that the $y_i^{(j)}$ are real and positive, and that

$$y_i^{(j)} > |x_i^{(j)}| \text{ for } t_0 \leqq t \gtreqless T. \tag{52}$$

The $x_i^{(0)}$ and $y_i^{(0)}$ are defined by

$$\left.\begin{aligned} x_i^{(0)} &= a_i + \int_{t_0}^{t} g_i\,(x_1^{(0)}, \ldots, x_n^{(0)}; t)\, dt, \\ y_i^{(0)} &= b_i + \int_{t_0}^{t} \varphi_i\,(y_1^{(0)}, \ldots, y_n^{(0)}; t)\, dt. \end{aligned}\right\} \tag{53}$$

Since, by hypothesis, the integrands of the second set of equations are real, positive, and greater than the maximum values of the moduli of the integrands in the first set of equations in the interval $t_0 \gtreqless t \leqq T$, it follows that in the whole interval $y_i^{(0)} > |x_i^{(0)}|$.

The $x_i^{(1)}$ and $y_i^{(1)}$ are defined by

$$\left.\begin{aligned} \frac{dx_i^{(1)}}{dt} &= \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_j^{(1)} + f_i\,(x_1^{(0)}, \ldots, x_n^{(0)}, 0; t), \\ \frac{dy_i^{(1)}}{dt} &= \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial y_j} y_j^{(1)} + \psi_i\,(y_1^{(0)}, \ldots, y_n^{(0)}, 0; t). \end{aligned}\right\} \tag{54}$$

It follows from (48) and (51) that $x_j^{(1)} = 0$ at $t = t_0$. Similarly $y_j^{(1)} = 0$ at $t = t_0$. Equations (54) can be solved by Picard's approximation process.* Let $x_{ik}^{(1)}$ and $y_{ik}^{(1)}$ be the k^{th} approximations to $x_i^{(1)}$ and $y_i^{(1)}$. Then

$$\left.\begin{aligned} x_{i1}^{(1)} &= \int_{t_0}^{t} f_i\,(x_1^{(0)}, \ldots, x_n^{(0)}, 0; t)\, dt \qquad (i=1, \ldots, n), \\ y_{i1}^{(1)} &= \int_{t_0}^{t} \psi_i\,(y_1^{(0)}, \ldots, y_n^{(0)}, 0; t)\, dt, \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} x_{i2}^{(1)} &= \int_{t_0}^{t} \Big[\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_{j1}^{(1)} + f_i\,(x_1^{(0)}, \ldots, x_n^{(0)}, 0; t) \Big] dt, \\ y_{i2}^{(1)} &= \int_{t_0}^{t} \Big[\sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial y_j} y_{j1}^{(1)} + \psi_i\,(y_1^{(0)}, \ldots, y_n^{(0)}, 0; t) \Big] dt, \end{aligned}\right\} \tag{56}$$

. . . .

$$\left.\begin{aligned} x_{ik}^{(1)} &= \int_{t_0}^{t} \Big[\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} x_{jk-1}^{(1)} + f_i\,(x_1^{(0)}, \ldots, x_n^{(0)}, 0; t) \Big] dt, \\ y_{ik}^{(1)} &= \int_{t_0}^{t} \Big[\sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial y_j} y_{jk-1}^{(1)} + \psi_i\,(y_1^{(0)}, \ldots, y_n^{(0)}, 0; t) \Big] dt, \end{aligned}\right\} \tag{57}$$

. . . .

*_Traité d'Analyse_, vol. II, edition of 1905, p. 340.

It follows from (55) and the relations between the f_i and the ψ_i that $y_{i1}^{(1)} > |x_{i1}^{(1)}|$ for $t_0 \gtreqless t \leqq T$. Then, making use of the relations between the coefficients of the expansions of the g_i and the φ_i, it follows from (56) that $y_{i2}^{(1)} \geqq |x_{i2}^{(1)}|$ for $t_0 \gtreqless t \leqq T$; and from the method of forming the successive approximations it is seen that, $y_{ik}^{(1)} > |x_{ik}^{(1)}|$ for $t_0 \gtreqless t \leqq T$, for all k.

Now Picard has shown* that $\lim_{k=\infty} x_{ik}^{(1)} = x_i^{(1)}$ for a sufficiently restricted range of values of t. But equations (44) being linear, the range of values for t is precisely that for which the differential equations are valid.† Therefore we conclude that $y_i^{(1)} > |x_i^{(1)}|$ for $t_0 \gtreqless t \leqq T$. The corresponding relation between $y_i^{(2)}$ and $x_i^{(2)}$ can be proved in the same manner, and the process can be continued step by step indefinitely. Consequently the inequalities (52) are established. Hence, if the series (42′) converge when $|\mu| < \rho'$, then the series (42) also converge when $|\mu| < \rho'$ for $t_0 \gtreqless t \leqq T$.

Since

$$\frac{dy_i^{(0)}}{dt} = \varphi_i(y_1^{(0)}, \ldots, y_n^{(0)};\ t),$$

it follows from the reference given in §11 that the conditions imposed upon (40′) can be satisfied by equations of the form‡

$$\frac{d\,(y_i - y_i^{(0)})}{dt} = M\,\frac{\left[\sum_{j=1}^{n}\frac{(y_j - y_j^{(0)})}{r} + \mu\right]\left[1 + \sum_{j=1}^{n}\frac{(y_j - y_j^{(0)})}{r} + \mu\right]}{\left(1 - \frac{\mu}{\rho}\right)\left\{1 - \left[\sum_{j=1}^{n}\frac{(y_j - y_j^{(0)})}{r} + \mu\right]\right\}}. \tag{58}$$

As a consequence of these equations $(y_i - y_i^{(0)}) = (y_j - y_j^{(0)}) + c_j$, where the c_j are constants. Since $y_i = y_i^{(0)}$ at $t = t_0$, it follows that $c_j = 0$. Now let

$$z = \sum_{j=1}^{n}\frac{(y_j - y_j^{(0)})}{r} + \mu. \tag{59}$$

Then, upon taking the sum of equations (58) with respect to i, we get

$$\frac{dz}{dt} = \frac{M\,n\,z\,(1+z)}{r\left(1 - \frac{\mu}{\rho}\right)(1-z)}. \tag{60}$$

On integrating this equation and determining the constant of integration by the condition that $z = \mu$ at $t = t_0$, it is found that

$$\log\frac{(1+\mu)^2}{\mu\,(1+z)^2} = \frac{M\,n\,(t - t_0)}{r\left(1 - \frac{\mu}{\rho}\right)}. \tag{61}$$

*Loc. cit. †*Traité d'Analyse*, vol. III, edition of 1894, p. 91.

‡See *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 60.

Solving this equation and determining the sign of the radical so that $z = \mu$ at $t = t_0$, the expression for z becomes

$$z = \frac{1 - \frac{2\mu}{(1+\mu)^2} e^{K(t-t_0)} - \sqrt{1 - \frac{4\mu}{(1+\mu)^2} e^{K(t-t_0)}}}{\frac{2\mu}{(1+\mu)^2} e^{K(t-t_0)}}, \tag{62}$$

where

$$K = \frac{Mn}{r\left(1 - \frac{\mu}{\rho}\right)}.$$

It follows from (62) that if $|\mu| < \rho$, $|\mu| < 1$, and $\left|\frac{4\mu e^{K(T-t_0)}}{(1+\mu)^2}\right| < 1$, then z can be expanded as a converging power series in μ for $t_0 \gtreqless t \leqq T$, and that in this range for t the values of z are such that the expansion of the right member of (60) as a power series in z also converges. Consequently, the y_i and x_i satisfying (40′) and (40) respectively can also be expanded as converging series in μ for all t in the interval $t_0 \gtreqless t \leqq T$.

The point to be noted in these results is that when the differential equations are of the Types I or II, as defined above, and when the interval $T - t_0$ has been chosen in advance and kept fixed, then the parameter μ, in which the solutions are developed, can be taken so small in absolute value that the series in which the solutions are expressed will all converge in the whole interval $t_0 \gtreqless t \leqq T$.

As in equations of Type I, there may be many parameters, $\mu_1, \mu_2, \ldots, \mu_k$, instead of a single one. The treatment can be reduced to the case of the single one, just as in the preceding case.

The parameter can be generalized precisely as was explained in §13. It is obvious that if there are many parameters they may all be generalized. Since the generalization can be made in an infinite number of ways, a great variety of possible expansions for these solutions is secured.

17. Case of Homogeneous Linear Equations.—While the linear equations are included in those already treated, they deserve some special attention for the reason that in their solutions the values of μ are not restricted by so many conditions. Consider the equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \theta_{ij}(t)\, x_j \qquad (i=1, \ldots, n), \tag{63}$$

where the θ_{ij} are expansible as power series in μ of the form

$$\theta_{ij} = \sum_{k=0}^{\infty} \theta_{ij}^{(k)} \mu^k,$$

which converge if $|\mu| < \rho$ for $t_0 \gtreqless t \leqq T$. Suppose $x_i = a_i$ at $t = t_0$. Then the solutions can be developed as power series of the form

$$x_i = \sum_{k=0}^{\infty} x_i^{(k)} \mu^k, \tag{64}$$

precisely as in §14.

To find the realm of convergence in μ of (64), consider a comparison set of differential equations

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \psi_{ij}(t)\, y_j, \tag{63'}$$

where the ψ_{ij} are expansible as power series in μ of the form

$$\psi_{ij} = \sum_{k=0}^{\infty} \psi_{ij}^{(k)} \mu^k,$$

which converge provided $|\mu| < \rho$ for $t_0 \gtreqless t \leqq T$. Suppose also that $\psi_{ij}^{(k)} > |\theta_{ij}^{(k)}|$ for $t_0 \gtreqless t \leqq T$. Develop the solutions of (63′) in the form

$$y_i = \sum_{k=0}^{\infty} y_i^{(k)} \mu^k. \tag{64'}$$

It can be shown by the method used in proving the inequalities given in (52) that if $y_i(t_0) = |a_i| = b_i$, then $y_i^{(k)} > |x_i^{(k)}|$ for $t_0 \gtreqless t \leqq T$.

The conditions imposed on (63′) are satisfied by the equations

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{M}{1 - \frac{\mu}{\rho}}\, y_j, \tag{63''}$$

in which M is the maximum value of the $|\theta_{ij}|$ for $t_0 \gtreqless t \leqq T$. It follows that $(y_i - b_i) = (y_j - b_j)$. Let the common value be $(y - b)$. Then (63″) becomes

$$\frac{d(y-b)}{dt} = \frac{nM}{1 - \frac{\mu}{\rho}}(y - b) + \frac{M}{1 - \frac{\mu}{\rho}} \sum_{j=1}^{n} b_j.$$

The solution of this equation satisfying the initial conditions is

$$(y - b) = \frac{(e^{K(t-t_0)} - 1)}{n} \sum_{j=1}^{n} b_j, \qquad K = \frac{nM}{1 - \frac{\mu}{\rho}}. \tag{64''}$$

Hence y, and therefore y_i and x_i, can be expanded as a power series in μ converging for $|\mu| < \rho$ for $t_0 \gtreqless t \leqq T$. That is, *when the differential equations are linear the realm of convergence of the solutions in the parameter μ is precisely the same as that of the coefficients of the differential equations.* Therefore, in those simple cases in which the original equations are polynomials in μ, the solutions converge for all finite values of μ.*

*See *Mémoire sur les Groupes des Équations Linéaires*, by Poincaré, *Acta Mathematica*, vol. IV, p. 212.

III. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS.

18. The Determinant of a Fundamental Set of Solutions.—Suppose

$$x_i' = \sum_{j=1}^{n} \theta_{ij}(t)\, x_j \qquad (i=1, \ldots, n), \tag{65}$$

where x_i' is the derivative of x_i with respect to t, is the set of linear homogeneous differential equations under consideration, and let

$$x_{i1} = \varphi_{i1}(t),\ x_{i2} = \varphi_{i2}(t),\ \ldots,\ x_{in} = \varphi_{in}(t) \qquad (i=1, \ldots, n), \tag{66}$$

be a fundamental set of its solutions. The determinant of this set of solutions may be denoted by

$$\Delta = |\varphi_{ij}|. \tag{67}$$

It will be shown that Δ can not vanish for any t for which the θ_{ii} are all regular. In the applications which follow, the θ_{ij} are analytic in t and in general regular for all finite values of t.

The result of differentiating Δ with respect to t is

$$\Delta' = \sum_{k=1}^{n} \overset{k}{|\varphi_{ij}|},$$

where the index k denotes that in the k^{th} column the φ_{ij} are replaced by the derivatives of the φ_{ik} with respect to t. But it follows from (65) that

$$\varphi_{ik}' = \sum_{j=1}^{n} \theta_{ij}\varphi_{jk}.$$

Hence Δ' can be written

$$\Delta' = \sum_{k=1}^{n} \begin{vmatrix} \varphi_{11}, & \varphi_{12}, & \ldots, & \sum_{j=1}^{n}\theta_{1j}\varphi_{jk}, & \ldots, & \varphi_{1n} \\ \varphi_{21}, & \varphi_{22}, & \ldots, & \sum_{j=1}^{n}\theta_{2j}\varphi_{jk}, & \ldots, & \varphi_{2n} \\ \varphi_{31}, & \varphi_{32}, & \ldots, & \sum_{j=1}^{n}\theta_{3j}\varphi_{jk}, & \ldots, & \varphi_{3n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \varphi_{n1}, & \varphi_{n2}, & \ldots, & \sum_{j=1}^{n}\theta_{nj}\varphi_{jk}, & \ldots, & \varphi_{nn} \end{vmatrix}. \tag{68}$$

The n determinants (68) can be expanded according to the elements θ_{ij}. The result is

$$\Delta' = \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_{ij} \sum_{k=1}^{n} (-1)^{k+i}\varphi_{jk}\Delta_{ik},$$

where Δ_{ik} is the minor of the element φ_{ik} in Δ. But it is known from the theory of determinants that $\sum_{k=1}^{n} (-1)^{k+i} \varphi_{jk} \Delta_{ik}$ is zero when $j \neq i$, and equal to Δ when $j=i$. Therefore*

$$\Delta' = \Delta \sum_{i=1}^{n} \theta_{ii},$$

whence

$$\Delta = \Delta_0 e^{\int_0^t \sum_{i=1}^{n} \theta_{ii} dt}, \tag{69}$$

where Δ_0 is the value of Δ at $t=0$. The initial conditions are taken so that $\Delta_0 \neq 0$. Thus Δ can vanish or become infinite only at the singularities of the coefficients of the main diagonal of the differential equations.

If $\sum_{i=1}^{n} \theta_{ii} = 0$ the exponent vanishes and the determinant reduces to the constant Δ_0. If the differential equations were originally of the second order, having the form usually arising in celestial mechanics

$$x_i'' = \sum_{j=1}^{n} \theta_{ij} x_j \qquad (i=1, \ldots, n),$$

they are equivalent to the system

$$x_i' = y_i, \qquad y_i' = \sum_{j=1}^{n} \theta_{ij} x_j \qquad (i=1, \ldots, n),$$

which has the form of equations (65). Since every θ_{ii} of this set of equations is zero, the determinant of any fundamental set of their solutions is a constant.

19. The Character of the Solutions of a Set of Linear Homogeneous Differential Equations with Uniform Periodic Coefficients.—Linear differential equations with simply periodic coefficients were first treated by Hill† in one of his celebrated memoirs on the lunar theory. About the same time Hermite‡ discovered the form of the solution of Lamé's equation, which has a doubly periodic coefficient. Starting from the results obtained by Hermite, Picard‖ showed that *in general* a fundamental set of solutions of a linear differential equation of the n^{th} order with doubly periodic coefficients of the first kind can be expressed by means of doubly periodic functions of the second kind. In 1883, Floquet§ published a complete discussion of the character of the solutions of a linear differential equation of the n^{th} order which has simply periodic coefficients. In this memoir Floquet gave not only the form of the solutions *in general*, but he considered in detail the forms of the solutions when the fundamental equation has multiple roots. The forms of the solutions being thus known, the efforts of later writers have been directed

*Equation (69) was first developed by Jacobi in a somewhat different connection, *Collected Works*, vol. IV, p. 403.

†*The Collected Works of* G. W. Hill, vol. I, p. 243; *Acta Mathematica*, vol. VIII, pp. 1—36; also published at Cambridge, Mass. in 1877.

‡*Comptes Rendus*, 1877 et seq.

‖*Comptes Rendus*, 1879–80; *Journal für Mathematik*, vol. 90 (1881).

§*Annales de l'École Normale Supérieure*, 1883–1884.

toward the discovery of practical methods for their actual construction, principally when the differential equation has the form

$$\frac{d^2x}{dt^2} + (a_0 + a_1 \cos t + a_2 \cos 2t + \cdots)x = 0. \tag{70}$$

Different methods for constructing solutions of this equation have been proposed by Lindemann,* Lindstedt,† Bruns,‡ Callandreau,|| Stieltjes,§ and Harzer.¶

In what follows there will arise only equations with simply periodic coefficients having the form

$$x_i' = \sum_{j=1}^{n} \theta_{ij} x_j \qquad (i=1, \ldots, n), \tag{71}$$

where the θ_{ij} are periodic functions of t with the period 2π. It will be assumed that the θ_{ij} are uniform analytic functions of t and are regular for $0 \gtreqless t \leqq 2\pi$. Let

$$x_{i1} = \varphi_{i1}(t), \qquad x_{i2} = \varphi_{i2}(t), \ldots, x_{in} = \varphi_{in}(t) \qquad (i=1, \ldots, n),$$

be a fundamental set of solutions which satisfy the initial conditions $\varphi_{ij}(0) = 0$ if $i \neq j$ and $\varphi_{ii}(0) = 1$. It is clear that n solutions can be constructed with these n sets of initial conditions, and since their determinant is unity at $t=0$, they constitute a fundamental set of solutions.

Now make the transformation

$$x_i = e^{\alpha t} y_i, \tag{72}$$

where α is an undetermined constant. The differential equations become

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \theta_{ij} y_j, \tag{73}$$

any solution of which can be written in the form

$$y_i = e^{-\alpha t} \sum_{j=1}^{n} A_j \varphi_{ij}(t) \qquad (i=1, \ldots, n), \tag{74}$$

where the A_j are suitably chosen constants.

The question arises whether it is possible to determine α and the A_j in such a manner that the y_i shall be periodic in t with the period 2π. From the form of equations (73) it is clear that sufficient conditions for the periodicity of the y_i are that $y_i(2\pi) = y_i(0)$ $(i=1, \ldots, n)$. On imposing these conditions upon (74), there results

$$0 = \sum_{j=1}^{n} A_j [\varphi_{ij}(2\pi) - e^{2\alpha\pi} \varphi_{ij}(0)] \qquad (i=1, \ldots, n). \tag{75}$$

Either all the A_j are zero or the determinant must vanish. The former case is trivial and we therefore impose the condition

$$|\varphi_{ij}(2\pi) - e^{2\alpha\pi} \varphi_{ij}(0)| = 0,$$

Mathematische Annalen, vol. XXII, (1883), p. 117–123.
†*Astronomische Nachrichten*, No. 2503 (1883). *Mémoires de l'Académie de St. Pétersbourg*, vol. XXI, No. 4.
‡*Astronomische Nachrichten*, No. 2533, 2553 (1884).
||*Astronomische Nachrichten*, No. 2547 (1884).
§*Astronomische Nachrichten*, No. 2602 (1884).
¶*Astronomische Nachrichten*, Nos. 2850 and 2851 (1888).

where $\varphi_{ij}(0)=0$ if $i \neq j$ and $\varphi_{ii}(0)=1$. Putting $e^{2\alpha\pi}=s$ and denoting $\varphi_{ij}(2\pi)$ simply by φ_{ij}, this determinant becomes

$$\begin{vmatrix} \varphi_{11}-s, & \varphi_{12}, & \varphi_{13}, & \dots, & \varphi_{1n} \\ \varphi_{21}, & \varphi_{22}-s, & \varphi_{23}, & \dots, & \varphi_{2n} \\ \varphi_{31}, & \varphi_{32}, & \varphi_{33}-s, & \dots, & \varphi_{3n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \varphi_{n1}, & \varphi_{n2}, & \varphi_{n3}, & \dots, & \varphi_{nn}-s \end{vmatrix} = 0. \tag{76}$$

This is an equation of the n^{th} degree in s, the constant term of which can not vanish since it is the determinant of a fundamental set of solutions. It is known as the *fundamental equation* for the period 2π. Its roots can be neither zero nor infinite, because the coefficient of s^n is unity and the term independent of s is Δ.

20. Solutions when the Roots of the Fundamental Equation are all Distinct.—Suppose the roots $s_1, s_2, \dots, s_n$ of (76) are all distinct. Then at least one of the first minors of (76) is distinct from zero when s is put equal to s_k, and therefore the ratios of the A_j are uniquely determined by (75). For each s_k a set of y_i is determined by (74) involving one arbitrary constant. Since this solution depends upon s_k it will be designated by y_{ik}, and the corresponding A_j by A_{jk}.

Since $s=e^{2\alpha\pi}$, the α is uniquely determined in terms of s except for the additive constant $\nu\sqrt{-1}$, where ν is an integer. In every case the principal value of α can be taken, for its other values simply remove periodic factors from the y_i. Consequently, for the n values of s there are n values α, and from equation (74) there are n solutions, one for each k from 1 to n,

$$y_{ik}=e^{-\alpha_k t}\sum_{j=1}^{n} A_{jk}\,\varphi_{ij}(t) \qquad (i=1,\dots,n), \tag{77}$$

where the ratios of the A_{jk} are determined by (75). From (72), n solutions of equations (71) are thus found, one for each k,

$$x_{ik}=e^{\alpha_k t}\,y_{ik} \qquad (i=1,\dots,n), \tag{78}$$

where the y_{ik} are periodic in t with the period 2π.

These solutions (78) form a fundamental set, for, if they did not, there would exist linear relations among the x_{ik} of the form

$$\sum_{k=1}^{n} C_k\,x_{ik}(t)\equiv 0 \qquad (i=1,\dots,n), \tag{79}$$

where not all the $C_k=0$. Increasing t by 2π, it follows from the conditions imposed upon the x_i and y_i that

$$\sum_{k=1}^{n} C_k\,x_{ik}(t+2\pi)=\sum_{k=1}^{n} C_k\,s_k\,x_{ik}(t)\equiv 0,$$

and similarly that

$$
\left.\begin{aligned}
&\sum_{k=1}^{n} C_k x_{ik}(t+4\pi) = \sum_{k=1}^{n} C_k s_k^2 x_{ik}(t) \equiv 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{k=1}^{n} C_k x_{ik}[t+2(n-1)\pi] = \sum_{k=1}^{n} C_k s_k^{n-1} x_{ik}(t) \equiv 0.
\end{aligned}\right\} \quad (80)
$$

Since the C_k are not all zero it follows that the determinant of these equations must vanish; that is,

$$
\left.\begin{aligned}
\prod_{k=1}^{n} x_{ik}(t) \begin{vmatrix} 1 & ,1 & ,1 & ,\ldots, & 1 \\ s_1 & ,s_2 & ,s_3 & ,\ldots, & s_n \\ s_1^2 & ,s_2^2 & ,s_3^2 & ,\ldots, & s_n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_1^{n-1} & ,s_2^{n-1} & ,s_3^{n-1} & ,\ldots, & s_n^{n-1} \end{vmatrix} &= (-1)^{\frac{n(n-4)}{4}} \prod_{k=1}^{n} x_{ik}(t) \prod_{j_1, j_2=1}^{n} \sqrt{(s_{j_1}-s_{j_2})} \equiv 0 \\
&\qquad\qquad (j_1 \neq j_2).
\end{aligned}\right\} \quad (81)
$$

Since, by hypothesis, the s_j are distinct this relation can not be satisfied unless some $x_{ik} \equiv 0$. For the sake of definiteness take first $i=1$, and suppose that $x_{1k_1} \equiv 0$, where k_1 is some particular value of the second subscript. But equation (79) becomes for $i=1$

$$
\sum_{k=1}^{n} C_k x_{1k}(t) \equiv 0 \qquad (k \neq k_1),
$$

and there is corresponding to (81) an identity of the form

$$
\prod_{k=1}^{n} x_{1k}(t) \prod_{j_1, j_2=1} \sqrt{(s_{j_1}-s_{j_2})} \equiv 0 \qquad (k \neq k_1,\ j_1 \neq k_1,\ j_2 \neq k_1,\ j_1 \neq j_2).
$$

From this it is inferred that another x_{1k}, say x_{1k_2}, is identically zero. Repeating the process n times, the final conclusion is

$$
x_{11} \equiv x_{12} \equiv \cdots \equiv x_{1n} \equiv 0.
$$

Upon starting from (79) for $i=2$, the conclusion is reached in a similar way that

$$
x_{21} \equiv x_{22} \equiv \cdots \equiv x_{2n} \equiv 0.
$$

On repeating the process, corresponding identities are obtained for all values of i from 1 to n.

Now from the identities $x_{ik} \equiv 0$ $(i=1, \ldots, n)$ and from (77) and (78), it follows that

$$
\left.\begin{aligned}
x_{1k} &\equiv A_{1k}\varphi_{11} + A_{2k}\varphi_{12} + \cdots + A_{nk}\varphi_{1n} \equiv 0, \\
x_{2k} &\equiv A_{1k}\varphi_{21} + A_{2k}\varphi_{22} + \cdots + A_{nk}\varphi_{2n} \equiv 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_{nk} &\equiv A_{1k}\varphi_{n1} + A_{2k}\varphi_{n2} + \cdots + A_{nk}\varphi_{nn} \equiv 0.
\end{aligned}\right\} \quad (82)
$$

These identities can not all be satisfied unless each $A_{ik}=0$, for, at $t=0$, $\varphi_{ij}=0$ if $i \neq j$ and $\varphi_{ii}=1$. The same result holds for each value of k from 1 to n, but by virtue of equations (75) and the hypothesis that (76) has simple roots, it follows that for each s_k there is a solution in which not all the A_{jk} are

zero. If these solutions are taken, the identities (82) can not be satisfied and consequently equations (79) can not be satisfied. *Therefore* (78) *constitute a fundamental set of solutions.*

21. Solutions when the Fundamental Equation has Multiple Roots.— Consider first the case where the fundamental equation has only two roots equal. The notation can be chosen so that $s_2 = s_1$. There are two cases according as all, or not all, of the first minors of (76) vanish when $s = s_1$. Suppose first that all the first minors vanish for this value of s. Since $s = s_1$ is only a double root, not all of the second minors can vanish. Hence two of the A_j can be taken arbitrarily and (75) can be solved for the remaining $(n-2)$ of them. Then equations (74) and (72) give the corresponding x_i. Since the φ_{ij} are linearly distinct two linearly distinct values of the y_i can be obtained by taking first one of the arbitrary A_j equal to zero, and then the other equal to zero. Therefore in this case there are two linearly distinct solutions of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \qquad x_{i2} = e^{\alpha_1 t} y_{i2} \qquad (i=1, \ldots, n),$$

where the y_{i1} and y_{i2} are periodic in t with the period 2π.

If, however, not all the first minors of (76) vanish for $s = s_1$, there is but a single solution of this form belonging to the root s_1 of the fundamental equation. Let $x_{i1} = e^{\alpha_1 t} y_{i1}$ be this solution; it will be shown that the other one belonging to this root has the form

$$x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1}) \qquad (i=1, \ldots, n), \tag{83}$$

where the y_{i2} are periodic in t with the period 2π.

Before proceeding to the demonstration a lemma pertaining to a certain type of transformation of a fundamental set of solutions will be proved. Suppose the φ_{ij} constitute a fundamental set of solutions. Then define new functions ψ_{ik} by the relations

$$\psi_{ik} = \sum_{j=k}^{n} A_{jk} \varphi_{ij} \qquad (i, k=1, \ldots, n). \tag{84}$$

The ψ_{ik} also constitute a fundamental set of solutions provided no $A_{kk} = 0$, for the determinant of the ψ_{ik} is

$$|\psi_{ik}| = \left|\sum_{j=k}^{n} A_{jk} \varphi_{ij}\right| = |A_{jk}|\,|\varphi_{ij}|,$$

where $|A_{jk}|$ and $|\varphi_{ij}|$ are the determinants of the A_{jk} and φ_{ij} respectively. The determinant $|\varphi_{ij}|$ is distinct from zero and

$$|A_{jk}| = \begin{vmatrix} A_{11}, & A_{21}, & A_{31}, & \ldots, & A_{n1} \\ 0, & A_{22}, & A_{32}, & \ldots, & A_{n2} \\ 0, & 0, & A_{33}, & \ldots, & A_{n3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & A_{nn} \end{vmatrix}, \tag{85}$$

which is distinct from zero unless some A_{kk} is zero. A special case, which will be used first, is that where all the elements except those in the first line and in the main diagonal are zero.

Now return to the point under discussion. By hypothesis not all the first minors of (76) vanish for $s = s_1$. Let the notation be chosen so that one of those which is distinct from zero is formed from the elements of the last $n-1$ columns. Then A_1 must be distinct from zero in order not to get the trivial case in which all the A_j are zero. Now in place of φ_{ij} $(i, j = 1, \ldots, n)$ as a fundamental set of solutions we can take, as a consequence of the lemma,

$$e^{\alpha_1 t}\, y_{i1}, \qquad \varphi_{ij} \qquad (i=1, \ldots, n;\ j=2, \ldots, n). \tag{86}$$

Any solution can be expressed in the form

$$x_i = B_1 e^{\alpha_1 t} y_{i1} + \sum_{j=2}^{n} B_j \varphi_{ij} \qquad (i=1, \ldots, n).$$

Now make the transformation $x_{i2} = e^{\alpha_1 t}(y_{i2} + t\, y_{i1})$; whence

$$y_{i2} = -\, t y_{i1} + B_1 y_{i1} e^{(\alpha_1 - \alpha_1)t} + e^{-\alpha_1 t} \sum_{j=2}^{n} B_j \varphi_{ij}.$$

Since by hypothesis the $x_{i1} = e^{\alpha_1 t} y_{i1}$ satisfy (71), it is found by substitution that, if (83) are to constitute a solution, the y_{i2} must satisfy the equations

$$y'_{i2} + \alpha_1 y_{i2} = \sum_{j=1}^{n} \theta_{ij} y_{j2} - y_{i1} \qquad (i=1, \ldots, n). \tag{87}$$

Since t enters only in the θ_{ij} and the y_{i1}, which are periodic with the period 2π, sufficient conditions that the y_{i2} shall be periodic with the period 2π are

$$y_{i2}(2\pi) - y_{i2}(0) = -\, 2\pi\, y_{i1}(0) + \sum_{j=2}^{n} B_j [\varphi_{ij}(2\pi)\, e^{-2\alpha_1 \pi} - \varphi_{ij}(0)] = 0.$$

On transforming from the exponential to s, these equations give

$$-2\pi\, s_1 y_{i1}(0) + \sum_{j=2}^{n} B_j [\varphi_{ij}(2\pi) - s_1 \delta_{ij}] = 0, \tag{88}$$

where $\delta_{ij} = 0$ if $j \neq i$ and $\delta_{ii} = 1$.

The condition that equations (88) shall be consistent is

$$D_1 = |y_{i1}(0),\ \varphi_{i2}(2\pi) - s_1 \delta_{i2}, \ldots, \varphi_{in}(2\pi) - s_1 \delta_{in}| = 0,$$

where D_1 is the determinant formed from their coefficients. This equation is satisfied, for if the fundamental equation is formed as usual from the fundamental set (86), it is found that $D = (s - s_1) D_1 = 0$. Since D is independent of the fundamental* set from which it is derived, and since $D = 0$ has the double root $s = s_1$, it follows that $D_1(s_1) = 0$. Therefore (88) can be solved uniquely for the $B_2, \ldots, B_n$. These equations determine the y_{i2}, and through them the x_{i2} in the form given in (83).

*Fuchs, *Journal für Mathematik*, vol. LXVI (1866), p. 133.

Now suppose $s=s_1$ is a triple root of the fundamental equation, but that it is not a quadruple root. If all its minors of the first and second order vanish for $s=s_1$, three of the A_j can be taken arbitrarily and three linearly distinct solutions of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \qquad x_{i2} = e^{\alpha_1 t} y_{i2}, \qquad x_{i3} = e^{\alpha_1 t} y_{i3}$$

can be determined, where the y_{i1}, y_{i2}, and y_{i3} are periodic in t with the period 2π.

If all of the minors of the first order of the fundamental determinant vanish, but not all of those of the second order, then two of the A_j can be taken arbitrarily, and two linearly distinct solutions of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \qquad x_{i2} = e^{\alpha_1 t} y_{i2}$$

will be obtained, where the y_{i1} and y_{i2} are again periodic.

In order to obtain a third solution associated with the root s_1 take as a new fundamental set of solutions

$$e^{\alpha_1 t} y_{i1}, \qquad e^{\alpha_1 t} y_{i2}, \qquad \varphi_{ij} \qquad (i=1, \ldots, n;\ j=3, \ldots, n),$$

so that any solution can be written in the form

$$x_i = B_1 e^{\alpha_1 t} y_{i1} + B_2 e^{\alpha_1 t} y_{i2} + \sum_{j=3}^{n} B_j \varphi_{ij} \qquad (i=1, \ldots, n).$$

Now make the transformation

$$x_{i3} = e^{\alpha_1 t} [y_{i3} + t(y_{i1} + y_{i2})];$$

whence

$$y_{i3} = -t y_{i1} - t y_{i2} + B_1 e^{(\alpha_1-\alpha_1)t} y_{i1} + B_2 e^{(\alpha_1-\alpha_1)t} y_{i2} + e^{\alpha_1 t} \sum_{j=3}^{n} B_j \varphi_{ij}.$$

In a manner similar to that in the case just treated the periodicity conditions on the y_{i3} lead to the equations

$$0 = -2\pi s_1 y_{i1}(0) - 2\pi s_1 y_{i2}(0) + \sum_{j=3}^{n} B_j [\varphi_{ij}(2\pi) - \delta_{ij} s_1].$$

As in the preceding case, it is found that for $s=s_1$ the $B_3, \ldots, B_n$ are uniquely determined and that the x_{i3} have the form

$$x_{i3} = e^{\alpha_1 t} [y_{i3} + t(y_{i1} + y_{i2})], \tag{89}$$

where the y_{i1}, y_{i2}, and y_{i3} are periodic in t with the period 2π.

Suppose now that not all of the first minors of the fundamental determinant vanish for $s=s_1$. Then there will be one solution $x_{i1} = e^{\alpha_1 t} y_{i1}$ and another $x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1})$. It will be shown that in this case the third solution belonging to s_1 is of the form

$$x_{i3} = e^{\alpha_1 t} [y_{i3} + t y_{i2} + \tfrac{1}{2} t^2 y_{i1}]. \tag{90}$$

Take as a new fundamental set of solutions

$$e^{\alpha_1 t}y_{i1}, \quad e^{\alpha_1 t}(y_{i2}+t\,y_{i1}), \quad \varphi_{ij} \qquad (i=1,\ldots,n;\ j=3,\ldots,n). \tag{90'}$$

After defining the x_i by

$$x_i = B_1 e^{\alpha_1 t} y_{i1} + B_2 e^{\alpha_1 t}(y_{i2}+t\,y_{i1}) + \sum_{j=3}^{n} B_j \varphi_{ij} \qquad (i=1,\ldots,n),$$

make the transformation (90). Then the expressions for y_{i3} are

$$y_{i3} = -t\,y_{i2} - \tfrac{1}{2}t^2 y_{i1} + B_1 e^{(\alpha_1-\alpha_1)t} y_{i1} + B_2 e^{(\alpha_1-\alpha_1)t}(y_{i2}+t\,y_{i1}) + e^{-\alpha_1 t}\sum_{j=3}^{n} B_j \varphi_{ij}.$$

If the x_{i3} constitute a solution of the original equations (71), the y_{i3} must satisfy the equations

$$y'_{i3} + \alpha_1 y_{i3} = \sum_{j=1}^{n} \theta_{ij} y_{j3} - y_{i2},$$

since the y_{i1} satisfy (73) and the y_{i2} satisfy (87). Hence sufficient conditions that the y_{i3} shall be periodic are that

$$y_{i3}(2\pi) - y_{i3}(0) = 0 \qquad (i=1,\ldots,n).$$

These conditions lead to the equations

$$0 = -2\pi s_1 y_{i2}(0) - 2\pi^2 s_1 y_{i1}(0) + 2\pi B_2 s_1 y_{i1}(0) + \sum_{j=3}^{n} B_j[\varphi_{ij}(2\pi) - s_1\delta_{ij}].$$

The terms $y_{i1}(0)$ in the second column of the determinant of the coefficients of these equations evidently may be suppressed. Let this determinant be denoted by D_2. In order that these equations shall be consistent it is necessary that $D_2=0$. This condition is satisfied; for if the fundamental equation be formed from (90′), it is found that

$$D = (s-s_1)^2 D_2.$$

But by hypothesis D admits $(s=s_1)$ as a triple root. Therefore $D_2(s_1)=0$ and the equations are consistent. Since $s=s_1$ is a simple root of D_2, not all of its first minors are zero. Therefore the $B_3,\ldots,B_n$ are uniquely determined, and the x_{i3} have the form (90).

Suppose $s=s_1$ is a root of multiplicity l. There is then a group of solutions, l in number, attached to this root. In general this group of solutions will have the following form

$$\left.\begin{aligned}
x_{i1} &= e^{\alpha_1 t} y_{i1} \qquad\qquad (i=1,\ldots,n),\\
x_{i2} &= e^{\alpha_1 t}[y_{i2} + t\,y_{i1}],\\
x_{i2} &= e^{\alpha_1 t}[y_{i3} + t\,y_{i2} + \tfrac{1}{2}t^2 y_{i1}],\\
&\ldots\ldots\ldots\ldots\\
x_{il} &= e^{\alpha_1 t}[y_{i,l} + t\,y_{il-1} + \cdots + \frac{1}{(l-1)!}t^{l-1} y_{i1}].
\end{aligned}\right\} \tag{91}$$

If all the minors of the fundamental equation $D=0$ up to the order $k-1$ $(k \overline{\gtrless} l)$, but not all of order k, vanish for $s=s_1$, then there are k solutions of the first form, *i. e.* of the form

$$x_{i1} = e^{\alpha_1 t}\, y_{i1}\,, \qquad x_{i2} = e^{\alpha_1 t}\, y_{i2}\,, \qquad \ldots\,, \qquad x_{ik} = e^{\alpha_1 t}\, y_{ik}\,.$$

If now the fundamental set

$$e^{\alpha_1 t}\, y_{i1}\,, \ \ldots\,, \ e^{\alpha_1 t}\, y_{ik}\,, \qquad \varphi_{i,k+1}\,, \ \ldots\,, \ \varphi_{in}\,,$$

be taken and the equation in s formed, it is found that

$$D = (s-s_1)^k D_k = 0.$$

Since the roots of the fundamental equation are not changed by adopting the new fundamental set of solutions, $D_k=0$ has $s=s_1$ as a root of multiplicity $l-k$. Suppose all the minors of D_k of order $g-1$, but not all of order g, vanish; then there are g solutions of the second form, viz.,

$$x_{i,k+1} = e^{\alpha_1 t}\Big[y_{i,k+1} + t\sum_{j=1}^{n} y_{ij}\Big]\,, \qquad \ldots\,, \qquad x_{i,k+g} = e^{\alpha_1 t}\Big[y_{i,k+g} + t\sum_{j=1}^{n} y_{ij}\Big].$$

If $k+g<l$, by a similar change of the fundamental set of solutions, it will be found that

$$D = (s-s_1)^{k+g} D_{k+g} = 0.$$

Now $D_{k+g}=0$ admits $s=s_1$ as a root of multiplicity $l-(k+g)$ and there is a certain number of solutions of the third type of (91), depending upon the order of the minors of D_{k+g} which do not all vanish for $s=s_1$. Continuing, there is obtained finally l linearly independent solutions associated with s_1, and in a similar way the solutions associated with the other roots of the fundamental equation can be found.

22. The Characteristic Equation when the Coefficients of the Differential Equations are Expansible as Power Series in a Parameter μ.—In the preceding discussions no explicit reference was made to the parameters upon which the θ_{ij} may depend. It will be assumed now that the θ_{ij} are expansible as power series in μ whose coefficients separately are periodic in t, and that the series converge for all finite values of t if $|\mu|<\rho$. It will be assumed further that $\theta_{ij}=a_{ij}$, where the a_{ij} are constants, for $\mu=0$. Under these conditions, which are often realized in practice and particularly in the applications which follow, the discussion of the character of the solutions can be made so as to lead to a convenient method for their practical construction. The discussion will depend upon the principles of §19 and the integration of the equations as power series in μ.

Consider now the equations

$$x_i' = \sum_{j=1}^{n} \theta_{ij}\, x_j = \sum_{j=1}^{n} \Big[a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k\Big] x_j \qquad (i=1,\ \ldots,\ n), \quad (92)$$

where the a_{ij} are constants, the $\theta_{ij}^{(k)}$ are periodic in t with the period 2π, and $\sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k$ converge for all real, finite values of t if $|\mu| < \rho$. For $\mu = 0$ equations (92) admit $x_i^{(0)} = c_i e^{\alpha^{(0)} t}$ as a solution, where the c_i are constants whose ratios depend upon the coefficients of the differential equations, and $\alpha^{(0)}$ is one of the roots of the *characteristic equation*

$$\begin{vmatrix} a_{11} - \alpha^{(0)}, & a_{12} & , \ldots, & a_{1n} \\ a_{21} & , a_{22} - \alpha^{(0)} & , \ldots, & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & , \ldots, & a_{nn} - \alpha^{(0)} \end{vmatrix} = 0. \tag{93}$$

This equation, which is of the n^{th} degree in $\alpha^{(0)}$, has n roots, $\alpha_1^{(0)}, \ldots, \alpha_n^{(0)}$. If these roots are all distinct there exists a fundamental set of solutions of the form

$$x_{ij}^{(0)} = c_{ij} e^{\alpha_j^{(0)} t} \qquad (i = 1, \ldots, n;\ j = 1, \ldots, n). \tag{94}$$

If two of the roots are equal, say $\alpha_1^{(0)} = \alpha_2^{(0)}$, a fundamental set of solutions is obtained by taking

$$x_{i1}^{(0)} = c_{i1} e^{\alpha_1^{(0)} t},\ x_{i2}^{(0)} = (c_{i2} + t c_{i1}) e^{\alpha_1^{(0)} t},\ x_{i3}^{(0)} = c_{i3} e^{\alpha_3^{(0)} t}, \ldots, x_{in}^{(0)} = c_{in} e^{\alpha_n^{(0)} t}.$$

Suppose the roots of (93) are all distinct and that the fundamental set of solutions is (94); then, for μ distinct from zero, the complete solutions of (92) are

$$x_i = \sum_{j=1}^{n} A_j \left[c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k \right] \qquad (i = 1, \ldots, n), \tag{95}$$

where, by §17, the series $\sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^{(k)}$ converge for any preassigned finite range for t if $|\mu| < \rho$. Without loss of generality the initial conditions can be taken so that the determinant of the c_{ij} is unity and $x_{ij}^{(k)}(0) = 0$. As before, the transformation

$$x_i = e^{\alpha t} y_i$$

is made, and the equations corresponding to (92) and (95) are respectively

$$\left.\begin{aligned} y_i' + \alpha y_i &= \sum_{j=1}^{n} \left[a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k \right] y_j \qquad (i = 1, \ldots, n), \\ y_i &= \sum_{j=1}^{n} A_j e^{-\alpha t} \left[c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)}(t) \mu^k \right]. \end{aligned}\right\} \tag{96}$$

The conditions that the y_i shall be periodic with the period 2π, viz., $y_i(2\pi) - y_i(0) = 0$, give

$$0 = \sum_{j=1}^{n} A_j e^{2\alpha_j^{(0)}\pi} \left[c_{ij} \left(1 - e^{2(\alpha - \alpha_j^{(0)})\pi}\right) + e^{-2\alpha_j^{(0)}\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k \right]. \tag{97}$$

Since the A_j must not all be zero, the determinant of their coefficients must vanish, whence

$$\Delta = \left| \left[c_{ij} \left(1 - e^{2(\alpha - \alpha_j^{(0)})\pi}\right) + e^{-2\alpha_j^{(0)}\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)} (2\pi) \mu^k \right] \right| = 0. \tag{98}$$

This equation has an infinite number of solutions, for if $\alpha = \alpha_j$ is a solution, then also is $\alpha = \alpha_j + \nu\sqrt{-1}$, ν any integer. The fundamental equation corresponding to (76) is obtained by the transformation $e^{2\alpha\pi} = s$. If the values of s satisfying the fundamental equation are distinct, the corresponding values of α are distinct but not the converse, for if two values of α differ by an imaginary integer the corresponding values of s are equal. Only those values of α will be taken which reduce to the $\alpha_j^{(0)}$ for $\mu = 0$, the $\alpha_j^{(0)}$ being uniquely determined by (93).

Suppose now that two of the roots of the characteristic equation, say $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, are equal. Then the solutions of (92) have in general the form

$$\left.\begin{aligned} x_i = {} & A_1\left[c_{i1} e^{\alpha_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i1}^{(k)} \mu^k\right] + A_2\left[(c_{i2} + t c_{i1}) e^{\alpha_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i2}^{(k)} \mu^k\right] \\ & + \sum_{j=3}^{n} A_j\left[c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k\right]. \end{aligned}\right\} \tag{99}$$

The exception to this general form is that tc_{i1} may be absent from the second term for $i = 1, \ldots, n$, and this possibility must be considered at those places where it makes differences in the discussion.

After making the transformation $x_i = e^{\alpha t} y_i$, the solutions for the y_i are

$$\left.\begin{aligned} y_i = {} & A_1\left[c_{i1} e^{(\alpha_1^{(0)} - \alpha)t} + e^{-\alpha t} \sum_{k=1}^{\infty} x_{i1}^{(k)} \mu^k\right] + A_2\left[(c_{i2} + t c_{i1}) e^{(\alpha_1^{(0)} - \alpha)t} + e^{-\alpha t} \sum_{k=1}^{\infty} x_{i2}^{(k)} \mu^k\right] \\ & + \sum_{j=3}^{n} A_j\left[c_{ij} e^{(\alpha_j^{(0)} - \alpha)t} + e^{-\alpha t} \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k\right]. \end{aligned}\right\} \tag{100}$$

The conditions for the periodicity of the y_i, viz., $y_i(2\pi) - y_i(0) = 0$, lead to the determinant

$$\left.\begin{aligned} \Delta = \Bigg| & \left[c_{i1}\left(1 - e^{2(\alpha - \alpha_1^{(0)})\pi}\right) + e^{-2\alpha_1^{(0)}\pi} \sum_{k=1}^{\infty} x_{i1}^{(k)} (2\pi) \mu^k\right], \\ & \left[c_{i2}\left(1 - e^{2(\alpha - \alpha_1^{(0)})\pi}\right) + 2\pi c_{i1} + e^{-2\alpha_1^{(0)}\pi} \sum_{k=1}^{\infty} x_{i2}^{(k)} (2\pi) \mu^k\right], \ldots \Bigg| = 0, \end{aligned}\right\} \tag{101}$$

where the elements which are not written are of the same form as those in (98).

If, for $\mu = 0$, the characteristic equation has a root of higher order of multiplicity, the fundamental equation is formed in a similar manner.

23. Solutions when $\alpha_1^{(0)}, \alpha_2^{(0)}, \ldots, \alpha_n^{(0)}$ are Distinct and their Differences are not Congruent to Zero mod. $\sqrt{-1}$.—The part of (98) independent of μ is

$$\Delta_0 = \left| c_{ij}\left(1 - e^{2(\alpha - \alpha_j^{(0)})\pi}\right) \right| = |c_{ij}| \prod_{j=1}^{n} \left(1 - e^{2(\alpha - \alpha_j^{(0)})\pi}\right), \tag{102}$$

and the determinant $|c_{ij}|$ is unity.

If, in any particular case, (98) were an identity in μ its n solutions would be simply $\alpha = \alpha_j^{(0)}$. In case it is not an identity, let

$$\alpha = \alpha_k^{(0)} + \beta_k \tag{103}$$

and Δ becomes

$$\left.\begin{array}{l} \Delta = \Delta_0 + \mu\, F_k(\beta_k, \mu) = (1 - e^{2\beta_k \pi}) \prod_{j=1}^{n} \left(1 - e^{2\left(\alpha_k^{(0)} - \alpha_j^{(0)} + \beta_k\right)\pi}\right) \\ \qquad + \mu F_k(\beta_k\,, \mu) = 0 \qquad (j \neq k), \end{array}\right\} \tag{104}$$

where $F_k(\beta_k, \mu)$ is a series in β_k and μ, converging for $|\beta_k|$ finite and $|\mu| < \rho$.

Since, by hypothesis, no $(\alpha_k^{(0)} - \alpha_j^{(0)})$ is an imaginary integer, the expansion of (104) as a power series in β_k and μ contains a term in β_k of the first degree and no term independent of both β_k and μ. Therefore (see §§1 and 2) it can be solved uniquely for β_k as a power series in μ of the form

$$\beta_k = \mu\, P_k(\mu). \tag{105}$$

Substituting this value of β_k in (103) and the resulting α in (97), n homogeneous linear relations among $A_1, \ldots, A_n$ are obtained whose determinant vanishes, but for μ sufficiently small not all of its first minors vanish, since the roots of the determinant set equal to zero were all distinct for $\mu = 0$. Therefore the ratios of the A_j are uniquely determined as power series in μ, converging for $|\mu|$ sufficiently small. When the ratios of the A_j have been determined, the y_{ik} are determined as power series in μ, and the coefficient of each power of μ separately is periodic in t. A solution is found similarly for each $\alpha_j^{(0)}$.

The origin of the singularities which determine the radii of convergence of the final solution series is known. If ρ is the smallest true radius of convergence of the original solutions (95) as t varies from 0 to 2π, then, in general, the final solutions will converge only if $|\mu| < \rho$. Consider the fundamental equation, $\Delta(s, \mu) = 0$, which is a polynomial in s of degree n and a power series in μ converging if $|\mu| < \rho$. From the algebraic character of Δ it follows that the only singularities introduced by solving for s in terms of μ are branch-points, which are determined by the simultaneous equations

$$\Delta(s, \mu) = 0, \qquad \frac{\partial \Delta}{\partial s} = 0. \tag{106}$$

The variable s can be eliminated from these equations by rational processes and the eliminant will converge if $|\mu| < \rho$. Its zeros are branch-points for s as defined by $\Delta(s, \mu) = 0$.

The zeros of the eliminant which lie within $|\mu| = \rho$ can be found in any particular numerical case by Picard's extension of Kronecker's method* provided the zeros are all simple. If there is a zero at $\mu = \mu_0$, then the solutions for s as a power series in μ converge only if $|\mu| < |\mu_0|$. If there is no μ_0 the limit remains ρ.

*Picard's *Traité d'Analyse*, vol. II, chap. 7.

Now consider β_k as a function of μ through its relation with s, viz., $s_k = e^{2(\alpha_k^{(0)}+\beta_k)\pi} = e^{2\alpha_k^{(0)}\pi} \cdot e^{2\beta_k\pi}$. If s_k has a branch-point for $\mu=\mu_0$, then β_k also has a branch-point at the same place since $\partial\beta/\partial s = 1/2\pi s$ is distinct from zero for all finite values of s. Therefore the series for β_k converges only if $|\mu|<|\mu_0|$.

The root s_k is $s_k = s_k^{(0)} + \sum_{i=1}^{\infty} s_k^{(i)}\mu^i$ and $\alpha_k^{(0)}+\beta_k = (1/2\pi)\log[s_k^{(0)} + \sum_{i=1}^{\infty} s_k^{(i)}\mu^i]$. If for any μ_1 such that $|\mu_1|<\rho$ we have $|s_k^{(0)}| = |\sum_{i=1}^{\infty} s_k^{(i)}\mu_1^i|$, then β_k has an essential singularity at $\mu=\mu_1$, and the series for it converges only if $|\mu|<|\mu_1|$. The zeros determining these singularities can also be found in a special numerical case by Picard's method. When $|\mu|$ satisfies the inequalities imposed by these various possible singularities, the solutions are convergent for all finite values of t.

24. Solutions when no two $\alpha_j^{(0)}$ are equal but when $\alpha_2^{(0)}-\alpha_1^{(0)}$ is Congruent to Zero mod. $\sqrt{-1}$.—Suppose two roots of the characteristic equation for $\mu=0$, say $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, differ only by an imaginary integer, and that there is no other such congruence among them. Then the equation corresponding to (104) becomes

$$(1-e^{2\beta_1\pi})^2 \prod_{j=3}^{n}\left(1-e^{2(\alpha_1^{(0)}-\alpha_j^{(0)}+\beta_1)\pi}\right)+\beta_1\mu F_1(\beta_1,\mu)+\mu^2F_2(\beta_1,\mu)=0. \quad (107)$$

The term of lowest degree in β_1 alone is $4\pi^2\beta_1^2$. The term of lowest degree in μ alone is at least of the second degree, and will in general be precisely of the second degree. This follows from the fact that every term in every element of the first two columns of the determinant (98) contains in this special case either β_1 or μ as a factor. In order to get the terms in μ alone, those involving β_1 are suppressed, and then the conclusion follows from the fact that every term in the expansion of the determinant contains one term from each of the first two columns. In a similar way if p of the $\alpha_j^{(0)}$ are congruent to zero mod. $\sqrt{-1}$, then the term of lowest degree in β_1 alone is exactly of degree p, and in μ alone it is at least of degree p.

Consider the expansion of (107), which may be written in the form

$$\beta_1^2+\gamma_{11}\beta_1\mu+\gamma_{02}\mu^2+\cdots=0,$$

where γ_{11}, γ_{02}, are constants. The quadratic terms can be factored, giving

$$(\beta_1-b_1\mu)(\beta_1-b_2\mu)+\text{ terms of higher degree}=0.$$

If b_1 and b_2 are distinct, as will in general be the case, the two solutions of (107) are then (see §6),

$$\beta_{11}=b_1\mu+\mu^2P_1(\mu), \qquad \beta_{12}=b_2\mu+\mu^2P_2(\mu), \quad (108)$$

where P_1 and P_2 are power series in μ. If $b_1=b_2$ the solutions are power series in $\sqrt{\mu}$ or μ, depending upon the terms of higher degree. If γ_{02} is

zero at least one of the solutions starts with a term of degree higher than the first in μ. If the first term in μ alone is μ^3 and if γ_{11} is zero, then the solution has the form

$$\beta_{11}=c_1\mu^{3/2}+\cdots, \qquad \beta_{12}=-c_1\mu^{3/2}+\cdots.$$

But in general the solutions are of the type (108), and no other special cases will be considered in detail; they can all be treated by the principles of §§6 and 7. Thus, starting from the root $\alpha_1^{(0)}$ of the characteristic equation, two solutions are obtained. But it follows from the form of equations (98) and (104) that if the start were made from the root $\alpha_2^{(0)}$, the same values for β_1 would be found.

The other $\beta_k (k=3, \ldots, n)$ are found as in the preceding case, the solutions from them are formed in the same way, and their realm of convergence is limited by possible singularities of the same types.

25. Solutions when, for $\mu=0$, the Characteristic Equation has a Multiple Root.—Suppose only two roots are equal, say $\alpha_2^{(0)}=\alpha_1^{(0)}$, and that there are none of the congruences treated above. Then, for $\mu=0$, equation (101) becomes

$$\Delta_0=\left|c_{i1}\left(1-e^{2(\alpha-\alpha_1^{(0)})\pi}\right),\ c_{i2}\left(1-e^{2(\alpha-\alpha_1^{(0)})\pi}\right)+2\pi c_{i1},\ldots,c_{ij}\left(1-e^{2(\alpha-\alpha_j^{(0)})\pi}\right),\ldots\right|,$$

which easily reduces to

$$\Delta_0=\left(1-e^{2(\alpha-\alpha_1^{(0)})\pi}\right)^2\prod_{j=3}^{n}\left(1-e^{2(\alpha-\alpha_j^{(0)})\pi}\right). \tag{109}$$

since the determinant $|c_{ij}|$ is unity.

After the substitution $\alpha=\alpha_1^{(0)}+\beta_1$ is made in (109) and the result expanded, it is found that the term of lowest degree in β_1 alone is $4\pi^2\beta_1^2$. If the determinant Δ for $\alpha_2^{(0)}=\alpha_1^{(0)}$ is of the special form (98), the term of lowest degree in μ alone is at least of the second; but if in this case Δ is of the general form (101), the term of lowest degree in μ alone is in general of the first. Except in the special cases the solutions of (101) in the vicinity of the double root $\alpha_1^{(0)}$ are of the form

$$\beta_{11}=\mu^{\frac{1}{2}}P(\mu^{\frac{1}{2}}), \qquad \beta_{12}=-\mu^{\frac{1}{2}}P(-\mu^{\frac{1}{2}}),$$

where P is a power series in $\mu^{\frac{1}{2}}$, containing a term independent of μ.

When the coefficient of μ is zero in the expansion of (101), the first term in μ alone is of at least the second degree, and the problem is of the type treated in the preceding article.

If, for $\mu=0$, p roots of the characteristic equation are equal, then for these roots the expansion of (101) starts with β_1^p as the term of lowest degree β_1 alone, and except in special cases the lowest degree of terms in μ alone is

the first. Consequently in general for $\alpha_1^{(0)} = \alpha_2^{(0)} = \cdots = \alpha_p^{(0)}$ the solutions of (101) are

$$\beta_{1j} = \epsilon^j \mu^{\frac{1}{p}} P(\epsilon^j \mu^{\frac{1}{p}}) \qquad (j=1, \ldots, p),$$

where ϵ is any p^{th} root of unity.

Another case is that in which $\Delta = 0$ has a double root identically in μ, the conditions for which are

$$\Delta(\alpha, \mu) = 0, \qquad \frac{\partial \Delta}{\partial \alpha}(\alpha, \mu) = 0$$

for all $|\mu|$ sufficiently small. Suppose $\alpha_2 = \alpha_1$. If, for $\alpha = \alpha_1$, all the first minors of Δ are zero, the solutions of (97) for the ratios of the A_j will carry two arbitraries, and the two solutions associated with α_1 will be obtained. If not all the first minors of Δ vanish for $\alpha = \alpha_1$, then in this way only one solution is found. But it is known from the general theory of §21 that the second solution has the form

$$x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1}) \qquad (i=1, \ldots, n).$$

On substituting these expressions in the differential equations and making use of the fact that $e^{\alpha_1 t} y_{i1}$ are a solution, it is found that

$$y'_{i2} + \alpha_1 y_{i2} - \sum_{j=1}^{n} \left[a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k \right] y_{i2} = -y_{i1} \qquad (i=1, \ldots, n).$$

If the left members of these equations are set equal to zero, they become precisely of the form of the equations satisfied by the y_{i1}. Consequently $y_{i2} = y_{i1}$ plus such particular integrals that the differential equations shall be satisfied when the right members are retained. The method of finding the particular integrals will be taken up in §§29–31.

26. Construction of the Solutions when, for $\mu = 0$, the Roots of the Characteristic Equation are Distinct and their Differences are not Congruent to Zero mod. $\sqrt{-1}$.—The knowledge of the properties of the solutions and their expansibility as power series in μ leads to convenient methods for constructing them. Under the conditions that for $\mu = 0$ the roots of the characteristic equation are distinct and that the difference of no two of them is congruent to zero mod. $\sqrt{-1}$, it has been shown that there are exactly n distinct values of α expansible as converging power series in μ, such that $x_{ik} = e^{\alpha_k t} y_{ik}$ $(i = 1, \ldots, n)$, where the y_{ik} are purely periodic, constitute a fundamental set of solutions. It will be assumed that for $\mu = 0$ no two $\alpha_j^{(0)}$ are equal and that the difference of no two of them is congruent to zero mod. $\sqrt{-1}$, and it will be shown that the coefficients of the expansions of the α_k and y_{ik} are determined, except for a constant factor, by the conditions that the differential equations shall be satisfied and that the y_{ik} shall be periodic in t with the period 2π.

Suppose the value of y_{ik} is $y_{ik} = \sum_{j=0}^{\infty} y_{ik}^{(j)} \mu^j$, where the series converge for all $|\mu|$ sufficiently small. It follows from the periodicity condition that

$$\sum_{j=0}^{\infty} y_{ik}^{(j)}(2\pi)\mu^j \underset{\mu}{\equiv} \sum_{j=0}^{\infty} y_{ik}^{(j)}(0)\mu^i \qquad (i, k=1, \ldots, n).$$

Since this relation is an identity, it follows that

$$y_{ik}^{(j)}(2\pi) = y_{ik}^{(j)}(0) \qquad (i, k=1, \ldots, n).$$

Therefore each y_{ik} separately is periodic with the period 2π.

Now the original differential equations (92) after making the transformation $x_i = e^{\alpha t} y_i$ become

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \left[a_{ij} + \sum_{l=1}^{\infty} \theta_{ij}^{(l)} \mu^l \right] y_j \qquad (i=1, \ldots, n). \quad (110)$$

For $\mu=0$ the roots of the characteristic equation belonging to these equations are $\alpha_1^{(0)}, \alpha_2^{(0)}, \ldots, \alpha_n^{(0)}$. Consider any one of them, as $\alpha_k^{(0)}$. It has been shown that for $\mu \neq 0$, but sufficiently small in absolute value, α_k and the y_{ik} are expansible in converging series of the form

$$\left.\begin{aligned} \alpha_k &= \alpha_k^{(0)} + \alpha_k^{(1)}\mu + \cdots = \sum_{\nu=0}^{\infty} \alpha_k^{(\nu)}\mu^\nu, \\ y_{ik} &= y_{ik}^{(0)} + y_{ik}^{(1)}\mu + \cdots = \sum_{\nu=0}^{\infty} y_{ik}^{(\nu)}\mu^\nu. \end{aligned}\right\} \quad (111)$$

On substituting (111) in (110), arranging as power series in μ, and equating coefficients of corresponding powers in μ, there results a series of sets of equations from which α_k and the y_{ik} can be determined so that the y_{ik} shall be periodic with the period 2π. The determination is unique except for an arbitrary constant factor of the y_{ik}. For simplicity of notation this constant factor will be determined so that $y_{1k}^{(0)}(0) = c_{1k}$, provided $c_{1k} \neq 0$, and it can be restored in the final results by multiplying this particular solution by an arbitrary constant.

Terms independent of μ. The terms of the solution independent of μ are defined by the differential equations

$$(y_{ik}^{(0)})' + \alpha_k^{(0)} y_{ik}^{(0)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(0)} = 0 \qquad (i=1, \ldots, n), \quad (112)$$

the general solution of which is

$$y_{ik}^{(0)} = \sum_{j=1}^{n} \eta_{jk}^{(0)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_k^{(0)})t} \qquad (i=1, \ldots, n), \quad (113)$$

where the $\eta_{jk}^{(0)}$ are the constants of integration. Since the $y_{ik}^{(0)}$ are periodic with the period 2π, and since, by hypothesis, $\alpha_j^{(0)} - \alpha_k^{(0)} \not\equiv 0 \bmod. \sqrt{-1}$, except when $j=k$, every $\eta_{jk}^{(0)} = 0$ if $j \neq k$. The initial value of $y_{1k}^{(0)}$ is c_{1k}; therefore $\eta_{kk}^{(0)} = 1$.

If c_{ik} were zero the initial condition would be imposed upon another $y_{ik}^{(0)}$, not all of which can be zero at $t=0$. The solution satisfying the conditions laid down is then

$$y_{ik}^{(0)}=c_{ik}. \tag{114}$$

Coefficients of μ. The differential equations for the terms in the first power of μ are

$$(y_{ik}^{(1)})'+\alpha_k^{(0)}y_{ik}^{(1)}-\sum_{j=1}^{n}a_{ij}y_{jk}^{(1)}=-\alpha_k^{(1)}y_{ik}^{(0)}+\sum_{j=1}^{n}\theta_{ij}^{(1)}y_{jk}^{(0)} \qquad (i=1,\ldots,n). \tag{115}$$

The general solution for the terms homogeneous in $y_{ik}^{(1)}$ is

$$y_{ik}^{(1)}=\sum_{j=1}^{n}\eta_{jk}^{(1)}c_{ij}e^{(\alpha_j^{(0)}-\alpha_k^{(0)})t}, \tag{116}$$

where the $\eta_{jk}^{(1)}$ are the as yet undetermined constants of integration, and the c_{ij} are the same as in (113).

Using the method of variation of parameters, we find

$$\sum_{j=1}^{n}(\eta_{jk}^{(1)})'c_{ij}e^{(\alpha_j^{(0)}-\alpha_k^{(0)})t}=-\alpha_k^{(1)}y_{ik}^{(0)}+\sum_{j=1}^{n}\theta_{ij}^{(1)}y_{jk}^{(0)}=g_{ik}^{(1)}(t), \tag{117}$$

where the $g_{ik}^{(1)}(t)$ are periodic in t with the period 2π. The determinant of the coefficients of the $(\eta_{jk}^{(1)})'$ is

$$\Delta=|c_{ij}|\,e^{\sum_{j=1}^{n}(\alpha_j^{(0)}-\alpha_k^{(0)})t}=e^{\sum_{j=1}^{n}(\alpha_j^{(0)}-\alpha_k^{(0)})t},$$

which can not vanish for any finite value of t. Therefore the solutions of equations (117) for $(\eta_{jk}^{(1)})'$ are

$$(\eta_{jk}^{(1)})'=e^{-(\alpha_j^{(0)}-\alpha_k^{(0)})t}\Delta_{jk}^{(1)}, \tag{118}$$

where the $\Delta_{jk}^{(1)}$ are periodic functions of t with the period 2π.

The solutions of (118) for $j\neq k$ have the form

$$\eta_{jk}^{(1)}=e^{-(\alpha_j^{(0)}-\alpha_k^{(0)})t}P_{jk}^{(1)}+B_{jk}^{(1)} \qquad (j\neq k), \tag{119}$$

where the $P_{jk}^{(1)}(t)$ are periodic with the period 2π, and the $B_{jk}^{(1)}$ are arbitrary constants. For $j=k$ equation (118) becomes

$$(\eta_{kk}^{(1)})'=\Delta_{kk}^{(1)}=-\alpha_k^{(1)}+\delta_{kk}^{(1)}, \tag{120}$$

where $\delta_{kk}^{(1)}$ is $\Delta_{kk}^{(1)}$ after the terms $-\alpha_k^{(1)}y_{ik}^{(0)}$ have been omitted from the k^{th} column. It is a periodic function of t with the period 2π, and has in general a term independent of t. It can be written in the form

$$\delta_{kk}^{(1)}=d_k^{(1)}+Q_k^{(1)}(t),$$

where $d_k^{(1)}$ is a constant and $Q_k^{(1)}(t)$ is a periodic function whose mean value is zero. Then

$$(\eta_{kk}^{(1)})'=(d_k^{(1)}-\alpha_k^{(1)})+Q_k^{(1)}(t).$$

It is clear that if $\eta_{kk}^{(1)}$ is to be periodic the right member of this equation must not contain any constant terms. Therefore

$$a_k^{(1)} = d_k^{(1)}, \tag{121}$$

and

$$\eta_{kk}^{(1)} = P_{kk}^{(1)} + B_{kk}^{(1)}, \tag{122}$$

where $P_{kk}^{(1)}$ is periodic with the period 2π and $B_{kk}^{(1)}$ is the constant of integration.

Upon substituting (119) and (122) in (116), the general solution with the value of $a_k^{(1)}$ determined by (121) becomes

$$y_{ik}^{(1)} = \sum_{j=1}^{n} B_{jk}^{(1)} c_{ij}\, e^{\left(a_j^{(0)} - a_k^{(0)}\right)t} + \sum_{j=1}^{n} c_{ij} P_{jk}^{(1)}(t). \tag{123}$$

In order that the y_{ik} shall be periodic with the period 2π, all the B_{jk} must vanish except B_{kk}. From the condition that $y_{1k}(0) = c_{1k}$ for all $|\mu|$ sufficiently small, it follows that $y_{1k}^{(0)}(0) = c_{1k}$ and $y_{1k}^{(j)}(0) = 0$ $(j = 1, \ldots \infty)$. From the condition that $y_{1k}^{(1)} = 0$ at $t = 0$ it follows that

$$B_{kk}^{(1)} = -\frac{1}{c_{1k}} \sum_{j=1}^{n} c_{1j} P_{jk}^{(1)}(0).$$

Therefore the solution satisfying all the conditions is

$$y_{ik}^{(1)} = \sum_{j=1}^{n} \left[c_{ij} P_{jk}^{(1)}(t) - \frac{c_{ik}}{c_{1k}} c_{1j} P_{jk}^{(1)}(0) \right]. \tag{124}$$

It remains to be shown that the integration of the coefficients of the higher powers of μ can be effected in a similar manner. Let it be supposed that $a_k^{(1)}, a_k^{(2)}, \ldots, a_k^{(m-1)}$ and the $y_{ik}^{(1)}, y_{ik}^{(2)}, \ldots, y_{ik}^{(m-1)}$ have been uniquely determined so that the $y_{ik}^{(l)}(t)$ are periodic with the period 2π and that $y_{1k}^{(l)} = 0$, $l = 1, \ldots, m-1$. It will be shown that the $y_{ik}^{(m)}$ can be determined so as to satisfy the same conditions.

From equations (96) it is found that

$$\left.\begin{aligned} (y_{ik}^{(m)})' + a_k^{(0)} y_{ik}^{(m)} - \sum_{j=1}^{k} a_{ij} y_{jk}^{(m)} &= -a_k^{(m)} y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(m)} y_{jk}^{(0)} \\ &+ \sum_{p=1}^{m-1} \left[-a_k^{(p)} y_{ik}^{(m-p)} + \sum_{j=1}^{n} \theta_{ij}^{(p)} y_{jk}^{(m-p)} \right]. \end{aligned}\right\} \tag{125}$$

Omitting the terms included under the sign of summation with respect to p, these equations are identical in form with equations (115) except for the superscripts (1) and (m). Obviously the integrations proceed with the index (m) just as with the index (1), and the character of the process is no wise altered by the inclusion of the terms under the sign of summation with respect to p, for they are all periodic with the period 2π and do not change the essential character of the $g_{ik}^{(m)}(t)$. Therefore $a_k^{(m)}$ and the $y_{ik}^{(m)}$ can be uniquely determined so that the $y_{ik}^{(m)}$ shall satisfy the differential equations and be periodic in t with the period 2π, and so that at the same time $y_{1k}^{(m)}(0) = 0$. The induction is complete and the process can be indefinitely continued. The solutions associated with the other $a_j^{(0)}$ are found in the same way.

27. Construction of the Solutions when the Difference of two Roots of the Characteristic Equation is Congruent to Zero mod. $\sqrt{-1}$.—It will be supposed now that $\alpha_2^{(0)}-\alpha_1^{(0)}$ is congruent to zero mod. $\sqrt{-1}$ and that this relation is not satisfied by any other pair of $\alpha_j^{(0)}$. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are computed by the method of § 26 without modification. It has been shown that in general α_1, α_2 and the y_{i1}, y_{i2} can be developed as converging series in integral powers of μ. It will be assumed further that the case under consideration is not an exceptional one.

The general solutions of (96) for the terms independent of μ is in this case

$$y_{i1}^{(0)}=\sum_{j=1}^{n}\eta_{j1}^{(0)}c_{ij}e^{(\alpha_j^{(0)}-\alpha_1^{(0)})t}.$$

Imposing the conditions that $y_{i1}^{(0)}$ shall be periodic with the period 2π and that $y_{11}^{(0)}(0)=c_{11}$, these equations become, since $\alpha_2^{(0)}-\alpha_1^{(0)}$ is an imaginary integer,

$$y_{i1}^{(0)}=\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)c_{i1}+\eta_{21}^{(0)}c_{i2}e^{(\alpha_2^{(0)}-\alpha_1^{(0)})t}\qquad(i=1,\ldots,n),\quad(126)$$

where $\eta_{21}^{(0)}$ is so far arbitrary.

Coefficients of μ. It follows from (96) that the coefficients of μ must satisfy the equations

$$(y_{i1}^{(1)})'+\alpha_1^{(0)}y_{i1}^{(1)}-\sum_{j=1}^{n}a_{ij}y_{j1}^{(1)}=-\alpha_1^{(1)}y_{i1}^{(0)}+\sum_{j=1}^{n}\theta_{ij}^{(1)}y_{j1}^{(0)}\qquad(i=1,\ldots,n).\quad(127)$$

The general solution of these equations when their right members are zero is

$$y_{i1}^{(1)}=\sum_{j=1}^{n}\eta_{j1}^{(1)}c_{ij}e^{(\alpha_j^{(0)}-\alpha_1^{(0)})t}\qquad(i=1,\ldots,n).\quad(128)$$

On considering the coefficients $\eta_{j1}^{(1)}$ as functions of t and imposing the conditions that (127) shall be satisfied, it is found that

$$\sum_{j=1}^{n}(\eta_{j1}^{(1)})'c_{ij}e^{(\alpha_j^{(0)}-\alpha_1^{(0)})t}=-\alpha_1^{(1)}y_{i1}^{(0)}+\sum_{j=1}^{n}\theta_{ij}^{(1)}y_{j1}^{(0)}\qquad(i=1,\ldots,n).$$

On substituting the values of the $y_{i1}^{(0)}$ from (126) and solving, there result

$$\left.\begin{aligned}(\eta_{11}^{(1)})'&=-\alpha_1^{(1)}\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)+\eta_{21}^{(0)}\Delta_{11}^{(1)}(t)+D_{11}^{(1)}(t),\\(\eta_{21}^{(1)})'&=-\alpha_1^{(1)}\eta_{21}^{(0)}+\eta_{21}^{(0)}\Delta_{21}^{(1)}(t)+D_{21}^{(1)}(t),\\(\eta_{j1}^{(1)})'&=e^{-(\alpha_j^{(0)}-\alpha_1^{(0)})t}\Delta_{j1}^{(1)}(t)\qquad(j=3,\ldots,n),\end{aligned}\right\}\quad(129)$$

where the $\Delta_{j1}^{(1)}$ and $D_{i1}^{(1)}$ are periodic functions of t with the period 2π depending upon the $\theta_{ij}^{(1)}$ and $e^{(\alpha_2^{(0)}-\alpha_1^{(0)})t}$. In the first two equations the undetermined constants $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ enter only as they are exhibited explicitly.

Equations (129) are to be integrated and the results substituted in (128). In order that the $y_{i1}^{(1)}$ shall be periodic the conditions must be imposed that

$$\left.\begin{aligned} 0 &= -\alpha_1^{(1)}\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)+\eta_{21}^{(0)}\,b_{11}^{(1)}+d_{11}^{(1)}, \\ 0 &= -\alpha_1^{(1)}\eta_{21}^{(0)}+\eta_{21}^{(0)}b_{21}^{(1)}+d_{21}^{(1)}, \\ 0 &= \quad B_{j1}^{(1)} \qquad\qquad (j=3,\ldots,n), \end{aligned}\right\} \tag{130}$$

where $b_{11}^{(1)}$, $b_{21}^{(1)}$, $d_{11}^{(1)}$, $d_{21}^{(1)}$ are the constant terms of $\Delta_{11}^{(1)}$, $\Delta_{21}^{(1)}$, $D_{11}^{(1)}$, and $D_{21}^{(1)}$ respectively, and where the $B_{j1}^{(1)}$ are the constants of integration obtained with the last $n-2$ equations. These equations determine two solutions for the arbitraries $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ except in those special cases where the existence shows the solutions are expansible in other forms.

Upon eliminating $\eta_{21}^{(0)}$ between the first two equations of (130), it is found that $\alpha_1^{(1)}$ must satisfy the equation

$$(\alpha_1^{(1)})^2-\left[d_{11}^{(1)}+b_{21}^{(1)}+\frac{c_{12}}{c_{11}}d_{21}^{(1)}\right]\alpha_1^{(1)}+\left[b_{21}^{(1)}d_{11}^{(1)}-b_{11}^{(1)}d_{21}^{(1)}\right]=0. \tag{131}$$

If the discriminant of this quadratic is not zero the case is that in the existence proof, equations (108), where b_1 and b_2 are distinct. In this case, which may be regarded as the general one, the solutions proceed according to integral powers of μ. If the discriminant is zero the character of the solutions depends upon the coefficients of terms of higher degree, and they may proceed according to powers of μ or $\pm\sqrt{\mu}$. It will be supposed that the discriminant is distinct from zero, and the method of constructing the solutions will be developed.

Choosing one of the pairs of values of $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ which satisfy (130), it will be shown that henceforth the solution is unique. Upon imposing the condition that $y_{11}^{(1)}(0)=0$, integrating (129), substituting the results in (128), and determining the constants of integration so that the solution shall be periodic, it is found that

$$\left.\begin{aligned} y_{i1}^{(1)} &= B_{21}^{(1)}\left[-\frac{c_{12}}{c_{11}}c_{i1}+c_{i2}\,e^{(\alpha_2^{(0)}-\alpha_1^{(0)})t}\right]+\sum_{j=1}^{n}\left[c_{ij}P_{j1}^{(1)}(t)-\frac{c_{1j}}{c_{11}}c_{i1}P_{j1}^{(1)}(0)\right] \\ &\qquad\qquad (i=1,\ldots,n), \end{aligned}\right\} \tag{132}$$

where $B_{21}^{(1)}$ is an undetermined constant, and the $P_{j1}^{(1)}$ are entirely known periodic functions of t, having the period 2π.

Coefficients of μ^2. The coefficients of μ^2 are defined by

$$\left.\begin{aligned} (y_{i1}^{(2)})'+\alpha_1^{(0)}y_{i1}^{(2)}-\sum_{j=1}^{n}a_{ij}y_{i1}^{(2)} &= -\alpha_1^{(2)}y_{i1}^{(0)}-\alpha_1^{(1)}y_{i1}^{(1)}+\sum_{j=1}^{n}\left[\theta_{ij}^{(2)}\,y_{j1}^{(0)}+\theta_{ij}^{(1)}\,y_{j1}^{(1)}\right] \\ &\qquad\qquad (i=1,\ldots,n). \end{aligned}\right\} \tag{133}$$

The general solution of these equations when the right members are neglected is the same as (128) except that the superscripts are (2) instead of (1). On varying the $\eta_{j1}^{(2)}$, the equations corresponding to (129) are

$$\left.\begin{aligned}
(\eta_{11}^{(2)})' &= -a_1^{(2)}\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)+\frac{c_{12}}{c_{11}}a_1^{(1)}B_{21}^{(1)}+B_{21}^{(1)}\Delta_{11}^{(1)}(t)+D_{11}^{(2)}(t),\\
(\eta_{21}^{(2)})' &= -a_1^{(2)}\eta_{21}^{(0)}-a_1^{(1)}B_{21}^{(1)}+B_{21}^{(1)}\Delta_{21}^{(1)}(t)+D_{21}^{(2)}(t),\\
(\eta_{j1}^{(2)})' &= e^{-(a_j^{(0)}-a_1^{(0)})t}\,\Delta_{j1}^{(2)}(t) \qquad (j=3,\ \ldots\ ,n).
\end{aligned}\right\}\tag{134}$$

The undetermined constants $a_1^{(2)}$ and $B_{21}^{(1)}$ are exhibited explicitly in the first two equations, and it is to be noted that $\Delta_{11}^{(1)}$ and $\Delta_{21}^{(1)}$ are precisely the same functions of t as those which appeared in (129).

In order that these equations shall lead to periodic values of the $y_{i1}^{(2)}$ the undetermined constants must satisfy the conditions

$$\left.\begin{aligned}
0 &= -a_1^{(2)}\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)+\frac{c_{12}}{c_{11}}a_1^{(1)}B_{21}^{(1)}+b_{11}^{(1)}B_{21}^{(1)}+d_{11}^{(2)},\\
0 &= -a_1^{(2)}\eta_{21}^{(0)}-a_1^{(1)}B_{21}^{(1)}+b_{21}^{(1)}B_{21}^{(1)}+d_{21}^{(2)},\\
0 &= B_{j1}^{(2)} \qquad (j=3,\ \ldots\ ,n),
\end{aligned}\right\}\tag{135}$$

The first two equations are linear in $a_1^{(2)}$ and $B_{21}^{(1)}$, and they determine these quantities uniquely unless their determinant is zero. The determinant is

$$\Delta=-\begin{vmatrix} 1-\eta_{21}^{(0)}\dfrac{c_{12}}{c_{11}}, & b_{11}^{(1)}+a_1^{(1)}\dfrac{c_{12}}{c_{11}} \\ \eta_{21}^{(0)} \quad , & b_{21}^{(1)}-a_1^{(1)} \end{vmatrix} = -b_{21}^{(1)}+a_1^{(1)}+\eta_{21}^{(0)}\left[b_{11}^{(1)}+\frac{c_{12}}{c_{11}}b_{21}^{(1)}\right].$$

On eliminating $\eta_{21}^{(0)}$ and $a_1^{(1)}$ by means of (130), it is found that

$$\Delta=\pm\sqrt{D},$$

where D is the discriminant of (131). Since by hypothesis D is not zero, the determinant Δ is not zero. Hence $a_1^{(2)}$ and $B_{21}^{(1)}$ are uniquely determined by (135). Having determined $B_{21}^{(1)}$ and $a_1^{(2)}$, equations (134) are integrated and the results are substituted in the equations corresponding to (128). Then the conditions that $y_{11}^{(2)}(0)=0$ are imposed and the final solution at this step becomes

$$\left.\begin{aligned}
y_{i1}^{(2)} = B_{21}^{(2)}\left[-\frac{c_{12}}{c_{11}}c_{i1}+c_{i2}\,e^{(a_2^{(0)}-a_1^{(0)})t}\right]+\sum_{j=1}^{n}\left[c_{ij}P_{j1}^{(2)}(t)-\frac{c_{1j}}{c_{11}}c_{i1}P_{j1}^{(2)}0)\right]\\
(i=1,\ \ldots\ ,n),
\end{aligned}\right\}\tag{136}$$

where $B_{21}^{(2)}$ is undetermined until the next step of the integration.

The next step is similar to the preceding and all the equations are the same except the superscripts are (3) and (2) in place of (2) and (1) respectively. The determinant of the equations corresponding to (135) is precisely the same. In fact all succeeding steps are the same, and the whole process can be repeated as many times as is desired. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are found as they were in §26.

For congruences of higher order similar methods can be used, and in the cases which are exceptions to this mode of treatment the existence discussion furnishes a sure guide for the construction of the solutions.

28. Construction of the Solutions when two Roots of the Characteristic Equation are Equal.—It will be supposed $\alpha_2^{(0)} = \alpha_1^{(0)}$ and that all the remaining $\alpha_j^{(0)}$ are mutually distinct and distinct from $\alpha_1^{(0)}$. The solutions depending upon $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ can be computed by the method of §26. It has been shown that the two solutions proceeding from $\alpha_1^{(0)}$ are in general expansible as power series in $\sqrt{\mu}$. The detailed discussion will be made only for the general case, where

$$\left.\begin{aligned} \alpha_1 &= \alpha_1^{(0)} + \alpha_1^{(1)}\mu^{\frac{1}{2}} + \alpha_1^{(2)}\mu + \cdots, \\ \alpha_2 &= \alpha_1^{(0)} - \alpha_1^{(1)}\mu^{\frac{1}{2}} + \alpha_1^{(2)}\mu - \cdots, \\ y_{i1} &= y_{i1}^{(0)} + y_{i1}^{(1)}\mu^{\frac{1}{2}} + y_{i1}^{(2)}\mu + \cdots, \\ y_{i2} &= y_{i1}^{(0)} - y_{i1}^{(1)}\mu^{\frac{1}{2}} + y_{i1}^{(2)}\mu - \cdots. \end{aligned}\right\} \tag{137}$$

Terms independent of μ. The terms independent of μ are defined by

$$(y_{i1}^{(0)})' + \alpha_1^{(0)} y_{i1}^{(0)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(0)} = 0 \qquad (i=1, \ldots, n). \tag{138}$$

The general solution of these equations is

$$y_{i1}^{(0)} = \eta_{11}^{(0)} c_{i1} + \eta_{21}^{(0)}(c_{i2} + tc_{i1}) + \sum_{j=3}^{n} \eta_{j1}^{(0)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t}. \tag{139}$$

In order that the $y_{i1}^{(0)}$ shall be periodic with the period 2π and the initial value c_{i1} of $y_{i1}^{(0)}$ shall be obtained, the $\eta_{j1}^{(0)}$ must satisfy the conditions

$$\eta_{j1}^{(0)} = 0 \qquad (j=2, \ldots, n).$$

The solution satisfying all the conditions is then

$$y_{i1}^{(0)} = c_{i1} \qquad (i=1, \ldots, n). \tag{140}$$

Coefficients of $\mu^{\frac{1}{2}}$. The coefficients of $\mu^{\frac{1}{2}}$ are defined by the equations

$$(y_{i1}^{(1)})' + \alpha_1^{(0)} y_{i1}^{(1)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(1)} = -\alpha_1^{(1)} y_{i1}^{(0)} = -\alpha_1^{(1)} c_{i1} \qquad (i=1, \ldots, n). \tag{141}$$

On neglecting the right members, the general solution of these equations is

$$y_{i1}^{(1)} = \eta_{11}^{(1)} c_{i1} + \eta_{21}^{(1)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} \eta_{j1}^{(1)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t}. \qquad (142)$$

The method of variation of parameters leads to the conditions

$$(\eta_{11}^{(1)})' c_{i1} + (\eta_{21}^{(1)})' (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} (\eta_{j1}^{(1)})' c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t} = -\alpha_1^{(1)} c_{i1} \qquad (i=1, \ldots, n).$$

On solving these equations for the $(\eta_{j1}^{(1)})'$, it is found that

$$(\eta_{11}^{(1)})' = -\alpha_1^{(1)}, \qquad (\eta_{j1}^{(1)})' = 0 \qquad (j=2, \ldots, n).$$

Consequently

$$\eta_{11}^{(1)} = B_{11}^{(1)} - \alpha_1^{(1)} t, \qquad \eta_{j1}^{(1)} = B_{j1}^{(1)} \qquad (j=2, \ldots, n). \quad (143)$$

On substituting the values from (143) in (142), the result becomes

$$y_{i1}^{(1)} = (B_{11}^{(1)} - \alpha_1^{(1)} t) c_{i1} + B_{21} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} B_{j1}^{(1)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t}.$$

To satisfy the conditions for periodicity and to make $y_{i1}^{(1)}(0) = 0$, the B_{j1} must fulfill the relations

$$B_{21}^{(1)} = \alpha_1^{(1)}, \qquad B_{11}^{(1)} c_{11} + B_{21}^{(1)} c_{12} = 0, \qquad B_{j1}^{(1)} = 0 \qquad (j=3, \ldots, n). \quad (144)$$

Then the solutions satisfying all the conditions become

$$y_{i1}^{(1)} = \left(-\frac{c_{12}}{c_{11}} c_{i1} + c_{i2} \right) \alpha_1^{(1)}, \qquad (145)$$

where the constant $\alpha_1^{(1)}$ remains as yet undetermined. It is to be observed that not all the coefficients of $\alpha_1^{(1)}$ can vanish, for otherwise the determinant $|c_{ij}|$ itself would vanish.

Coefficients of μ. The coefficients of μ are determined by the equations

$$(y_{i1}^{(2)})' + \alpha_1^{(0)} y_{i1}^{(2)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(2)} = -\alpha_1^{(2)} y_{i1}^{(0)} - \alpha_1^{(1)} y_{i1}^{(1)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)} \qquad (i=1, \ldots, n). \quad (146)$$

The solution of the homogeneous terms is of the same form as (142), and by varying the constants of integration, it is found that

$$\left.\begin{aligned} &(\eta_{11}^{(2)})' c_{i1} + (\eta_{21}^{(2)})' (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} (\eta_{j1}^{(2)})' c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t} = \\ &\qquad -\alpha_1^{(2)} c_{i1} - \left(-\frac{c_{12}}{c_{11}} c_{i1} + c_{i2} \right) (\alpha_1^{(1)})^2 + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)} \end{aligned}\right\} (147)$$

The solutions of these equations for the $(\alpha_{j1}^{(2)})'$ are

$$\left.\begin{aligned}
(\eta_{11}^{(2)})' &= -\alpha_1^{(2)} + \left(t + \frac{c_{12}}{c_{11}}\right)(\alpha_1^{(1)})^2 + t\Delta_{11}^{(2)}(t) + D_{11}^{(2)}(t), \\
(\eta_{21}^{(2)})' &= \qquad\qquad -(\alpha_1^{(1)})^2 - \Delta_{11}^{(2)}(t), \\
(\eta_{j1}^{(2)})' &= +e^{-\left(\alpha_j^{(0)}-\alpha_1^{(0)}\right)t}\Delta_{j1}^{(2)}(t) \qquad (j=3, \ldots, n),
\end{aligned}\right\} \tag{148}$$

where the $\Delta_{j1}^{(2)}(t)$ and $D_{11}^{(2)}(t)$ are known periodic functions of t. The first equation gives rise to integrals of the type

$$-a_j \int t \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} jt\,dt = \mp \frac{a_j t}{j} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jt + \frac{a_j}{j^2} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} jt.$$

The second equation gives rise to the corresponding integral

$$-a_j \int \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} jt\,dt = \pm \frac{a_j}{j} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jt.$$

When these results are substituted in equations (142) the terms of the type $t \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jt$ destroy each other. Hence at this step

$$\left.\begin{aligned}
y_{11}^{(2)} &= B_{11}^{(2)}c_{11} + B_{21}^{(2)}(c_{12} + tc_{11}) + \sum_{j=3}^{n} B_{j1}^{(2)}c_{1j}e^{\left(\alpha_j^{(0)}-\alpha_1^{(0)}\right)t} \\
&\quad + \left[\left(-\alpha_1^{(2)} + \frac{c_{12}}{c_{11}}\alpha_1^{(1)2} + d_{11}^{(2)}\right)t + \tfrac{1}{2}(\alpha_1^{(1)2} + b_{11}^{(2)})t^2 + P_{11}^{(2)}(t)\right]c_{11} \\
&\quad + [-(\alpha_1^{(1)2} + b_{11}^{(2)})t + P_{21}^{(2)}(t)]c_{12} + \sum_{j=3}^{n} c_{1j}P_{j1}^{(2)}(t),
\end{aligned}\right\} \tag{149}$$

where the $P_{j1}^{(2)}(t)$ are periodic functions of t, and $b_{11}^{(2)}$ and $d_{11}^{(2)}$ are the constant terms in $\Delta_{11}^{(2)}$ and $D_{11}^{(2)}$. Equations (149) are the general solutions of equations (147). In order to satisfy the conditions for periodicity and the initial condition $y_{11}^{(2)}(0)=0$, the constants $\alpha_1^{(1)}$ and $B_{j1}^{(2)}$ must fulfill the relations

$$\left.\begin{aligned}
\alpha_1^{(1)} &= \pm\sqrt{-b_{11}^{(2)}}, & 0 &= B_{11}^{(2)}c_{11} + B_{21}^{(2)}c_{12} + \sum_{j=1}^{n} P_{j1}^{(2)}(0), \\
B_{21}^{(2)} &= \alpha_1^{(2)} + \frac{c_{12}}{c_{11}} b_{11}^{(2)} - d_{11}^{(2)}, & 0 &= B_{j1}^{(2)} \qquad (j=3, \ldots, n).
\end{aligned}\right\} \tag{150}$$

The constant $\alpha_1^{(2)}$ still remains undetermined. The solutions now are

$$y_{11}^{(2)} = \left(-\frac{c_{12}}{c_{11}}c_{11} + c_{12}\right)\alpha_1^{(2)} + \Phi_{11}^{(2)}(t), \tag{151}$$

where the $\Phi_{11}^{(2)}$ are known periodic functions of t having the period 2π. After making a choice as to $\alpha_1^{(1)}$, provided $b_{11}^{(2)} \neq 0$, it is found that $\alpha_1^{(2)}$ is determined uniquely by the periodicity conditions for $y_{11}^{(3)}$. The process can be continued indefinitely and the constants are determined uniquely. The other solution associated with $\alpha_1^{(0)}$ is obtained by taking the other determination of $\alpha_1^{(1)}$, and the solutions depending on $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ by the method of § 26.

The chief types of cases have been treated, and the exceptions to them are developed similarly according to the forms indicated in the existence proofs.

IV. NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.

29. Case where the Right Members are Periodic with the Period 2π and the α_j are Distinct.—Take the set of differential equations

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = g_i(t) \qquad (i=1, \ldots, n), \tag{152}$$

where the θ_{ij} and the $g_i(t)$ are periodic in t with the period 2π. For the left members set equal to zero the form of the solution of (152) is

$$x_i = \sum_{j=1}^{n} \eta_j e^{\alpha_j t} y_{ij}\,, \tag{153}$$

where the η_j are arbitrary constants, the α_j are the characteristic exponents which are supposed distinct, and the y_{ij} are periodic in t with the period 2π.

By the method of variation of parameters, it is found that

$$\sum_{j=1}^{n} \eta_j' e^{\alpha_j t} y_{ij} = g_i(t). \tag{154}$$

The determinant of the coefficients of the η_j' is the determinant of the fundamental set of solutions. Since the θ_{ij} are assumed to be regular for all finite values of t, it follows from §18 that this determinant can not vanish for any finite value of t. This determinant is

$$\Delta e^{\sum_{j=1}^{n} \alpha_j t},$$

where Δ is the determinant of the y_{ij}. If Δ_j denotes that which Δ becomes when the j^{th} column is replaced by the $g_i(t)$, the solutions of (154) for the η_j' are

$$\eta_j' = \frac{\Delta_j}{\Delta} e^{-\alpha_j t}, \tag{155}$$

and consequently

$$\eta_j = \int \frac{\Delta_j}{\Delta} e^{-\alpha_j t}\, dt + B_j\,. \tag{156}$$

The quotient Δ_j/Δ is a periodic function of t, continuous and finite in the interval $0 \overline{\gtrless} t \leqq 2\pi$. Therefore it can be expanded into the Fourier series

$$\frac{\Delta_j}{\Delta} = a_0^{(j)} + \sum_{m=1}^{\infty} [a_m^{(j)} \cos mt + b_m^{(j)} \sin mt].$$

If $\alpha_j^2 + m^2 \neq 0$ $(j=1, \ldots, n;\ m=1, \ldots, \infty)$, the integral becomes

$$\int e^{-\alpha_j t} \frac{\Delta_j}{\Delta} dt = -\frac{a_0^{(j)}}{\alpha_j} e^{-\alpha_j t} + e^{-\alpha_j t} \sum_{m=1}^{\infty} \left[-\frac{\alpha_j a_m^{(j)} + m b_m^{(j)}}{\alpha_j^2 + m^2} \cos mt + \frac{m a_m^{(j)} - \alpha_j b_m^{(j)}}{\alpha_j^2 + m^2} \sin mt \right] \tag{157}$$

so that

$$\eta_j = e^{-\alpha_j t} P_j(t) + B_j\,,$$

where the $P_j(t)$ are periodic with the period 2π, and the B_j are constants of integration. On substituting these values of the η_j in (153), the general solutions of (152) become

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + \sum_{j=1}^{n} P_j(t) y_{ij} \qquad (i=1, \ldots, n). \tag{158}$$

Now suppose $\alpha_i = k\sqrt{-1}$, where k is an integer. Then the term

$$\int e^{-k\sqrt{-1}t}[a_k \cos kt + b_k \sin kt]\, dt$$

becomes, after the integration has been carried out,

$$\tfrac{1}{2}(a_k - b_k\sqrt{-1})\,t + \frac{1}{4k}(a_k\sqrt{-1} - b_k)(\cos 2kt - \sqrt{-1}\sin 2kt).$$

Therefore the expression corresponding to (158) becomes in this case

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + \overline{P}_i(t) + \tfrac{1}{2}(a_k - b_k\sqrt{-1})\,t\, e^{\alpha_i t} y_{ii}, \tag{159}$$

where the $\overline{P}_i(t)$ are periodic with the period 2π.

Therefore, *if the characteristic exponents are distinct and none of them is congruent to zero mod.* $\sqrt{-1}$, *and if the* $g_i(t)$ *are periodic with the period* 2π, *then the particular integrals are also periodic with the period* 2π. *But if some of the characteristic exponents are congruent to zero mod.* $\sqrt{-1}$, *then the particular integrals in general contain, in addition to periodic terms, the corresponding parts of the complementary function multiplied by a constant times* t.

30. Case where the Right Members are Periodic Terms Multiplied by an Exponential, and the α_j are Distinct.—Consider the case where the $g_i(t)$ have the form

$$g_i(t) = e^{\lambda t} f_i(t),$$

the $f_i(t)$ being periodic with the period 2π. When $\lambda = l\sqrt{-1}$ is a pure imaginary, this form includes such cases as

$$g_i(t) = \sum_k [a_k \cos(k+l)t + b_k \sin(k+l)t].$$

If the differential equations, which are now of the form

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = e^{\lambda t} f_i(t), \tag{160}$$

are transformed by $x_i = e^{\lambda t} z_i$, they become

$$z_i' + \lambda z_i - \sum_{j=1}^{n} \theta_{ij} z_j = f_i(t), \tag{161}$$

and have the same character as those treated in §29. If the characteristic exponents α_j of (160) are distinct, then the characteristic exponents of (161) are $\alpha_j - \lambda$, and are also distinct. Applying the results of the preceding case, it is seen that if no $\alpha_j - \lambda$ is congruent to zero mod. $\sqrt{-1}$, then the solutions of (161) are

$$z_i = \sum_{j=1}^{n} B_j e^{(\alpha_j - \lambda)t} y_{ij} + Q_i(t),$$

where the $Q_i(t)$ are periodic with the period 2π. Therefore the solutions of (160) are

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + e^{\lambda t} Q_i(t) \qquad (i = 1, \ldots, n). \tag{162}$$

But if one of the α_j, say α_k, is congruent to λ mod. $\sqrt{-1}$, then the z_j have the form

$$z_i = \sum_{j=1}^{n} B_j e^{(\alpha_j-\lambda)t} y_{ij} + \overline{Q}_i(t) + c_k t\, e^{(\alpha_k-\lambda)t} y_{ik} ;$$

and therefore the expressions for the x_i become

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + e^{\lambda t}\overline{Q}_i(t) + c_k t\, e^{\alpha_k t} y_{ik} . \tag{163}$$

These results may be stated as follows: *If the $g_i(t)$ have the form $g_i(t) = e^{\lambda t} f_i(t)$, where $f_i(t) \equiv f_i(t+2\pi)$, and if none of the characteristic exponents is congruent to λ mod. $\sqrt{-1}$, then the particular solution has the form*

$$x_i = e^{\lambda t} Q_i(t) \qquad (i=1, \ldots, n),$$

where the $Q_i(t)$ are periodic with the period 2π; but if one of the characteristic exponents, α_k, is congruent to λ mod. $\sqrt{-1}$, then the particular solution has the form

$$x_i = e^{\lambda t}\overline{Q}_i(t) + c_k t\, e^{\alpha_k t} y_{ik} \qquad (i=1, \ldots, n),$$

where the $\overline{Q}_i(t)$ are periodic with the period 2π.

31. Case where two Characteristic Exponents are Equal and the Right Members are Periodic.—Suppose $\alpha_2 = \alpha_1$. Then the solutions of

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = 0 \qquad (i=1, \ldots, n),$$

in general have the form

$$x_i = \eta_1 e^{\alpha_1 t} y_{i1} + \eta_2 e^{\alpha_1 t}(y_{i2} + t y_{i1}) + \sum_{j=3}^{n} \eta_j e^{\alpha_j t} y_{ij} \qquad (i=1, \ldots, n). \tag{164}$$

For the associated non-homogeneous equations

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = g_i(t) \tag{165}$$

it is found by applying the method of the variation of parameters that

$$e^{\alpha_1 t} y_{i1} \eta_1' + e^{\alpha_1 t}(y_{i2} + t y_{i1})\eta_2' + \sum_{j=3}^{n} e^{\alpha_j t} y_{ij} \eta_j' = g_i(t) \qquad (i=1, \ldots, n).$$

On solving these equations for the η_j', the results are found to be

$$\begin{aligned}
\Delta\eta_1' &= |g_i(t),\ (y_{i2}+t y_{i1}),\ y_{i3},\ \ldots,\ y_{in}|\, e^{-\alpha_1 t},\\
\Delta\eta_2' &= |y_{i1},\ g_i(t),\ y_{i3},\ \ldots,\ y_{in}|\, e^{-\alpha_1 t},\\
\Delta\eta_j' &= |y_{i1},\ (y_{i2}+t y_{i1}),\ y_{i3},\ \ldots,\ y_{in}|\, e^{-\alpha_j t} \qquad (j=3, \ldots, n),
\end{aligned}$$

where Δ is the determinant $|y_{ij}|$. The expressions of these determinants have the form

$$\left.\begin{aligned}
\eta_1' &= e^{-\alpha_1 t} P_1(t) - e^{-\alpha_1 t} t P_2(t), \qquad & \eta_2' &= e^{-\alpha_1 t} P_2(t),\\
\eta_j' &= e^{-\alpha_j t} P_j(t) & & (j=3, \ldots, n),
\end{aligned}\right\} \tag{166}$$

where the $P_j(t)$ $(j=1, \ldots, n)$ are periodic with the period 2π.

Suppose no α_j is congruent to zero mod. $\sqrt{-1}$. Then it is easily found that

$$\left.\begin{aligned}\eta_1&=e^{-\alpha_1 t}R_1(t)-e^{-\alpha_1 t}t\,R_2(t)+B_1,\\ \eta_2&=e^{-\alpha_1 t}R_2(t)+B_2,\\ \eta_j&=e^{-\alpha_j t}R_j(t)+B_j \qquad (j=3,\ldots,n),\end{aligned}\right\}\tag{167}$$

where $R_1(t),\ \ldots,\ R_n(t)$ are periodic with the period 2π. On substituting these values in (164), the solution becomes

$$x_i=B_1e^{\alpha_1 t}y_{i1}+B_2e^{\alpha_1 t}(y_{i2}+t\,y_{i1})+\sum_{j=3}^{n}B_je^{\alpha_j t}y_{ij}+\sum_{j=1}^{n}R_j(t)\,y_{ij}.$$

The $\sum_{j=1}^{n}R_jy_{ij}$ are periodic with the period 2π. Hence, *if two of the characteristic exponents are equal but none of them is congruent to zero mod.* $\sqrt{-1}$, *then the particular solution also is periodic with the period* 2π.

The case where one $\alpha_j\,(j=3,\ \ldots,\ n)$ is congruent to zero mod. $\sqrt{-1}$ is a combination of the present case with the second part of that treated in §29, and that where $\alpha_2=\alpha_1$ is congruent to zero mod. $\sqrt{-1}$ does not differ in any essentials from that where $\alpha_2=\alpha_1=0$.

Now suppose $\alpha_2=\alpha_1=0$. Then the equations which correspond to (166) become

$$\eta_1'=P_1(t)-tP_2(t),\qquad \eta_2'=P_2(t),\qquad \eta_j'=e^{-\alpha_j t}P_j(t)\qquad (j=3,\ \ldots,\ n).$$

The $P_j(t)$ are periodic with the period 2π and can be written in the form

$$P_j=a_j+\sum_{k=1}^{\infty}[a_{kj}\cos kt+b_{kj}\sin kt].$$

Hence the η_j are found, by integrating, to have the form

$$\begin{aligned}\eta_1&=+R_1(t)+a_1t-\tfrac{1}{2}a_2t^2-tR_2(t),\\ \eta_2&=+R_2(t)+a_2t,\\ \eta_j&=e^{-\alpha_j t}R_j(t)+B_j \qquad (j=3,\ \ldots,\ n),\end{aligned}$$

where

$$R_j(t)\equiv R_j(t+2\pi)\qquad (j=1,\ \ldots,\ n).$$

These values substituted in (164) give for the complete solutions

$$x_i=B_1y_{i1}+B_2(y_{i2}+ty_{i1})+\sum_{j=1}^{n}B_je^{\alpha_j t}y_{ij}+[a_1t+\tfrac{1}{2}a_2t^2]\,y_{i1}+a_2ty_{i2}+\sum_{j=3}^{n}R_jy_{ij}\,.$$

Hence, *when the* $g_i(t)$ *are periodic with the period* 2π, *and when two of the characteristic exponents are not only equal, but are zero, then the particular integral involves not only* t *but, in general, also* t^2 *outside of the trigonometric symbols.* Of course it may happen that a_1 and a_2 are either or both zero.

Those particular cases have been treated which will be most useful in in the applications which follow. Any others which may arise can be discussed in a similar way.

V. EQUATIONS OF VARIATION AND THEIR CHARACTERISTIC EXPONENTS.*

32. The Equations of Variations.—The preceding considerations find immediate application in dynamics in the study of small variations from known periodic solutions. Suppose there are given the equations

$$\frac{dx_i}{dt}=X_i \qquad (i=1,\ .\ .\ .\ ,\ n), \tag{168}$$

where the X_i are functions of the x_i, and that they are satisfied by

$$x_i=\varphi_i(t), \tag{169}$$

the $\varphi_i(t)$ being periodic functions of t with the period 2π. These are called the *generating solutions.*

Let the initial conditions be varied slightly and put

$$x_i(0)=\varphi_i(0)+\beta_i, \tag{170}$$

where the β_i are small arbitrary constants. The value of the x_i for any t will be

$$x_i=\varphi_i(t)+\xi_i(t), \tag{171}$$

the ξ_i being functions of t which for at least a short interval of time will remain small. On substituting (171) in (168) and expanding the right members as power series in the ξ_j, it is found that

$$\frac{d\xi_i}{dt}=\sum_{j=1}^{n}\frac{\partial X_i}{\partial x_j}\xi_j+\text{higher degree terms} \qquad (i=1,\ .\ .\ .\ ,\ n), \tag{172}$$

the x_i being replaced by $\varphi_i(t)$ in the partial derivatives. Since the $\varphi_i(t)$ are periodic in t, so also are all the coefficients of (172). The ξ_i are expansible as power series in the β_i which, by the Cauchy-Poincaré theorem, §16, converge for any preassigned interval of time provided the $|\beta_i|$ are sufficiently small.

The differential equations for the linear terms in the β_i are the linear terms of (172), or

$$\xi_i'=\frac{\partial X_i}{\partial x_1}\xi_1+\frac{\partial X_i}{\partial x_2}\xi_2+\ \cdot\ \cdot\ \cdot\ +\frac{\partial X_i}{\partial x_n}\xi_n \qquad (i=1,\ .\ .\ .\ ,\ n). \tag{173}$$

These equations are known as the *equations of variation.*

Suppose the solution (169) contains an arbitrary constant c, that is, one not contained in the differential equations (168). If $c=c_0+\gamma$, the $\varphi_i(t)$ are expansible as power series in γ of the form

$$\varphi_i(t)=\varphi_i^{(0)}(t)+\frac{\partial\varphi_i^{(0)}}{\partial c}\gamma+\tfrac{1}{2}\frac{\partial^2\varphi_i^{(0)}}{\partial c^2}\gamma^2+\ \cdot\ \cdot\ \cdot \qquad (i=1,\ .\ .\ .\ ,\ n).$$

Obviously

$$\xi_i=\frac{\partial\varphi_i^{(0)}}{\partial c}\gamma+\tfrac{1}{2}\frac{\partial^2\varphi_i^{(0)}}{\partial c^2}\gamma^2+\ \cdot\ \cdot\ \cdot \qquad (i=1,\ .\ .\ .\ ,\ n)$$

*The subject of this section and other related questions have been treated by Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, chap. 4.

is a solution of equations (172), and consequently

$$\xi_i = \frac{\partial \varphi_i^{(0)}}{\partial c} \qquad (i=1, \ldots, n) \qquad (174)$$

is a solution of equations (173).

One such constant is always present when the X_i do not contain t explicitly and the $\varphi_i(t)$ are not mere constants, for then the origin of time is arbitrary. Hence in this case

$$\xi_i = \frac{\partial \varphi_i}{\partial t_0} = -\frac{\partial \varphi_i}{\partial t} \qquad (i=1, \ldots, n), \qquad (175)$$

is a solution of (173). Another such constant usually present is the scale factor which determines the size of the generating orbit. If there are p such arbitrary constants in the generating solution, the equations of variation have p solutions of the form (174).

If the equations (168) admit an integral which is independent of t,

$$F_1(x_1, \ldots, x_n) = c_1,$$

where c_1 is an arbitrary constant, the x_i can be replaced by $\varphi_i(t)+\xi_i$ in the integral and the integral can be expanded in powers of ξ_i. The result is

$$\gamma = \frac{\partial F_1}{\partial x_1}\xi_1 + \frac{\partial F_1}{\partial x_2}\xi_2 + \cdots + \frac{\partial F_1}{\partial x_n}\xi_n + \text{higher degree terms.}$$

The constant γ is a power series in the β_i, and therefore the linear terms are

$$\gamma_1 = \frac{\partial F_1}{\partial x_1}\xi_1 + \frac{\partial F_1}{\partial x_2}\xi_2 + \cdots + \frac{\partial F_1}{\partial x_n}\xi_n, \qquad (176)$$

which is therefore an integral of equations (173). The coefficients $\partial F_1/\partial x_j$ are periodic functions of t, the x_i having been replaced by $\varphi_i(t)$ after differentiation.

33. Theorems on the Characteristic Exponents.—The existence of arbitrary constants in the generating solutions and the existence of integrals of equations (168) have an intimate connection with the characteristic exponents of the solutions. These solutions are in general of the form

$$\xi_i = e^{\alpha t} f_i(t) \qquad (i=1, \ldots, n), \qquad (177)$$

the f_i being periodic with the period 2π. The solutions (175) have the form (177), but since the φ_i are periodic so also are their derivatives, and the characteristic exponent of this solution is zero. There is an exception only if the φ_i are constants, in which case the solution (175) disappears.

The solution obtained by differentiating with respect to the scale constant, which will be denoted by a, will, in general, have the form

$$\frac{\partial \varphi_i}{\partial a} = \psi_i(t) + t\bar{\psi}_i(t),$$

ψ_i and $\bar{\psi}_i$ being periodic. The characteristic exponent of this solution is zero. If the generating solutions have p distinct arbitrary constants, the equations of variation will have at least p characteristic exponents equal to zero.

From the existence of the integral (176) it follows also that at least one of the characteristic exponents is zero; for all solutions have the form

$$\xi_{ij} = e^{\alpha_j t} f_{ij}(t) \qquad (i, j = 1, \ldots, n),$$

and substituting them successively with respect to the index j in (176), we get

$$\sum_{i=1}^{n} \frac{\partial F_1}{\partial x_i} f_{ij} = e^{-\alpha_j t} \gamma_1^{(j)} \qquad (j = 1, \ldots, n). \qquad (178)$$

The left members of these equations are periodic with the period 2π, except, perhaps, for coefficients which are polynomials in t. It follows, therefore, that either the $\alpha_j \equiv 0 \mod \sqrt{-1}$, or all the $\gamma_1^{(j)} = 0$. In this connection a congruence has the same properties as an equality, and they need not be distinguished from each other. If all the α_j are distinct from zero, then $\gamma_1^{(j)} = 0$ $(j = 1, \ldots, n)$ and (178) becomes

$$\sum_{i=1}^{n} \frac{\partial F_1}{\partial x_i} f_{ij} = 0 \qquad (j = 1, \ldots, n). \qquad (179)$$

Since the determinant $|f_{ij}| \neq 0$, these equations can be satisfied only if

$$\frac{\partial F_1}{\partial x_i} \underset{t}{\equiv} 0 \qquad (i = 1, \ldots, n).$$

Therefore, unless the integral (176) vanishes identically at least one of the characteristic exponents is zero. Suppose that $\alpha_1 = 0$ and that $\gamma_1^{(1)} \neq 0$. It is possible then to solve the equations corresponding to (178) uniquely for the $\partial F_1/\partial x_i$ in terms of $\gamma_1^{(1)}$ and the f_{ij}.

Suppose now there is a second integral $F_2(x_1, \ldots, x_n) = c_2$. Then

$$\sum_{i=1}^{n} \frac{\partial F_2}{\partial x_i} \xi_i = \text{const.}$$

On substituting in this equation successively the n fundamental solutions for the ξ_i it follows, since $\alpha_1 = 0$, that

$$\sum_{i=1}^{n} \frac{\partial F_2}{\partial x_i} f_{i1} = \delta_1^{(1)}, \qquad \sum_{i=1}^{n} \frac{\partial F_2}{\partial x_i} f_{ij} = e^{-\alpha_j t} \delta_1^{(j)} \qquad (j = 2, \ldots, n). \qquad (180)$$

If $\alpha_j \neq 0$, and therefore $\delta_1^{(j)} = 0$ $(j = 2, \ldots, n)$, these equations can be solved uniquely for $\partial F_2/\partial x_i$ in terms of f_{ij} and $\delta_1^{(1)}$. It results that, aside from a constant factor,

$$\frac{\partial F_2}{\partial x_i} \underset{t}{\equiv} \frac{\partial F_1}{\partial x_i},$$

and the second integral is identical with the first. But if F_1 and F_2 are distinct, then there must be at least two characteristic exponents, say α_1 and α_2, which are zero. Proceeding in this manner it follows that if the equations of variation admit of p linearly distinct integrals not identically zero, then there are p characteristic exponents equal to zero.

If the original differential equations have the form

$$\frac{d^2x_i}{dt^2}=\frac{\partial V}{\partial x_i} \qquad (i=1, \ldots, n),$$

which is the case usually in celestial mechanics, they may be reduced to equations involving only first derivatives by writing

$$\frac{dx_i}{dt}=y_i, \qquad \frac{dy_i}{dt}=\frac{\partial V}{\partial x_i} \qquad (i=1, \ldots, n).$$

If the generating solution is

$$x_i=\varphi_i(t), \qquad y_i=\varphi_i'(t),$$

and the equations of variation are formed by putting

$$x_i=\varphi_i(t)+\xi_i, \qquad y_i=\varphi_i'(t)+\eta_i,$$

there will result

$$\left.\begin{aligned}\frac{d\xi_i}{dt}&=\eta_i \qquad (i=1, \ldots, n),\\ \frac{d\eta_i}{dt}&=\frac{\partial^2 V}{\partial x_i\partial x_1}\xi_1+\frac{\partial^2 V}{\partial x_i\partial x_2}\xi_2+\cdots+\frac{\partial^2 V}{\partial x_i\partial x_n}\xi_n.\end{aligned}\right\} \tag{181}$$

The main diagonal of the right members of these equations (considered as a determinant matrix) contains only zero elements. Therefore, by §18, the determinant of any fundamental set of solutions of these equations is a constant. But the determinant of the fundamental set of solutions

$$\xi_{ij}=e^{a_jt}f_{ij}(t), \qquad \eta_{ij}=e^{a_jt}g_{ij}(t) \qquad (j=1, \ldots, n).$$

has the form $\Delta=e^{\sum_{j=1}^{2n} a_jt}P(t)$. This must therefore be a constant, from which it follows that the sum of the characteristic exponents is zero since $P(t)$ has the period 2π.

Suppose $\xi_i^{(1)}$, $\eta_i^{(1)}$ and $\xi_i^{(2)}$, $\eta_i^{(2)}$ $(i=1, \ldots, n)$ are any two solutions of equations (181). Then

$$\frac{d\xi_i^{(1)}}{dt}=\eta_i^{(1)}, \qquad \frac{d\eta_i^{(1)}}{dt}=\frac{\partial^2V}{\partial x_i\partial x_1}\xi_1^{(1)}+\frac{\partial^2V}{\partial x_i\partial x_2}\xi_2^{(1)}+\cdots+\frac{\partial^2V}{\partial x_i\partial x_n}\xi_n^{(1)}, \tag{182}$$

and also

$$\frac{d\xi_i^{(2)}}{dt}=\eta_i^{(2)}, \qquad \frac{d\eta_i^{(2)}}{dt}=\frac{\partial^2V}{\partial x_i\partial x_1}\xi_1^{(2)}+\frac{\partial^2V}{\partial x_i\partial x_2}\xi_2^{(2)}+\cdots+\frac{\partial^2V}{\partial x_i\partial x_n}\xi_n^{(2)}. \tag{183}$$

From these equations it follows that

$$\sum_{i=1}^{n}\left(\eta_i^{(2)}\frac{d\xi_i^{(1)}}{dt}-\eta_i^{(1)}\frac{d\xi_i^{(2)}}{dt}\right)=0, \qquad \sum_{i=1}^{n}\left(\xi_i^{(1)}\frac{d\eta_i^{(2)}}{dt}-\xi_i^{(2)}\frac{d\eta_i^{(1)}}{dt}\right)=0. \tag{184}$$

The sum of these two equations is

$$\sum_{i=1}^{n}\left[\left(\eta_i^{(2)}\frac{d\xi_i^{(1)}}{dt}+\xi_i^{(1)}\frac{d\eta_i^{(2)}}{dt}\right)-\left(\eta_i^{(1)}\frac{d\xi_i^{(2)}}{dt}+\xi_i^{(2)}\frac{d\eta_i^{(1)}}{dt}\right)\right]=0, \tag{185}$$

which can be written

$$\frac{d}{dt}\sum_{i=1}^{n}(\xi_i^{(1)}\eta_i^{(2)}-\xi_i^{(2)}\eta_i^{(1)})=0.$$

Consequently

$$\sum_{i=1}^{n} (\xi_i^{(1)}\eta_i^{(2)} - \xi_i^{(2)}\eta_i^{(1)}) = \text{const.} \tag{186}$$

The relation (186) between any two solutions leads to important conclusions respecting the characteristic exponents. Suppose the $\xi_i^{(j)}$ and $\eta_i^{(j)}$ are

$$\xi_i^{(j)} = e^{\alpha_j t} f_{ij}(t), \qquad \eta_i^{(j)} = e^{\alpha_j t} g_{ij}(t) \qquad (i=1, \ldots, n; j=1, \ldots, 2n),$$

where f_{ij} and g_{ij} are polynomials in t with periodic coefficients, and that they constitute a fundamental set of solutions. On substituting any two of these solutions in (186) and dividing through by the exponential, there results

$$\sum_{i=1}^{n} (f_{ij}g_{ik} - f_{ik}g_{ij}) = \gamma_{jk}\, e^{-(\alpha_j+\alpha_k)t}. \tag{187}$$

It follows from the character of the left member of this equation that either $\alpha_j + \alpha_k = 0$, or $\gamma_{jk} = 0$. It will be shown, however, that γ_{jk} can not be zero for every k.

Suppose j is kept fixed and give to k all the values from $1, \ldots, 2n$. Suppose $\gamma_{jk} = 0 \quad (k=1, \ldots, 2n)$. Then one equation of (187) is an identity and the others are linear in the f_{ij} and the g_{ij}. The determinant of this set of linear equations is the determinant of the fundamental set and is not zero. Hence they can be satisfied only by $f_{ij} \equiv g_{ij} \equiv 0$. But this also is impossible since f_{ij} and g_{ij} are a solution of the fundamental set. Therefore not all the γ_{jk} can be zero. Hence for some k

$$\alpha_j + \alpha_k = 0, \qquad \gamma_{jk} \neq 0. \tag{188}$$

But since α_j is any one of the characteristic exponents, it follows that corresponding to each characteristic exponent there is another one which differs from it only in sign.

If two of the α_j are equal but not equal to zero, then there are two others which are also equal and which differ from the first two only in sign. In order to show this suppose $\alpha_j = \alpha_{j+1} = -\alpha_m$. Then $\alpha_m = \alpha_{m+1}$, because from (188) it follows that $\alpha_j + \alpha_m = 0$, $\gamma_{jm} \neq 0$. If $\gamma_{jk} = 0$ $(k=1, \ldots, 2n, k \neq m)$, then (187) can be solved for f_{ij} and g_{ij} uniquely in terms of f_{ik} and g_{ik} $(k=1 \ldots, 2n)$. Now the corresponding equations for $f_{i,j+1}$ and $g_{i,j+1}$ will differ from (187) only in that j is replaced by $j+1$. They can be solved uniquely for $f_{i,j+1}$ and $g_{i,j+1}$, but this solution will differ from the solution for f_{ij} and g_{ij} only by a constant factor. Since this is impossible it follows that $\gamma_{j+1,m+1} \neq 0$, and consequently $\alpha_{j+1} + \alpha_{m+1} = 0$. In the same manner it can be shown that if p of the α_j are equal, then p other α_j are also equal and differ from the first set only in sign.

CHAPTER II.

ELLIPTIC MOTION.*

34. The Differential Equations of Motion.—Consider two spheres whose materials are arranged in homogeneous spherical layers concentric with their centers. Then they attract each other as material points, their orbits are plane curves, and the differential equations which the motion of one relative to the other must satisfy are, in polar coördinates,

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2(m_1+m_2)}{r^2} = 0, \qquad \frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = 0. \tag{1}$$

In writing these equations the origin has been placed at one of the bodies and the variables r and v are measured in the plane of motion.

Equations (1) are easily integrated, and the integrals show that the relative motion is in a conic section for any initial conditions. If the initial velocity is not too great the orbit is an ellipse, and the discussion will be limited to this case. While the ordinary integration of (1) shows that under certain conditions the orbits are ellipses, it does not express the coördinates explicitly in terms of the time. The explicit developments are obtained through solving Kepler's equation, generally by Lagrange's method or by means of Bessel's functions. In treating elliptic motion as periodic motion the expressions for r and v in terms of t will be derived directly from the differential equations.

On integrating the second equation of (1), and by means of this integral eliminating dv/dt from the first, it is found that

$$r^2\frac{dv}{dt} = c, \qquad \frac{d^2r}{dt^2} - \frac{c^2}{r^3} + \frac{k^2(m_1+m_2)}{r^2} = 0. \tag{2}$$

Assume that the conditions for an elliptical orbit are satisfied, and let

$a =$ the major semi-axis of the orbit;
$e =$ the eccentricity of the orbit;
$\omega =$ the mean angular velocity in the orbit;
$T =$ the time the body passes its nearest apse;
$\tau = \omega(t-T) =$ the mean anomaly.

It is found from the integrals of (2) that†

$$\omega^2a^3 = k^2(m_1+m_2), \qquad c^2 = k^2(m_1+m_2)a(1-e^2) = \omega^2a^4(1-e^2). \tag{3}$$

*This chapter was written in 1900 and a brief account of it was published in the Astronomical Journal Vol. XXV, May, 1907.

†Moulton's *Introduction to Celestial Mechanics*, pp. 173–8.

On making use of (3) and using τ as the independent variable, the second equation of (2) becomes

$$\frac{d^2r}{d\tau^2}-\frac{a^4(1-e^2)}{r^3}+\frac{a^3}{r^2}=0. \tag{4}$$

Equation (4) is satisfied by the circular solution $r=a$. Let the radius in the elliptic orbit be

$$r=a(1-\rho e), \tag{5}$$

where $\rho=1$, $d\rho/d\tau=0$, at $\tau=0$. At the half period $\tau=\pi$, $r=a(1+e)$. Therefore $\rho=-1$ at $\tau=\pi$. These are the extreme values of r in elliptical motion, and therefore $+1 \geqq \rho \geqq -1$.

Upon substituting (5) in (4), the latter becomes

$$\frac{d^2\rho}{d\tau^2}+\frac{\rho-e}{(1-\rho e)^3}=0. \tag{6}$$

The second term in this equation can be expanded as a power series in e for all the values of ρ if $|e|<1$, as is explicitly assumed, giving

$$\frac{d^2\rho}{d\tau^2}+\rho=\tfrac{1}{2}\sum_{i=1}^{\infty}(i+1)[i-(i+2)\rho^2]\rho^{i-1}e^i, \tag{7}$$

and the first equation of (2) becomes by the same substitutions

$$\frac{dv}{d\tau}=\frac{\sqrt{1-e^2}}{(1-\rho e)^2}=\sqrt{1-e^2}\sum_{i=0}^{\infty}(i+1)\rho^i e^i. \tag{8}$$

35. Form of the Solution.—The solution of equation (7) will first be considered. After it has been found, v is determined from (8) by a simple quadrature.

Equation (7) belongs to the type treated in §§ 14-16, and therefore can be integrated as a power series in e, and $|e|$ can be taken so small that the series will converge for $0 \leqq \tau \leqq 2\pi$. Since the body moves so that the law of areas is satisfied and completes a revolution in 2π, ρ is periodic with the period 2π. Consequently, if the series converges for $0 \leqq \tau \leqq 2\pi$, it converges for all real values of τ. It is, indeed, possible to find the precise limits for $|e|$ within which the series will converge for all values of τ, and outside of which they will diverge for some values of τ. The problem was first solved by Laplace,* who found that the series converge for all τ if $e<0.6627\ldots$, which is far above the eccentricity of the orbit of any planet or satellite in the solar system.

*_Mécanique Céleste_, vol. V, Supplement; see also Tisserand's _Mécanique Céleste_, vol. I, chapter 16, and a demonstration by Hermite, _Cours a la Fac. des Sci. de Paris_, 3d edition (1886), p. 167.

The solution of (7) can be written in the form

$$\rho = \sum_{j=0}^{\infty} \rho_j(\tau)e^j, \tag{9}$$

where the $\rho_j(\tau)$ are functions of τ. According to §15 and the initial conditions, the constants of integration which arise are to be determined by the conditions

$$\rho_0(0)=1,\ \rho_i(0)=0 \quad (i=1,\ \ldots\ \infty); \quad \frac{d\rho_i}{dt}(0)=0 \quad (i=1,\ \ldots\ \infty). \tag{10}$$

As ρ is periodic with the period 2π, it follows that $\rho(\tau+2\pi)\equiv\rho(\tau)$; whence

$$\sum_{j=0}^{\infty} \rho_j(\tau+2\pi)e^j \underset{\tau,e}{\equiv} \sum_{j=0}^{\infty} \rho_j(\tau)e^j. \tag{11}$$

Since (11) is an identity in e it follows that $\rho_j(\tau+2\pi)\equiv\rho_j(\tau)$. But this is simply the definition of periodicity. Therefore each ρ_j separately is periodic.

The body is at its nearest apse when $\tau=0$, and the orbit is symmetrical with respect to the line of apses. Therefore it follows that ρ is an even function of τ. Since ρ is periodic in τ identically with respect to e, each $\rho_j(\tau)$ is expressible as a sum of cosines of integral multiples of τ.

If the sign of e in (6) is changed, then the body is at its farthest apse when $\tau=0$. Consequently changing the sign of e and increasing τ by π does not change the value of r. Since $r=a(1-\rho e)$, it follows that

$$\rho_j(\tau)e^{j-1}\equiv\rho_j(\tau+\pi)\ (-e)^{j-1}. \tag{12}$$

Therefore when j is even, $\rho_j(\tau)$ involves cosines of only odd multiples of τ; and when j is odd, $\rho_j(\tau)$ involves cosines of only even multiples of τ.

If we substitute (9) in (8), we get

$$\left.\begin{aligned} \frac{dv}{d\tau} &= \sqrt{1-e^2}\sum_{i=0}^{\infty}(i+1)\left[\sum_{j=0}^{\infty}\rho_j e^j\right]^i e^i, \\ \frac{dv}{d\tau} &= \sqrt{1-e^2}\sum_{i=0}^{\infty}(i+1)\sum_{j_1,\ldots,j_\kappa} C_{i_1,\ldots,i_\kappa}\rho_{j_1}^{i_1}\ldots\rho_{j_\kappa}^{i_\kappa}e^{i_1j_1+\cdots+i_\kappa j_\kappa+i}, \end{aligned}\right\} \tag{13}$$

where

$$C_{i_1,\ldots,i_\kappa}=\frac{i\,!}{i_1\,!\,i_2\,!\ldots i_\kappa\,!}, \qquad (i_1+i_2+\ \cdots\ +i_k=i). \tag{14}$$

Suppose $i_1j_1+\ \cdots\ +i_\kappa j_\kappa+i$ is even; then there are two cases to be considered, viz. (*a*) when i is even, and (*b*) when i is odd:

(*a*) When i is even an even number of $i_1,\ \ldots\ ,\ i_\kappa$ must be odd, and the number of odd i_λ multiplied by odd j_λ must be even. Therefore the number of odd i_λ multiplied by even j_λ must be even. All those factors $\rho_{j_\lambda}^{i_\lambda}$ in (13) for which i_λ is even involve only even multiples of τ, and those for which j_λ is even and i_λ odd involve only odd multiples of τ. Since there must be an even number of these terms involving only odd multiples of τ, their product involves only even multiples of τ.

(*b*) When i is odd it follows from (14) that an odd number of $i_1, \ldots, i_\kappa$ are odd, and from the hypothesis that $i_1 j_1 + \ldots + i_\kappa j_\kappa + i$ is even it follows that an odd number of $i_1 j_1, \ldots, i_\kappa j_\kappa$ are odd. A term $i_\lambda j_\lambda$ can be odd only if both i_λ and j_λ are odd. When j_λ is odd the term involves only even multiples of τ whether raised to an odd or even power. Since the whole number of odd i_λ is odd, and an odd number of them are multiplied by an odd j_λ, it follows that there is an even number of terms $\rho_{j_\lambda}^{i_\lambda}$, where j_λ is even and i_λ is odd. Therefore their product will be cosines of even multiples of τ. That is, in the right member of (13) the coefficients of even powers of e involve only even multiples of τ.

It is easily proved in a similar way that the coefficients of odd powers of e in (13) are odd multiples of τ.

Upon integrating (13), it is found that v is expansible as a power series in e of the form

$$v = c\tau + \sum_{i=0}^{\infty} v_i e^i, \tag{15}$$

where v_i is a sum of sines of even multiples of τ when i is even, and of odd multiples of τ when i is odd. The coefficient c is unity because, the ellipse being fixed in space, v increases by precisely 2π in a period.

36. Direct Construction of the Solution.—Upon substituting equation (9) in (7) and arranging in powers of e, we obtain

$$\left.\begin{aligned} \sum_{j=0}^{\infty} \rho_j'' e^j + \sum_{j=0}^{\infty} \rho_j e^j = {} & [1 - 3\rho_0^2]e + [3\rho_0 - 6\rho_0\rho_1 - 6\rho_0^3]e^2 \\ & + [-6\rho_0\rho_2 + 3\rho_1(1 - \rho_1 - 6\rho_0^2) + 2\rho_0^2(3 - 5\rho_0^2)]e^3 \\ & + [-6\rho_0(\rho_3 + 3\rho_1^2 + 3\rho_0\rho_2 - 2\rho_1) + 3\rho_2(1 - 2\rho_1) - 5\rho_0^3(8\rho_1 - 2 - 3\rho_0^2)]e^4 \ldots, \end{aligned}\right\} \tag{16}$$

where ρ_j'' is the second derivative of ρ_j with respect to τ. Upon equating coefficients of like powers of e, the differential equations which define the several coefficients become

$$\begin{aligned} &(a)\quad \rho_0'' + \rho_0 = 0, \\ &(b)\quad \rho_1'' + \rho_1 = 1 - 3\rho_0^2, \\ &(c)\quad \rho_2'' + \rho_2 = 3\rho_0(1 - 2\rho_1 - 2\rho_0^2), \\ &(d)\quad \rho_3'' + \rho_3 = -6\rho_0\rho_2 + 3\rho_1(1 - \rho_1 - 6\rho_0^2) + 2\rho_0^2(3 - 5\rho_0^2), \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The only solution of (*a*) satisfying (10) is

$$\rho_0 = \cos\tau. \tag{17}$$

Then equation (*b*) becomes

$$\rho_1'' + \rho_1 = -\frac{1}{2} - \frac{3}{2}\cos 2\tau.$$

The solution of this equation satisfying (10) is

$$\rho_1 = -\frac{1}{2} + \frac{1}{2}\cos 2\tau. \tag{18}$$

In a similar manner equations (*c*), (*d*), . . . can be integrated in order, and their solutions are found to be

$$\rho_2 = \frac{3}{8}(-\cos\tau + \cos 3\tau), \qquad \rho_3 = -\frac{1}{2}(\cos 2\tau - \cos 4\tau), \qquad (19)$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

The general term of the solution is defined by an equation of the form

$$\rho_j'' + \rho_j = A_0^{(j)} + A_1^{(j)}\cos\tau + \cdots + A_\kappa^{(j)}\cos\kappa\tau, \qquad (20)$$

where the $A_\lambda^{(j)}$ are known constants. Since ρ_j is periodic, $A_1^{(j)}$ is zero for all values of j, and since ρ_j involves only even or odd multiples of τ according as j is odd or even, all the $A_\lambda^{(j)}$ with even subscripts are zero if j is odd, and all the $A_\lambda^{(j)}$ with odd subscripts are zero if j is even. On putting $A_1^{(j)}$ equal to zero, the solution of (20) satisfying the initial conditions (10) is

$$\rho_j = A_0^{(j)} + \left[-A_0^{(j)} + \sum_{\lambda=2}^{\kappa}\frac{A_\lambda^{(j)}}{\lambda^2-1}\right]\cos\tau - \sum_{\lambda=2}^{\kappa}\frac{A_\lambda^{(j)}}{\lambda^2-1}\cos\lambda\tau. \qquad (21)$$

On substituting (17), (18), (19), . . . in (9) and (5), the final expression for r becomes

$$\left.\begin{aligned} r = a\Big\{1 - \left[\cos\tau\right]e + \tfrac{1}{2}\left[1-\cos 2\tau\right]e^2 + \tfrac{3}{8}\left[\cos\tau - \cos 3\tau\right]e^3 \\ + \tfrac{1}{3}\left[\cos 2\tau - \cos 4\tau\right]e^4 + \cdots\Big\}. \end{aligned}\right\} \qquad (22)$$

On making the same substitutions in (8) and integrating, the explicit value of v is found to be

$$\left.\begin{aligned} v = \tau + \left[2\sin\tau\right]e + \left[\tfrac{5}{4}\sin 2\tau\right]e^2 + \left[-\tfrac{1}{4}\sin\tau + \tfrac{13}{12}\sin 3\tau\right]e^3 \\ + \left[-\tfrac{11}{24}\sin 2\tau + \tfrac{103}{96}\sin 4\tau\right]e^4 + \cdots. \end{aligned}\right\} \qquad (23)$$

37. Additional Properties of the Solution.—It will be proved that no ρ_j carries a higher multiple of τ than $j+1$. It has been seen that it is true for $j=0, 1, 2, 3$. It will be assumed that it is true up to $j-1$, and then it will be proved that it is true for the next step.

The general term in the right member of (7) is, apart from its numerical coefficient, $\rho^{i\pm1}e^i$. After substituting the series (9) for ρ, any term of degree j in e arising from this term has the form $\rho_0^{\lambda}\rho_1^{\lambda_1}\rho_2^{\lambda_2}\cdots\rho_\kappa^{\lambda_\kappa}e^j$, where

$$\lambda + \lambda_1 + \lambda_2 + \cdots + \lambda_\kappa = i \pm 1, \qquad \lambda_1 + 2\lambda_2 + \cdots + \kappa\lambda_\kappa + i \pm 1 = j.$$

After eliminating i from equations, it is found that

$$\lambda + 2\lambda_1 + 3\lambda_2 + \cdots + (\kappa+1)\lambda_\kappa = j \pm 1, \qquad (24)$$

where obviously $\kappa < j$.

By hypothesis the highest multiples of τ in $\rho_0, \rho_1, \ldots, \rho_\kappa$ are respectively $1, 2, \ldots, \kappa+1$. Therefore the highest multiples in $\rho_0^{\lambda}, \rho_1^{\lambda_1}, \ldots, \rho_\kappa^{\lambda_\kappa}$ are respectively $\lambda, 2\lambda_1, \ldots, (\kappa+1)\lambda_\kappa$. Consequently the highest multiple in the product $\rho_0^{\lambda}\rho_1^{\lambda_1} \ldots \rho_\kappa^{\lambda_\kappa}$ is $\lambda+2\lambda_1+\cdots+(\kappa+1)\lambda_\kappa$, which is $j\pm1$ by (24). Therefore the highest multiple in the expression for ρ^j is $j+1$.

A similar discussion of (8), (13), and (15) shows that v_j does not involve multiples of τ greater than j.

38. Problem of the Rotating Ellipse.—In certain cases where the motion is not strictly elliptical, it is convenient to suppose the body moves in an ellipse whose position and form are constantly changing. This conception is at the foundation of the theory of perturbations originated by Newton, and has been essential in the work of most writers on celestial mechanics.

One of the historically interesting and important problems has been the theory of revolution of the line of apsides of the moon's orbit. When Clairaut first made a computation of the rate of this revolution under the supposition that it was due to perturbations of the moon's motion by the sun, he obtained an amount about half as great as that furnished by observations.* Later work by himself and others has shown that the discrepancy was due to imperfections in his theory, but at first he sought to relieve the difficulties by supposing that gravitation does not vary simply as the inverse square of the distance, but that it also depends upon a term in the inverse third power of the distance. We shall solve the problem of the motion for this law of force as a further illustration of the power and simplicity of the methods which are being used here. The steps in the solution are almost exactly parallel to those used above, and it will be noted in the course of the work that they would not be fundamentally different if the added term were not such as to make the peculiar simplicity of this problem.

The differential equations of motion in this case are

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2(m_1+m_2)}{r^2} = -\frac{k^2(m_1+m_2)}{r^3}a\mu, \qquad \frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = 0, \tag{25}$$

where, for simplicity in the final formulas, the constant coefficient of the term in the inverse third power of r is given the form $k^2(m_1+m_2)a\mu$. The μ is an arbitrary parameter.

The integral of the second equation of (25) is

$$r^2\frac{dv}{dt} = c, \tag{26}$$

by means of which the first equation reduces to

$$\frac{d^2r}{dt^2} - \frac{c^2}{r^3} + \frac{k^2(m_1+m_2)}{r^2} = -\frac{k^2(m_1+m_2)}{r^3}a\mu. \tag{27}$$

*See Tisserand's *Mécanique Céleste,* vol. III, chap. 4, and particularly articles 24 and 27.

39. The Circular Solution.—We shall first find a solution of (27) with an arbitrary constant of areas, c, for which r is constant. Let a and ω be defined by

$$k^2(m_1+m_2)=\omega^2 a^3, \qquad c=k\sqrt{m_1+m_2)a}. \tag{28}$$

Let $r=a(1+p)$, where p is a constant. Then (27) becomes

$$-\frac{1}{(1+p)^3}+\frac{1}{(1+p)^2}=-\frac{1}{(1+p)^3}\mu;$$

or, expanding as a power series in p,

$$p-3p^2+6p^3-\cdots=-\mu(1-3p+\cdots). \tag{29}$$

By §§1 and 2 this equation can be solved for p as a power series in μ, converging for $|\mu|$ sufficiently small. In fact, the additional term has been chosen in such a way that the power series reduces to a single term, but it is evident that this condition is in no way essential to the process. It is found at once that

$$p=-\mu. \tag{30}$$

In this case the solutions of equations (25) and (26) are

$$r=a(1-\mu), \qquad v-v_0=\frac{c}{a^2(1-\mu)^2}(t-T)=\frac{k\sqrt{m_1+m_2}}{a^{\frac{3}{2}}(1-\mu)^2}(t-T), \tag{31}$$

involving the three constants of integration a, v_0, and T.

40. Existence of the Non-Circular Solutions.—We shall now derive a solution of equations (25) corresponding to the elliptic solution in the ordinary two-body problem. It will involve *four* constants of integration which are arbitrary except for the restriction that the orbit shall not deviate too widely from a circle, a condition which is imposed to secure convergence of the series. The solution is thus seen to be the general solution.

Now let

$$\left.\begin{aligned} c&=k\sqrt{(m_1+m_2)a(1-e^2)}, & k^2(m_1+m_2)&=\omega^2a^3,\\ r&=a(1-\mu-\rho e), & \omega(t-T)&=\sqrt{1+\delta}\,\tau, \end{aligned}\right\} \tag{32}$$

where T is the time of passing the nearest apse, and $a(1-\mu-e)$ is the arbitrary initial value of r. Therefore the initial value of ρ is unity. The constants a and e are defined by the first and third equations at $t=T$, ω by the second, and δ is a parameter to be determined later.* There are so far three arbitrary constants of integration a, e, and T; the fourth is introduced in integrating equation (26).

With these substitutions equation (27) becomes

$$\frac{d^2\rho}{d\tau^2}+\frac{(1+\delta)(\rho-e)}{(1-\mu-\rho e)^3}=0. \tag{33}$$

*Poincaré introduces a parameter τ somewhat analogous to this. *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 61.

Equation (33) admits a periodic solution, as is known from the fact that the orbit is a rotating ellipse. However, it will be shown directly by forming the first integral of (27), viz.,

$$\left(\frac{dr}{dt}\right)^2 = c_4 - \frac{c^2}{r^2} + 2\frac{k^2(m_1+m_2)}{r} + \frac{k^2(m_1+m_2)}{r^2}a\mu = \varphi(r). \tag{34}$$

Suppose the initial conditions are real; then $\varphi(r_0)>0$. For $\mu=0$ the equation $\varphi(r)=0$ has two roots, viz., $r_1=a(1-e)$, $r_2=a(1+e)$ between which r must vary. For μ small it also has two roots, r_1' and r_2', near r_1 and r_2 respectively. Suppose $r_1'<r_2'$. Then $\varphi(r)<0$ if $r<r_1'$ or $r>r_2'$. Consequently it follows from $dr/dt=\sqrt{\varphi(r)}$ that if r is increasing at $t=0$ it will increase until $r=r_2'$ when the radical changes sign, after which it will decrease until the radical changes sign again at $r=r_1'$. The period of a complete oscillation is

$$P = 2\int_{r_1'}^{r_2'} \frac{dr}{\sqrt{\varphi(r)}}. \tag{35}$$

If r is periodic then ρ is periodic also.

The existence of the periodic solution can be established directly from (33) and a proof made of the possibility of a construction similar to that used in treating the elliptic motion. Equation (33) can be expanded in the form

$$\left.\begin{aligned} \rho''+(1+\delta)\,\rho = (1+\delta)\Big\{&\left[(1-3\rho^2)e+3\rho(1-2\rho^2)e^2+\cdots\right] \\ &+\left[-3\rho+3(1-4\rho^2)\,e+\cdots\right]\mu+\left[-6\rho+6e+\cdots\right]\mu^2+\cdots\Big\}, \end{aligned}\right\} \tag{36}$$

where the right member converges so long as $|\mu+\rho e|<1$. By §§ 14–16 this equation can be integrated so as to express ρ as a power series in δ, μ, and e of the form

$$\rho = P(\delta,\, \mu,\, e;\; \tau). \tag{37}$$

The series will converge for all τ in the interval $0 \gtreqless \tau \leqq 2\pi$ if $|e|$, $|\mu|$, $|\delta|$ are sufficiently small.

We now avail ourselves of the arbitrary parameter δ to determine the period. We will determine δ so that the period shall be 2π in τ. Since equation (36) does not involve τ explicitly, sufficient conditions for periodicity with the period 2π are

$$\rho(2\pi)=\rho(0), \qquad \rho'(2\pi)=\rho'(0). \tag{38}$$

These equations are not independent, for (36) has an integral corresponding to (34), which is a relation between ρ' and ρ that is always satisfied.* This integral has the form

$$\rho'^2+(1+\delta)\,\rho^2 = (1+\delta)\,P(\rho,\, e,\, \mu)+C. \tag{39}$$

*Compare *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 87.

Suppose the particle is projected from an apse so that $\rho'=0$ at $\tau=0$. Then C is a power series in e, μ, and the initial value of ρ, which is unity. Let the general value of ρ be $1+\sigma$. Then the value of the integral at any time minus its value at $\tau=0$ will be ρ'^2 plus a power series in σ, e, and μ, vanishing for $\sigma=0$ whatever e and μ may be. The terms coming from the right member all involve e or μ as a factor, but there is a term coming from ρ^2 which involves σ alone to the first degree. Therefore (39) can be solved uniquely for σ as a power series in ρ'^2, e, μ, vanishing with $\rho'=0$. Hence if $\rho'=0$ at $\tau=2\pi$, then will the first equation of (38) necessarily be satisfied, and consequently it may be suppressed.

This result can also be shown from a certain symmetry which is particularly simple in the present problem. Equation (36) can be written in the form

$$\frac{d\rho}{d\tau}=\rho', \qquad \frac{d\rho'}{d\tau}=F(\rho,\ e,\ \mu). \tag{40}$$

Suppose $\rho=1$, $\rho'=0$ at $\tau=0$ and that these equations have the solution $\rho=f_1(\tau)$, $\rho'=f_2(\tau)$.

Now consider the differential equations obtained when (40) are transformed by the substitution $\rho=\rho_1$, $\rho'=-\rho_1'$, $\tau=-\tau_1$. The equations in the new variables are the same as in the old, and consequently if the initial conditions are the same ($\rho_1=1$, $\rho_1'=0$ at $\tau=0$), the solution is

$$\rho_1=f_1(\tau_1)=f(-\tau)=\rho, \qquad \rho_1'=f_2(\tau_1)=f_2(-\tau)=-\rho'.$$

Therefore ρ is an even function of τ and ρ' is an odd function of τ.

Suppose $\rho'=0$ at $\tau=\pi$. Since it is an odd function it must also have been zero at $\tau=-\pi$. Since ρ is even in τ it has the same value at $\tau=-\pi$ as it has at $\tau=\pi$. Consequently the system is the same at $\tau=\pi$ as it was at $\tau=-\pi$, and the motion is periodic with the period 2π. Hence if $\rho'=0$ at $\tau=0$, it is sufficient to satisfy the condition $\rho'=0$ at $\tau=\pi$ in order to secure a periodic solution of (33) with the period 2π.

It will now be shown that the second equation of (38) can be solved uniquely for δ as a power series in μ and e, vanishing with $\mu=0$. It is found by integrating (36) and imposing the initial conditions $\rho \underset{\mu,e}{\equiv} 1$, $\rho' \underset{\mu,e}{\equiv} 0$ that

$$\rho=\rho_{00}+\rho_{10}\mu+\rho_{01}e+\cdots,$$

where

$$\left.\begin{aligned}
\rho_{00}&=\cos\sqrt{1+\delta}\,\tau,\\
\rho_{10}&=-\tfrac{3}{2}\sqrt{1+\delta}\,\tau\sin\sqrt{1+\delta}\,\tau,\\
\rho_{01}&=-\tfrac{1}{2}+\tfrac{1}{2}\cos 2\sqrt{1+\delta}\,\tau.\\
&\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{41}$$

On substituting these results in the second equation of (38) and expanding in powers of δ also, we get

$$\left.\begin{aligned}\rho'(2\pi)-\rho'(0)=\delta\left[-\tfrac{1}{2}\,\pi+(\text{terms in }\delta,\,\mu,\,e)\right]\\ +\mu\left[-3\,\pi+(\text{terms in }\delta,\,\mu,\,e)\right]=0.\end{aligned}\right\}\qquad(42)$$

There are no terms in e alone, for when $\mu=0$, the orbit becomes a fixed ellipse and $\delta=0$ satisfies the periodicity condition. Hence (42) can be solved uniquely for δ as a power series in μ and e, vanishing with $\mu=0$. When the value of δ obtained from (42) is substituted in (37), ρ becomes periodic in τ with the period 2π, and is expanded as a power series in μ and e which converges provided $|\mu|$ and $|e|$ are sufficiently small. It can be written

$$\rho=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\rho_{ij}\,\mu^i e^j,\qquad\qquad \delta=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\delta_{ij}\,\mu^i e^j.\qquad(43)$$

From the reasoning of §35 it follows that each ρ_{ij} separately is periodic.

The range of convergence of (43) is limited in two ways. In the first place the inequalities $|\delta|<\delta_0$, $|\mu|<\mu_0$, $|e|<e_0$ must be satisfied in order that (37) may converge for $0\overline{\overline{<}}\tau\leqq 2\pi$. Then the inequalities $|\mu|<\mu_1$, $|e|<e_1$ must be satisfied in order that the solution of (42) shall converge and give for $|\delta|$ a value less than δ_0. When $|\mu|$ and $|e|$ satisfy both of these sets of inequalities the convergence of (43) is assured for all τ.

After the explicit development of equations (43) has been made the results can be substituted in (26), when v will be determined by a quadrature. The final form of v is

$$v=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_{ij}\,\mu^i e^j.\qquad(44)$$

The constant parts of the v_{ij} are independent of e since for $\mu=0$ it was found that $v=\tau$ + periodic terms.

41. Direct Construction of the Non-Circular Solution.—In carrying out the practical construction of the solution, we shall make use of the facts that (*a*) $\rho\underset{\mu,e}{\equiv}1$, $\rho'\underset{\mu,e}{\equiv}0$ at $\tau=0$, (*b*) ρ is expansible in the form (43), and (*c*) each ρ_{ij} separately is periodic with the period 2π.

On substituting (43) in (36) and equating coefficients of equal powers of μ and e, it is found that the several coefficients must satisfy

$$\begin{aligned}
&(A)\ \ \rho_{00}''+\rho_{00}=0,\\
&(B)\ \ \rho_{01}''+\rho_{01}=(1-3\rho_{00}^2),\\
&(C)\ \ \rho_{10}''+\rho_{10}=-(\delta_{10}+3)\rho_{00},\\
&(D)\ \ \rho_{02}''+\rho_{02}=3\rho_{00}(1-2\rho_{01}-2\rho_{00}^2),\\
&(E)\ \ \rho_{11}''+\rho_{11}=-\delta_{11}\rho_{00}-\delta_{10}\rho_{01}+\delta_{10}(1-3\rho_{00}^2)-6\rho_{00}\rho_{10}-3\rho_{01}+3(1-4\rho_{00}^2),\\
&(F)\ \ \rho_{20}''+\rho_{20}=-\delta_{20}\rho_{00}-\delta_{10}\rho_{10}-3\delta_{10}\rho_{00}-3\rho_{10}-6\rho_{00},
\end{aligned}$$

. . . .

The solution of (A) satisfying (a) is $\rho_{00}=\cos\tau$. The solution of (B) is given in equation (18). The solution of (C) is not periodic unless the coefficient of ρ_{00} is zero. Imposing also the condition (a), $\delta_{10}=-3$, $\rho_{10}=0$. The term ρ_{02} is given in equation (19). Equation (E) becomes explicitly $\rho_{11}''+\rho_{11}=-3/2-\delta_{11}\cos\tau-3/2\cos 2\tau$. Upon imposing conditions (a) and (c), the solution of this equation is found to be

$$\delta_{11}=0, \qquad \rho_{11}=-\frac{3}{2}+\cos\tau+\frac{1}{2}\cos 2\tau. \tag{45}$$

The explicit form of (F) now becomes $\rho_{20}''+\rho_{20}=(-\delta_{20}+3)\cos\tau$, whose solution satisfying the conditions (a) and (c) is

$$\delta_{20}=3, \qquad \rho_{20}=0. \tag{47}$$

Hence the final expressions for ρ and δ as power series in e and μ are

$$\left.\begin{aligned} \rho&=\cos\tau+\left[-\frac{1}{2}+\frac{1}{2}\cos 2\tau\right]e+\left[-\frac{3}{2}+\cos\tau+\frac{1}{2}\cos 2\tau\right]\mu e \\ &\qquad +\frac{3}{8}\left[-\cos\tau+\cos 3\tau\right]e^2+\cdots, \\ \delta&=-3\mu+3\mu^2+\cdots, \end{aligned}\right\} \tag{48}$$

where, from (32), τ is to be replaced throughout by $\omega(t-T)/\sqrt{1+\delta}$.

The differential equation defining the general term is

$$\rho_{ij}''+\rho_{ij}=-\delta_{ij}\rho_{00}+F_{ij}(\delta_{\kappa\lambda}\,,\ \rho_{\kappa\lambda}) \qquad (\kappa=1,\ldots,i-1;\ \lambda=1,\ldots,j-1).$$

Suppose all the $\rho_{\kappa\lambda}$ and $\delta_{\kappa\lambda}$ for which $\kappa<i$, $\lambda<j$ have been found. Then this equation can be written in the form

$$\rho_{ij}''+\rho_{ij}=A_0^{(ij)}+(-\delta_{ij}+A_1^{(ij)})\cos\tau+A_2^{(ij)}\cos 2\tau+\cdots+A_\kappa^{(ij)}\cos\kappa\tau, \tag{49}$$

there being no sine terms. Its solution satisfying conditions (a) and (c) is

$$\left.\begin{aligned} \delta_{ij}&=A_1^{(ij)}, \\ \rho_{ij}&=A_0^{(ij)}+\left(-A_0+\frac{A_2^{(ij)}}{3}+\cdots+\frac{A_\kappa^{(ij)}}{\kappa^2-1}\right)\cos\tau-\frac{A_2^{(ij)}}{3}\cos 2\tau-\cdots-\frac{A_\kappa^{(ij)}}{\kappa^2-1}\cos\kappa\tau. \end{aligned}\right\} \tag{50}$$

The δ_{ij} and all the coefficients are uniquely determined. Therefore the process can be continued without modification as far as may be desired.

After the transformations (32) equation (26) becomes

$$\frac{dv}{d\tau}=\sqrt{1+\delta}\left\{1+2\mu+2\rho e+\left(-\frac{1}{2}+3\rho^2\right)e^2+6\rho\mu e+3\mu^2+\cdots\right\}.$$

Upon substituting the value of ρ given in (48) and integrating, it is found that

$$v-v_0=\left[1+2\mu+3\mu^2+\cdots\right]\tau+\left[2\sin\tau\right]e+\left[\frac{5}{4}\sin 2\tau\right]e^2+\cdots. \tag{51}$$ $\times\sqrt{1+\delta}$

where τ is to be replaced by $\omega(t-T)/\sqrt{1+\delta}$. The four arbitrary constants of integration are a, e, T, and v_0.

42. Properties of the Solution.—It has been proved that each ρ_{ij} (except ρ_{00}) and ρ'_{ij} vanish at $\tau=0$, that each ρ_{ij} is periodic in τ with the period 2π, and that ρ is an even function in τ. Hence each ρ_{ij} involves only cosines of multiples of τ. It has also been noted that δ depends upon no terms independent of μ.

There is no term ρ_{i0} distinct from zero, for when $e=0$ the differential equations are satisfied for the same initial values of the variables by $\rho=0$. Or, it is seen, from the development of the second term of (33), that the only terms which are independent of e involve ρ to the first degree alone. Consequently the right members of the equations corresponding to (49), which define ρ_{i0}, will be $[-\delta_{i0}+f(\mu)]\cos\tau$ alone. The periodicity condition makes it necessary to put $\delta_{i0}=f(\mu)$, and the initial conditions then make $\rho_{i0}=0$.

The expression for the right member of (51) has no term independent of μ, except unity, in the non-periodic part, for when $\mu=0$ the ellipse is fixed and this part reduces simply to τ. On making use of all of these facts and some simple artifices, the labor of actually constructing the series can be very much reduced.

The radius r completes its period in $\tau=2\pi$, or $t=2\pi\sqrt{1+\delta}/\omega$. It follows from (51) that the longitude of the radius has increased in this interval by $\sqrt{1+\delta}\,[1+2\mu+3\mu^2+\cdots]\,2\pi$. Therefore the line of apsides has moved forward in this interval through the angle $\sqrt{1+\delta}\,[1+2\mu+3\mu^2+\cdots]\,2\pi-2\pi$. Hence its average rate of angular motion in t is

$$\frac{d\tilde{\omega}}{dt}=\frac{\sqrt{1+\delta}\,[1+2\mu+3\mu^2+\cdots]\,2\pi-2\pi}{\dfrac{2\pi\sqrt{1+\delta}}{\omega}}=\tfrac{1}{2}\mu\left[1+\tfrac{9}{4}\mu+\cdots\right]\omega, \quad (52)$$

where $\tilde{\omega}$ is the longitude of the nearest apse. The parameter μ can be determined so as to secure any rate of revolution of the apsides not too great.

CHAPTER III.

THE SPHERICAL PENDULUM.

I. SOLUTION OF THE Z-EQUATION.

43. The Differential Equations.—The problem of the spherical pendulum falls in the class of those which can be treated by the methods of periodic orbits. Its simplicity makes it particularly well suited to illustrating these processes, and its value as an introduction to the subject is increased by the fact that it is easy to verify the results experimentally. It is doubtful whether there is a problem which is superior in these respects.

Let us take a rectangular system of axes with the positive z-axis directed upward and with the origin at the fixed point of the pendulum. The pendulum is subject to gravity and the normal reaction, N, which we shall take with the positive sign when directed outward. If we represent the radius of gyration by l, the motion of the pendulum satisfies

$$x^2+y^2+z^2=l^2, \qquad mx''=N\frac{x}{l}, \qquad my''=N\frac{y}{l}, \qquad mz''=N\frac{z}{l}-mg, \qquad (1)$$

where the accents indicate derivatives with respect to the time.

The last three equations of (1) admit the integral

$$m(x'^2+y'^2+z'^2)=mv^2=mg(-2z+c_1), \qquad (2)$$

where c_1 is the constant of integration.

The normal reaction exactly balances the centrifugal acceleration of the pendulum due to its motion and the component of mg along the normal to the surface; hence

$$N=-\frac{mv^2}{l}-F_N=-\frac{mv^2}{l}+mg\frac{z}{l},$$

where F_N is the normal component of mg. On making use of (2), we get

$$N=\frac{mg}{l}(3z-c_1). \qquad (3)$$

Hence equations (1) become

$$\left.\begin{aligned} &x^2+y^2+z^2=l^2, &\qquad y''&=\frac{g}{l^2}(3z-c_1)y, \\ &x''=\frac{g}{l^2}(3z-c_1)x, &\qquad z''&=\frac{g}{l^2}(3z-c_1)z-g. \end{aligned}\right\} \qquad (4)$$

The last equation is independent of the others and is therefore solved first. After it is solved the second gives x in terms of t, and then y can be found from the first.

44. Transformation of the z-Equation.—It will be convenient to transform the last equation of (4). It admits the integral

$$z'^2 = \frac{g}{l^2}(2z - c_1)z^2 - g(2z - c_2) = f(z), \tag{5}$$

where c_2 is a constant of integration which is independent of c_1. If we subtract this equation from (2) and reduce the result by (2), we find

$$g(c_1 - c_2) = (x'^2 + y'^2)\left(1 - \frac{z^2}{l^2}\right) - \frac{z^2}{l^2} z'^2. \tag{6}$$

Now $z^2 \leqq l^2$. In the case where the spherical pendulum reduces to the simple pendulum z takes the value $= l$, and at the same time $z' = 0$. In this case $c_1 - c_2 = 0$. In the spherical pendulum $z > -l$ when $z' = 0$; consequently in this case $c_1 - c_2 > 0$. Hence in all cases of the physical problem $c_1 - c_2 \geqq 0$.

Now consider equation (5). If the initial conditions are real, $z_0'^2 = f(z_0)$ is zero or positive and

$$\begin{array}{ll} f(-\infty) = -\infty, & f(-l) = -g(c_1 - c_2) \leqq 0, \\ f(z_0) = 0 & (-l \leqq z_0 \leqq +l), \\ f(+l) = -g(c_1 - c_2) \leqq 0, & f(+\infty) = +\infty. \end{array}$$

Therefore the equation $f(z) = 0$ has always three real roots. Let them be α_1, α_2, and α_3, where the notation is chosen so that $\alpha_1 \geqq \alpha_2 \geqq \alpha_3$. Then equation (5) becomes

$$z'^2 = \frac{2g}{l^2}(z - \alpha_1)(z - \alpha_2)(z - \alpha_3).$$

On comparing this equation with (5), it is seen that

$$2(\alpha_1 + \alpha_2 + \alpha_3) = c_1, \qquad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = -l^2, \qquad 2\alpha_1\alpha_2\alpha_3 = -c_2 l^2. \tag{7}$$

It will now be shown that α_3 satisfies the inequalities $-l \gtreqqless \alpha_3 \leqq 0$. It follows from the last of (7) that c_2 is negative if α_3 is positive. On putting $z' = 0$ and $z = \alpha_3$ in (5), we get

$$c_2 = 2\alpha_3 + \frac{1}{l^2}(-2\alpha_3 + c_1).$$

By hypothesis the first term on the right is positive, and by (2) the second can not be negative. Therefore c_2 must be positive, which contradicts the implication from the last of (7). Therefore $\alpha_3 \leqq 0$.

Some special cases may be indicated:

(1) It follows from (5), (6), and (7) that $\alpha_3 = -l$ implies that $\alpha_1 = +l$, $2\alpha_2 = c_1 = c_2$, $\alpha_1 - \alpha_3 = 2l$. The constant α_2 is not determined, and we shall suppose it is less than $+l$. This case is that of the ordinary simple pendulum making finite oscillations. In the sub-case where $\alpha_2 = -l$, we have $c_1 = c_2 = -2l$ and the solutions of (4) are $x = y = 0$, $z = -l$.

(2) If $a_2=+l$, it follows from the same equations that $a_3=-l$, $2a_1=c_1=c_2$. The constant a_1 is not determined. If $a_1>l$, we have the case of the simple pendulum swinging round and round, and $a_1-a_3>2l$. In the sub-case where $a_1=l$, we have $c_1=c_2=+2l$, and the solutions of (4) are $x=y=0$, $z=+l$.

(3) If $a_3=0$, it follows that $a_2\geqq 0$. Therefore the second of (7) can not be satisfied except by $a_2=0$, $a_1=\infty$. Then, from the first equation we get $c_1=\infty$. This is the case of revolution in the xy-plane with infinite speed, and of course can not be realized physically. Excluding this case and that of the simple pendulum, the constants a_1, a_2, a_3 satisfy the inequalities $-l<a_3<0$, $-l<a_2<+l$, $a_1>+l$.

Now make the transformation

$$z-a_3=(a_2-a_3)u^2. \tag{8}$$

Then equation (5) becomes

$$u'^2=\frac{g(a_1-a_3)}{2l^2}(1-u^2)\left(1-\frac{a_2-a_3}{a_1-a_3}u^2\right). \tag{9}$$

Also let

$$\mu=\frac{a_2-a_3}{a_1-a_3}, \qquad \tau=\sqrt{\frac{g(a_1-a_3)}{2l^2(1+\delta)}}\,(t-t_0), \tag{10}$$

where t_0 is an arbitrary initial time and δ is a constant as yet undefined. The constant μ satisfies the inequalities $0\gtreqqless\mu\leqq 1$. Then (9) becomes

$$\dot{u}^2=(1+\delta)(1-u^2)\ (1-\mu\, u^2)=F(u), \tag{11}$$

where $\dot{u}$ is the first derivative of u with respect to the new independent variable τ. The first derivative of (11) is

$$\ddot{u}=(1+\delta)\,[-(1+\mu)u+2\mu\, u^3]. \tag{12}$$

45. First Demonstration that the Solution of (12) is Periodic, and that u and the Period are Expansible as Power Series in μ.—It will first be shown that, for any initial conditions belonging to the physical problem, except when $\mu=1$, the solution of (12) is periodic. By the fundamental existence theorem of the solutions of differential equations* the solutions of (12) are regular in τ for all finite values of τ and u. For real initial conditions the coefficients of u and its derivatives expanded as power series in τ are real, and by analytic continuation they remain real for all finite real values of τ provided u does not become infinite. Now consider the curve $F=F(u)$. Suppose $u=u_0$ at $\tau=0$ and that $\dot{u}_0$ is positive. Then u is increasing at a rate which is proportional to the square root of $F(u_0)$, and it continues to increase until $u=1$. It can not increase beyond $+1$ for then $\dot{u}$ would become a pure imaginary, and it has just been shown that it always remains real. It can not remain constantly equal to 1 unless $\mu=1$, for otherwise $u=1$ does not satisfy (12). Therefore, unless $\mu=1$, u will increase

*Picard's *Traité d'Analyse*, vol. II, chap. 11, §III.

to 1 and then decrease to -1; then, in a similar way, it changes at $u=-1$ from a decreasing to an increasing function. That is, at $u=\pm 1$ the function $F(u)$ changes sign and u varies periodically between $+1$ and -1.

This result follows, of course, from the fact that in the present problem u is the sine amplitude of τ, one of whose properties is that of having a real period, but the argument given above applies to much more general cases, and the result can be read from the diagram for $F=F(u)$. It may be mentioned in passing that the imaginary period of the elliptic function is associated in a similar way with the portions of the curve between $+1$ and $+1/\sqrt{\mu}$, and between $-1/\sqrt{\mu}$ and -1.

The period of a complete oscillation is found from (11) to be

$$P=\frac{2}{\sqrt{1+\delta}}\int_{-1}^{+1}\frac{du}{\sqrt{(1-u^2)(1-\mu u^2)}}, \tag{13}$$

which is finite unless $\mu=1$. We shall exclude this exceptional case. It is well known that

$$P=\frac{2\pi}{\sqrt{1+\delta}}\left[1+\left(\frac{1}{2}\right)^2\mu+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\mu^2+\cdots+\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2\mu^n+\cdots\right]. \tag{14}$$

In t the period is

$$T=\sqrt{\frac{2l^2(1+\delta)}{g(a_1-a_3)}}\,P=\frac{\sqrt{2}\,l\,2\pi}{\sqrt{g(a_1-a_3)}}\left[1+\left(\frac{1}{2}\right)^2\mu+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\mu^2+\cdots\right]. \tag{15}$$

That is, the period is expansible as a power series in μ, and in the present simple case the series converges provided $|\mu|<1$.

The constant δ has so far remained undetermined. If we let

$$\sqrt{1+\delta}=\left[1+\left(\frac{1}{2}\right)^2\mu+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\mu^2+\cdots\right],$$

the period in τ will be simply 2π. On solving this equation, we find that the required value of δ is

$$\delta=\tfrac{1}{2}\mu+\tfrac{11}{32}\mu^2+\cdots, \tag{16}$$

which is a power series in μ.

Now consider equation (12). By §§14–16, this equation can be integrated as a power series in μ, and $|\mu|$ can be taken so small that the series will converge for all τ in the interval $0\overline{\gtreqless}\tau\leqq\tau_1$ chosen arbitrarily in advance. If τ_1 is greater than P and the constant of integration is chosen so that (11) is the first integral of (12), that is, so that the period of the motion is P, then it follows from the periodicity of the solution that the series will converge for all finite values of τ. That is, both u and T are expansible as power series in μ.

46. Second Demonstration that the Solution of Equation (12) is Periodic.—While the proof of §45 is sufficient for the construction of the solution, it will be instructive to give another demonstration of its periodicity. In many problems the former methods can not be applied.

By §§14–16, equation (12) can be integrated as a power series in the two parameters μ and δ of the form

$$u = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij}(\tau)\, \delta^i \mu^j, \tag{17}$$

where the u_{ij} are functions of τ to be determined by the conditions that (17) shall satisfy (12) and the initial conditions, and where $|\delta|$ and $|\mu|$ can be taken so small that the series will converge for any interval $0 \gtreqqless \tau \leqq \tau_1$, chosen arbitrarily in advance. We may take the initial values $u(0)=0$, $\dot{u}(0)=a$, and from (11) we see that in order to get the same solution as before we must put $a=\sqrt{1+\delta}$ at the end. The subsequent steps of this demonstration would not be essentially modified if we took general initial conditions. From these initial conditions we get

$$0 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij}(0)\, \delta^i \mu^j, \qquad a = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \dot{u}_{ij}(0)\, \delta^i \mu^j,$$

from which it follows that

$$u_{ij}(0)=0 \quad (i, j=0, \cdots \infty), \qquad \dot{u}_{00}(0)=a, \quad \dot{u}_{ij}(0)=0 \quad (i+j>0). \tag{18}$$

On substituting (17) in (12) and equating coefficients of corresponding powers of δ and μ, we get

$$\ddot{u}_{00} + u_{00} = 0, \qquad \ddot{u}_{10} + u_{10} = -u_{00}, \qquad \cdots \tag{19}$$

The solution of the first of these equations satisfying (18) is

$$u_{00} = a \sin \tau. \tag{20}$$

Upon substituting this result in the right member of the second equation of (19), integrating, and imposing the conditions (18), we find

$$u_{10} = -\frac{a}{2} \sin \tau + \frac{a}{2} \tau \cos \tau. \tag{21}$$

Hence we have

$$u = a \sin \tau + \frac{a}{2}[-\sin \tau + \tau \cos \tau]\, \delta + \text{higher powers of } \delta \text{ and } \mu. \tag{22}$$

Since the right member of (12) does not contain τ explicitly, sufficient conditions that u shall be periodic with the period 2π in τ are

$$u(2\pi) - u(0) = 0, \qquad \dot{u}(2\pi) - \dot{u}(0) = 0. \tag{23}$$

It will now be shown that the second one of these equations is necessarily satisfied when the first is fulfilled. Let $u=0+v$, $\dot{u}=a+\dot{v}$, where v and $\dot{v}$ vanish at $\tau=0$. Then (11) becomes

$$(a+\dot{v})^2=(1+\delta)(1-v^2)(1-\mu v^2).$$

On making use of the fact that $1+\delta=a$, we have

$$2a\dot{v}+\dot{v}^2+a(1+\mu)v^2-a\mu v^4=0.$$

There are two solutions of this equation for $\dot{v}$, but the one which vanishes at $\tau=0$ must be used. It has the form

$$\dot{v}=vp(v), \tag{24}$$

where $p(v)$ is a power series in v. Since $u(0)=v(0)=0$, it follows from the first equation of (23) that $u(2\pi)=0$. Then, from (24) we have $\dot{v}(2\pi)=\dot{u}(2\pi)-\dot{u}(0)=0$. That is, by virtue of the existence of the integral (11), the second equation of (23) is a consequence of the first.

Now let us consider the solution of the first equation of (23). Upon substituting u from (22), we get

$$0=\pi a\delta+\text{terms of higher degree in } \delta \text{ and } \mu. \tag{25}$$

It follows from the theorems of §§1–3 that this equation can be solved for δ uniquely in the form

$$\delta=\mu\, p_1(\mu), \tag{26}$$

where $p_1(\mu)$ is a power series in μ, which converges if $|\mu|$ is sufficiently small. On substituting this result in (17), we have

$$u=\sum_{j=0}^{\infty} u_j(\tau)\mu^j, \tag{27}$$

which converges for all $0 \eqqgtr \tau \leqq 2\pi$ for $|\mu|$ sufficiently small. It is sufficient that $|\mu|$ and $|\delta|$ satisfy the conditions necessary to insure the convergence of (17) and the solution of (25). These conditions can both be satisfied by values of μ different from zero because the expression for δ, given in equation (26), carries μ as a factor.

Hence, the periodicity conditions having been satisfied, we have proved that the solution is periodic. It has been found to be expansible as a power series in μ, and the period, which in t is

$$T=\frac{2\pi l\sqrt{2(1+\delta)}}{\sqrt{g(\alpha_1-\alpha_3)}}$$

is also expansible uniquely as a power series in μ. It is clear that this mode of demonstration applies to a wide class of equations, for the explicit values of only the first terms of the right member of the differential equation, the general properties of its convergence, and the existence of a first integral have been used.

47. Third Proof that the Solution of Equation (12) is Periodic.—There is a certain symmetry property of the solutions which can be used to simplify the demonstration that the motion is periodic. It will be shown that the motion is symmetrical in τ with respect to the value $u=0$, which, by (8), corresponds to $z=\alpha_3$, or the lowest point reached by the pendulum.

Suppose $u=0$, $\dot{u}=a$ at $\tau=0$ and that the solution of (12) for these initial conditions is

$$u=f_1(\tau), \qquad f_1(0)=0, \qquad \dot{u}=f_2(\tau), \qquad f_2(0)=a. \tag{28}$$

Now make the transformation

$$u=-v, \qquad \dot{u}=\dot{v}, \qquad \tau=-\sigma. \tag{29}$$

Then (12) becomes

$$\ddot{v}=(1+\delta)\,[-(1+\mu)\,v+2\mu\,v^3]. \tag{30}$$

Hence if $v=0$, $\dot{v}=a$ at $\sigma=0$, the solution of (30) is

$$v=f_1(\sigma), \qquad f_1(0)=0, \qquad \dot{v}=f_2(\sigma), \qquad f_2(0)=a, \tag{31}$$

where f_1 and f_2 are the same functions of σ that f_1 and f_2, of (28), are of τ. On substituting (28) and (31) in (29), we get

$$u=f_1(\tau)=-f_1(-\tau), \qquad \dot{u}=f_2(\tau)=+f_2(-\tau).$$

Therefore u is an odd function of τ when $u=0$ at $\tau=0$, and hence it follows that sufficient conditions that the solution of (12) shall be periodic with the period 2π are

$$u(0)=0, \qquad u(\pi)=0. \tag{32}$$

The solution (22) was obtained with the initial condition $u(0)=0$. Hence the second equation of (32) becomes

$$0=-\frac{\pi}{2}a\delta+\text{higher powers of } \delta \text{ and } \mu. \tag{33}$$

The solution of this equation for δ and the further discussion are precisely like the treatment of (25), and lead to the same results.

48. Direct Construction of the Solution.—It has been proved that δ can be expanded as a power series of the form

$$\delta=\delta_1\,\mu+\delta_2\,\mu^2+\cdots \tag{34}$$

such that, when (12) is integrated as a power series in μ of the form

$$u=u_0+u_1\,\mu+u_2\,\mu^2+\cdots \tag{35}$$

with the initial conditions $u(0)=0$, $\dot{u}(0)=a$, u will be periodic with the period 2π. In fact, the value of δ is given in (16), but we shall make use only of its expansibility in this form; in more complicated problems its explicit value would not, in general, be known. Since u is periodic with the period 2π, we have

$$u(2\pi+\tau)-u(\tau)\equiv \sum_{j=0}^{\infty}[u_j(2\pi+\tau)-u_j(t)]\,\mu^j=0,$$

and this equation holds for all $|\mu|$ sufficiently small. Therefore

$$u_j(2\pi+\tau)-u_j(\tau)=0, \qquad (j=0,\ 1,\ \ldots\ \infty). \tag{36}$$

Hence each u_j separately is periodic with the period 2π.

Instead of determining the solution by the initial conditions $u(0)=0$, $\dot{u}(0)=0$, we may use $u(0)=0$, $u(\pi/2)=1$. Or, by (35),

$$\sum_{j=0}^{\infty} u_j(0)\,\mu^j=0, \qquad \sum_{j=0}^{\infty} u_j\left(\frac{\pi}{2}\right)\mu^j=1.$$

Therefore we have

$$\left.\begin{aligned} &u_0(0)=0, \qquad u_0\left(\frac{\pi}{2}\right)=1,\\ &u_j(0)=0, \qquad u_j\left(\frac{\pi}{2}\right)=0, \qquad (j=1,\ 2,\ \ldots\ \infty). \end{aligned}\right\} \tag{37}$$

Now we determine u and δ by the conditions that (12) shall be satisfied identically in μ and τ, and that the conditions (36) and (37) shall be fulfilled. By direct substitution and equating of coefficients, we find

$$\left.\begin{aligned} &\ddot{u}_0+u_0=0,\\ &\ddot{u}_1+u_1=-(1+\delta_1)\,u_0+2u_0^3,\\ &\ddot{u}_2+u_2=-(\delta_1+\delta_2)\,u_0-(1+\delta_1)\,u_2+6u_0^2\,u_1+2\delta_1\,u_0^3,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{38}$$

The solution of the first equation of (38) satisfying (37) is $u_0=\sin\tau$. Then the second equation of (38) becomes

$$\ddot{u}_1+u_1=-(1+\delta_1)\sin\tau+\frac{3}{2}\sin\tau-\frac{1}{2}\sin 3\tau. \tag{39}$$

In order that the solution of this equation shall be periodic we must set the coefficient of $\sin\tau$ equal to zero. Then the solution satisfying (37) is

$$\delta_1=\frac{1}{2}, \qquad u_1=\frac{1}{16}[\sin\tau+\sin 3\tau]. \tag{40}$$

Upon substituting the results already obtained in the third equation of (38), we find

$$\ddot{u}_2+u_2=-\left(\frac{1}{2}+\delta_2\right)\sin\tau+\frac{27}{32}\sin\tau-\frac{1}{4}\sin 3\tau-\frac{3}{72}\sin 5\tau. \tag{41}$$

Upon setting the coefficient of $\sin\tau$ equal to zero, as before, and integrating subject to the conditions (37), we find

$$\delta_2 = \frac{11}{32}, \qquad u_2 = \frac{1}{256}\,[7\sin\tau + 8\sin 3\tau + \sin 5\tau]. \tag{42}$$

The induction to the general term can now be made. We assume that $u_0, \ldots, u_{i-1}$; $\delta_1, \ldots, \delta_{i-1}$ have been determined and that it has been found that u_j is a sum of sines of odd multiples of τ, of which the highest is $2j+1$. The differential equation for the coefficient of μ^i is

$$\ddot{u}_i + u_i = -\delta_i u_0 + F_i(\delta_\kappa, u_\lambda) \qquad (k, \lambda = 0, \ldots, i-1), \tag{43}$$

where the F_i are linear in the δ_κ and of the third degree in the u_λ. The general term of F_i is

$$T_i = \delta_\kappa^m\, u_{\lambda_1}^{\nu_1} u_{\lambda_2}^{\nu_2} u_{\lambda_3}^{\nu_3},$$

where

$$\left.\begin{aligned} &m = 0 \text{ or } 1,\\ &\nu_1+\nu_2+\nu_3 = 1 \text{ or } 3,\\ &m\kappa + \nu_1\lambda_1 + \nu_2\lambda_2 + \nu_3\lambda_3 = i \text{ or } i-1 \qquad (\nu_1+\nu_2+\nu_3 = 1 \text{ or } 3). \end{aligned}\right\} \tag{44}$$

It follows from the second of these equations that there are an odd number of odd ν_j. Consequently T_i is a sum of sines of odd multiples of τ. The highest multiple of τ is

$$N_i = \nu_1(2\lambda_1+1) + \nu_2(2\lambda_2+1) + \nu_3(2\lambda_3+1).$$

On reducing this expression by the third of equation (44), it is found that $N_i = -2m\kappa + 2i + 1$, the greatest value of which is, by (44),

$$N_i = 2i+1. \tag{45}$$

Hence (43) may be written

$$\ddot{u} + u_i = [-\delta_i + A_1^{(i)}]\sin\tau + A_3^{(i)}\sin 3\tau + \cdots + A_{2i+1}^{(i)}\sin(2i+1)\tau, \tag{46}$$

where the $A_{2\lambda+1}^{(i)}$ are known constants.

In order that the solution of (46) shall be periodic, the condition

$$\delta_i = A_1^{(i)} \tag{47}$$

must be satisfied, which uniquely determines δ_i. Then the solution of equation (46) satisfying the conditions (37) is

$$\left.\begin{aligned} &u_i = a_1^{(i)}\sin\tau + a_3^{(i)}\sin 3\tau + \cdots + a_{2i+1}^{(i)}\sin(2i+1)\tau,\\ &a_{2\lambda+1}^{(i)} = -\frac{A_{2\lambda+1}^{(i)}}{4\lambda(\lambda+1)} \qquad (\lambda = 1, \ldots, i),\\ &a_1^{(i)} = \sum_{\lambda=1}^{i} (-1)^\lambda a_{2\lambda+1}^{(i)} = \sum_{\lambda=1}^{i} \frac{(-1)^{\lambda+1} A_{2\lambda+1}^{(i)}}{4\lambda(\lambda+1)}. \end{aligned}\right\} \tag{48}$$

The solution at this step has the same form as that which was assumed for $u_0, \ldots, u_{i-1}$, and the induction is therefore complete.

On collecting results, we have for the first terms of the solution

$$\left.\begin{aligned} u&=[\sin\tau]+\tfrac{1}{16}[\sin\tau+\sin 3\tau]\mu+\tfrac{1}{256}[7\sin\tau+8\sin 3\tau+\sin 5\tau]\mu^2+\cdots, \\ \delta&=0+\tfrac{1}{2}\mu+\tfrac{11}{32}\mu^2+\cdots. \end{aligned}\right\} \tag{49}$$

And substituting these results in (8), we get for the final result

$$\left.\begin{aligned} z&=a_3+\tfrac{1}{2}(a_1-a_3)[1-\cos 2\tau]\mu+\tfrac{1}{16}(a_1-a_3)[1-\cos 4\tau]\mu^2 \\ &\quad+\tfrac{1}{512}(a_1-a_3)[16+3\cos 2\tau-16\cos 4\tau-3\cos 6\tau]\mu^3+\cdots; \\ T&=\frac{2\pi l\sqrt{2(1+\delta)}}{\sqrt{g(a_1-a_3)}}=\sqrt{\frac{2}{g(a_1-a_3)}}\,2\pi l\left[1+\left(\tfrac{1}{2}\right)^2\mu+\left(\tfrac{3}{8}\right)^2\mu^2+\cdots\right], \end{aligned}\right\} \tag{50}$$

the expression for T agreeing with that found in (15).

49. Construction of the Solution from the Integral.—In the direct construction of the solution we have made no explicit use of the existence of the integral (11). We shall show that it can be used to check the computations, or to furnish the solution itself.

Equation (11) can be written in the form

$$\varphi(\dot{u}, u, \mu)=0.$$

Since $\dot{u}$, u, and δ can be expanded as converging power series in μ, we have

$$\varphi=\varphi_0+\varphi_1\mu+\varphi_2\mu^2+\cdots=0. \tag{51}$$

Since this equation is an identity in μ, it follows that

$$\varphi_i=\varphi_i(\dot{u}_0, u_0, \delta_2, \ldots, \dot{u}_i, u_i, \delta_i)=0 \qquad (i=0, \ldots \infty), \tag{52}$$

where φ_i is a polynomial in $\dot{u}_0, \ldots, \delta_i$. It follows from (11) that φ_i is linear in the δ_λ, of the second degree in $\dot{u}_{\lambda_1}$ and $\dot{u}_{\lambda_2}$, and of the second or fourth degree in u_{λ_1} and u_{λ_2}. Therefore φ_i is a sum of cosines of even multiples of τ. It is seen without difficulty that the highest multiple of τ in φ_i is $2i+2$. Hence we have

$$\varphi_i=B_0^{(i)}+B_2^{(i)}\cos 2\tau+\cdots+B_{2i+2}^{(i)}\cos(2i+2)\tau=0. \tag{53}$$

Since this equation holds for all values of τ, it follows that

$$B_{2\lambda}^{(i)}=0 \qquad (i=1, \ldots \infty;\ \lambda=0, \ldots, i+1). \tag{54}$$

The $B_{2\lambda}^{(i)}$ are functions of the $a_{2\lambda+1}^{(i)}$ and δ_i, and equations (54) constitute a searching check upon the computation of these quantities. If by some numerical accident an error were not indicated at any particular step, it would be revealed by the failure to satisfy (54) at some later step.

But the $a_{2\lambda+1}^{(i)}$ can be computed from (54), as will now be shown. Suppose $u_0, \ldots, u_{i-1}; \delta_2, \ldots, \delta_{u-1}$ have been computed. Then we find from the explicit expression (11) that

$$\varphi_i = 2\dot{u}_0\dot{u}_i + 2u_0u_i - \delta_i(1-u_0^2) + \psi_i(\dot{u}_\lambda, u_\lambda, \delta_{2\lambda}) \qquad (\lambda = 0, \ldots, i-1),$$

the ψ_i being known functions. Hence, using the notation of the first equation of (48), we get

$$\left.\begin{aligned}
&B_0^{(i)} = 2a_1^{(i)} + 0 - \tfrac{1}{2}\delta_i + C_0^{(i)} = 0,\\
&B_2^{(i)} = 0 + 4a_3^{(i)} - \tfrac{1}{2}\delta_i + C_2^{(i)} = 0,\\
&B_{2\lambda}^{(i)} = 2(\lambda-1)a_{2\lambda-1}^{(i)} + 2(\lambda+1)a_{2\lambda+1}^{(i)} + C_{2\lambda}^{(i)} = 0 \qquad (\lambda = 2, \ldots, i),\\
&B_{2i+2}^{(i)} = 2ia_{2i+1}^{(i)} + 0 + C_{2i+2}^{(i)} = 0,
\end{aligned}\right\} \quad (55)$$

where the $C_{2\lambda}^{(i)}$ $(\lambda = 0, \ldots, i+1)$ are known constants. On solving these equations beginning with the last, we find

$$\left.\begin{aligned}
&a_{2i+1}^{(i)} = -\frac{C_{2i+2}^{(i)}}{2i}, \qquad a_{2\lambda-1}^{(i)} = -\frac{\lambda+1}{\lambda-1}a_{2\lambda}^{(i)} - \frac{C_{2\lambda}^{(i)}}{2(\lambda-1)} \qquad (\lambda = i, \ldots, 2),\\
&\delta_i = 8a_3^{(i)} + 2C_2^{(i)}, \qquad a_1^{(i)} = \frac{1}{4}\delta_i - \frac{1}{2}C_0^{(i)},
\end{aligned}\right\} \quad (56)$$

which uniquely determine δ_i and the $a_{2\lambda+1}^{(i)}$.

Let us apply these equations to the computation of the first terms of u. Suppose $u_0 = \sin\tau$ and take $i = 1$. Then we find from (11) that

$$\varphi_i = 2\dot{u}_0\dot{u}_1 + 2u_0u_i - \delta_i(1-u_0^2) + u_0^2 - u_0^4;$$

whence

$$B_0^{(1)} = 2a_1^{(1)} - \frac{1}{2}\delta_i + \frac{1}{8} = 0, \qquad B_2^{(1)} = 4a_3^{(1)} - \frac{1}{2}\delta_1 = 0, \qquad B_4^{(1)} = 2a_3^{(1)} - \frac{1}{8} = 0.$$

Therefore

$$a_3^{(1)} = \frac{1}{16}, \qquad \delta_1 = \frac{1}{2}, \qquad a_1^{(1)} = \frac{1}{16},$$

agreeing with the results already found. The process can be continued as far as may be desired.

Two different methods of computing the solutions have been developed. They are both reduced by the general discussion to the mere routine of handling trigonometric terms. When they are both used each serves as a check on the other. There is little advantage with either over the other, so far as their convenience is concerned; the integral has a slight advantage in that when using it the computations are made with cosines, and the disadvantage of involving u to the fourth degree instead of only to the third.

II. DIGRESSION ON HILL'S EQUATION.

50. The x-Equation.—The value of z was given explicitly in (50), and since it is periodic the second equation of (4), after transforming to the independent variable τ, has the form

$$\ddot{x}+[a^2+\theta_1\mu+\theta_2\mu^2+\cdots]x=0, \tag{57}$$

where a is a constant independent of μ, and $\theta_1, \theta_2, \ldots$ are periodic functions of τ, having in this case the period π. Since the period can always be made 2π by a linear transformation on the independent variable, we shall suppose for the sake of uniformity that the period is 2π.

This problem belongs to the class which was treated in Chapter I, and can be transformed to the form considered there. But it is now in the form used first by Hill, and later by Bruns, Stieltjes, Harzer, Callandreau, etc., and because of its historical interest and for the sake of comparison with this earlier work, it will be treated directly in the form (57).

51. The Characteristic Equation.—Suppose that with the initial conditions $x(0)=1$, $\dot{x}(0)=0$ the solution of (57) is

$$x=\varphi(\tau), \qquad \dot{x}=\dot{\varphi}(\tau), \qquad \varphi(0)=1, \qquad \dot{\varphi}(0)=0,$$

and that with the initial conditions $x(0)=0$, $\dot{x}(0)=1$, the solution of (57) is

$$x=\psi(\tau), \qquad \dot{x}=\dot{\psi}(\tau), \qquad \psi(0)=0, \qquad \dot{\psi}(0)=1.$$

The determinant

$$\Delta=\begin{vmatrix}\varphi(\tau), & \dot{\varphi}(\tau)\\ \psi(\tau), & \dot{\psi}(\tau)\end{vmatrix}$$

being equal to unity at $\tau=0$, these solutions form a fundamental set. In fact, Δ is independent of τ for an equation of the form (57), as was shown in §18. It follows from the initial conditions that its value is unity. Hence the general solution of (57) is

$$x=c_1\varphi(\tau)+c_2\psi(\tau), \qquad \dot{x}=c_1\dot{\varphi}(\tau)+c_2\dot{\psi}(\tau). \tag{58}$$

where c_1 and c_2 are arbitrary constants.

Now let us make the transformation

$$x=e^{\alpha\tau}\xi, \tag{59}$$

where α is an undetermined constant. Then equation (57) becomes

$$\ddot{\xi}+2\alpha\dot{\xi}+\alpha^2\xi+[a^2+\theta_1\mu+\theta_2\mu^2+\cdots]\xi=0. \tag{60}$$

The general solution of this equation is, by (58) and (59),

$$\xi=e^{-\alpha\tau}[c_1\varphi(\tau)+c_2\psi(\tau)], \qquad \dot{\xi}=e^{-\alpha\tau}[c_1\dot{\varphi}(\tau)+c_2\dot{\psi}(\tau)]-\alpha e^{-\alpha\tau}[c_1\varphi(\tau)+c_2\psi(\tau)]. \tag{61}$$

We now raise the question whether it is possible to determine α in such a way that ξ shall be periodic with the period 2π. It follows from the form of (60) that sufficient conditions for the periodicity of ξ with the period 2π are

$$\xi(2\pi)-\xi(0)=0, \qquad \dot{\xi}(2\pi)-\dot{\xi}(0)=0.$$

On substituting from (61), we get, after making use of the initial values of φ, $\dot{\varphi}$, ψ, and $\dot{\psi}$,

$$\left.\begin{aligned}&[e^{-2\alpha\pi}\varphi(2\pi)-1]c_1+e^{-2\alpha\pi}\psi(2\pi)c_2=0,\\&e^{-2\alpha\pi}\dot{\varphi}(2\pi)c_1+[e^{-2\alpha\pi}\dot{\psi}(2\pi)-1]c_2=0.\end{aligned}\right\} \tag{62}$$

In order that these equations may have a solution other than the trivial one $c_1=c_2=0$, the determinant of the coefficients of c_1 and c_2 must vanish; or,

$$D=e^{-4\alpha\pi}\begin{vmatrix}\varphi(2\pi)-e^{2\alpha\pi}, & \psi(2\pi)\\ \dot{\varphi}(2\pi), & \dot{\psi}(2\pi)-e^{2\alpha\pi}\end{vmatrix}=0. \tag{63}$$

Since Δ is equal to unity, equation (63) is a reciprocal equation, and becomes

$$D=(e^{2\alpha\pi})^2-[\varphi(2\pi)+\dot{\psi}(2\pi)]\,e^{2\alpha\pi}+1=0, \tag{64}$$

of which the roots are $e^{2\alpha_1\pi}$ and $e^{-2\alpha_1\pi}$. (This is not the α_1 of §44.)

When the value of $e^{2\alpha_1\pi}$ which satisfies (64) is substituted in (62) the ratio of c_1 to c_2 is determined. Then equations (61) give $\xi^{(1)}$ and $\dot{\xi}^{(1)}$ periodic with the period 2π. We get a second solution $\xi^{(2)}$ and $\dot{\xi}^{(2)}$ by using the other root $e^{-2\alpha_1\pi}$. The $\xi^{(1)}$ and $\xi^{(2)}$ will each carry an arbitrary factor. We shall determine these factors so that $\xi^{(1)}(0)=\xi^{(2)}(0)=1$, and multiply the solutions by arbitrary constants at the end.

Consider equation (64). If $|\varphi(2\pi)+\dot{\psi}(2\pi)|<2$, α_1 is a pure imaginary; if $\varphi(2\pi)+\dot{\psi}(2\pi)>2$, α_1 is real; and if $\varphi(2\pi)+\dot{\psi}(2\pi)<-2$, α_1 is complex. In the first case x remains finite for all real values of τ; in the second case x becomes infinite as τ becomes infinite through real values; and in the third, $x=\infty$ for $\tau=\infty$ except for special initial conditions. It is found from (57) that $\varphi(2\pi)=\dot{\psi}(2\pi)=\cos a\pi$ for $\mu=0$. Therefore the part of α_1 which is independent of μ is the pure imaginary $a\sqrt{-1}$. Suppose a is not an integer; then α_1 is a pure imaginary for all real values of μ whose modulus is sufficiently small. If a is an integer, the value of α_1 for real values of μ whose modulus is small may be purely imaginary, real, or complex according to the values of $\varphi(2\pi)$ and $\dot{\psi}(2\pi)$.

Some of the more important properties of $\xi^{(1)}$ and $\xi^{(2)}$ will be derived. There are two particular solutions of (57) of the form $x=e^{\alpha_1\tau}\,\xi$ such that α_1 is a constant reducing to $\pm a\sqrt{-1}$ for $\mu=0$, and such that ξ is periodic with the period 2π, viz. $e^{\alpha_1\tau}\xi^{(1)}$ and $e^{-\alpha_1\tau}\xi^{(2)}$. The coefficients of (57) by hypothesis are all real, and the θ_j are sums of cosines of multiples of τ. Therefore, if the sign of $\sqrt{-1}$ be changed in a solution the result will be a solution. Suppose

a and μ have such values that α_1 is a pure imaginary. Then it follows that $\xi^{(1)}(\sqrt{-1})=\xi^{(2)}(-\sqrt{-1})$. Similarly, since (57) is unchanged by changing the sign of τ, it follows that $\xi^{(1)}(\tau)=\xi^{(2)}(-\tau)$. And finally, since (57) is unchanged by changing the sign of both $\sqrt{-1}$ and τ, it follows that $\xi^{(1)}(\sqrt{-1},\tau)=\xi^{(1)}(-\sqrt{-1},-\tau)$, $\xi^{(2)}(\sqrt{-1},\tau)=\xi^{(2)}(-\sqrt{-1},-\tau)$. Therefore in the expressions for $\xi^{(1)}$ and $\xi^{(2)}$ the coefficients of the cosine terms are real and of the sine terms pure imaginaries, and $\xi^{(1)}$ and $\xi^{(2)}$ differ only in the sign of $\sqrt{-1}$. Hence, writing them as Fourier series, we have

$$\xi^{(1)}=\Sigma\,[a_j\cos j\tau+\sqrt{-1}\,b_j\sin j\tau],$$
$$\xi^{(2)}=\Sigma\,[a_j\cos j\tau-\sqrt{-1}\,b_j\sin j\tau],$$

where the a_j and the b_j are real constants. It follows from this that it is sufficient to compute $\xi^{(1)}$.

Any solution of (57) can be expressed linearly in terms $e^{\alpha_1\tau}\,\xi^{(1)}$ and $e^{-\alpha_1\tau}\xi^{(2)}$. It follows from the initial values of φ, $\dot{\varphi}$, $\xi^{(1)}$, and $\xi^{(2)}$ and the equations above that

$$\varphi=\frac{1}{2}\,e^{\alpha_1\tau}\,\xi^{(1)}+\frac{1}{2}\,e^{-\alpha_1\tau}\,\xi^{(2)}.$$

Since $\xi^{(1)}$ and $\xi^{(2)}$ are periodic with the period 2π, and since their initial values are unity, we have

$$\frac{1}{2}\,(e^{2\alpha_1\pi}+e^{-2\alpha_1\pi})=\cosh 2\alpha_1\pi=\varphi(2\pi).$$

But by (64), $e^{2\alpha_1\pi}+e^{-2\alpha_1\pi}=\varphi(2\pi)+\dot{\psi}(2\pi)$. Therefore $\varphi(2\pi)=\dot{\psi}(2\pi)$.

When $\alpha_1=\beta\sqrt{-1}$ is a pure imaginary, as it is in many physical problems, we have

$$\cos 2\beta\pi=\varphi(2\pi). \tag{65}$$

This equation has the same form as that developed by Hill in his memoir on the motion of the Lunar Perigee, *Acta Mathemetica*, vol. VIII, pp. 1–36, and *Collected Works*, vol. I, pp. 243–270. It is also derived differently in Tisserand's *Mécanique Céleste*, vol. III, chap. 1.

Equation (64) or (65) furnishes a means of computing the transcendental α_1 because, under the hypotheses on (57), φ can always be found, for example as a power series in μ, with any desired degree of accuracy. Though this constitutes a complete solution of the problem and there are no difficulties in carrying it out except those of the lengthy computations, we shall find it convenient to make use of more of the properties of equation (57), and to find both α_1 and ξ_1 otherwise.

When $\mu=0$ the general solution of (57) is known to be

$$x_0=a_1\,e^{\alpha\sqrt{-1}\tau}+a_2\,e^{-\alpha\sqrt{-1}\tau},\qquad \dot{x}_0=\alpha\sqrt{-1}\,[a_1\,e^{\alpha\sqrt{-1}\tau}-a_2\,e^{-\alpha\sqrt{-1}\tau}], \tag{66}$$

where a_1 and a_2 are arbitrary constants.

It follows from the form of (57) that x can be expanded as a power series in μ which will converge for $0 \overline{\leqq} \tau \leqq 2\pi$ if $|\mu|$ is sufficiently small. Hence

$$\left.\begin{aligned}
\varphi &= \varphi_0 + \varphi_1\mu + \varphi_2\mu^2 + \cdots, & \varphi_0 &= \tfrac{1}{2}[e^{a\sqrt{-1}\tau} + e^{-a\sqrt{-1}\tau}],\\
\dot\varphi &= \dot\varphi_0 + \dot\varphi_1\mu + \dot\varphi_2\mu^2 + \cdots, & \dot\varphi_0 &= \frac{a\sqrt{-1}}{2}[e^{a\sqrt{-1}\tau} - e^{-a\sqrt{-1}\tau}],\\
\psi &= \psi_0 + \psi_1\mu + \psi_2\mu^2 + \cdots, & \psi_0 &= -\frac{\sqrt{-1}}{2a}[e^{a\sqrt{-1}\tau} - e^{-a\sqrt{-1}\tau}],\\
\dot\psi &= \dot\psi_0 + \dot\psi_1\mu + \dot\psi_2\mu^2 + \cdots, & \dot\psi_0 &= \tfrac{1}{2}[e^{a\sqrt{-1}\tau} + e^{-a\sqrt{-1}\tau}].
\end{aligned}\right\} \quad (67)$$

Therefore equation (63) becomes

$$D = e^{-4\alpha\pi}\begin{vmatrix} \cos 2a\pi - e^{2\alpha\pi} + \sum_{\lambda=1}^{\infty} \varphi_\lambda(2\pi)\mu^\lambda, & \frac{1}{a}\sin 2a\pi + \sum_{\lambda=1}^{\infty} \psi_\lambda(2\pi)\mu^\lambda \\ -a\sin 2a\pi + \sum_{\lambda=1}^{\infty} \dot\varphi_\lambda(2\pi)\mu^\lambda, & \cos 2a\pi - e^{2\alpha\pi} + \sum_{\lambda=1}^{\infty} \dot\psi_\lambda(2\pi)\mu^\lambda \end{vmatrix} = 0. \quad (68)$$

This equation expresses the condition that (57) shall have a periodic solution of the form (59), where ξ is periodic with the period 2π. If it is satisfied by $\alpha = \alpha_0$ it is also satisfied by $\alpha = \alpha_0 + \nu\sqrt{-1}$, where ν is any integer. These different values of α do not, however, lead to distinct values of x. We shall use only those values which reduce to $\pm a\sqrt{-1}$ for $\mu = 0$.

52. The Form of the Solution.—For $\mu = 0$ the principal solutions of (68) are $\alpha_1^{(0)} = +a\sqrt{-1}$ and $\alpha_2^{(0)} = -a\sqrt{-1}$. There are three cases depending upon the value of a:

Case I. $a \neq 0$ and $2a$ not an integer.
Case II. $a \neq 0$ and $2a$ an integer.
Case III. $a = 0$ and therefore $\alpha_1^{(0)} = \alpha_2^{(0)} = 0$.

Case I. This may be regarded as being the general case, and is that actually found and discussed by Hill and later writers on the same subject.

It follows from the form of (68) that

$$D = P(\alpha, \mu), \quad (69)$$

where P is a power series in α and μ which vanishes for $\mu = 0$, $\alpha = \pm a\sqrt{-1}$. It is also easily found for $\mu = 0$ and $\alpha = \pm a\sqrt{-1}$ that

$$\frac{\partial P}{\partial \alpha} = \pm 4\pi\sqrt{-1}\sin 2a\pi \cdot e^{\pm 2a\sqrt{-1}\pi},$$

which is distinct from zero under the conditions of Case I. Therefore it follows from the theory of implicit functions that (68) can be solved for α in the form

$$\left.\begin{aligned}
\alpha_1 &= +a\sqrt{-1} + \alpha_1^{(1)}\mu + \alpha_1^{(2)}\mu^2 + \cdots,\\
\alpha_2 &= -a\sqrt{-1} + \alpha_2^{(1)}\mu + \alpha_2^{(2)}\mu^2 + \cdots,
\end{aligned}\right\} \quad (70)$$

where the series converge for $|\mu|$ sufficiently small. Since the equation for α is a reciprocal equation in $e^{2\alpha\pi}$, it follows that $\alpha_1^{(j)} = -\alpha_2^{(j)}$ $(j = 1, 2, \ldots, \infty)$.

If we substitute either of the series (70) in (62), we shall have the ratio of c_1 and c_2 expressed as a power series in μ. If this result and equations (67) are substituted in (61), ξ will be expressed as a power series in μ, converging for $|\mu|$ sufficiently small, and carrying one arbitrary constant factor. We shall omit the superfix and adopt the notation

$$\xi=\xi_0+\xi_1\,\mu+\xi_2\,\mu^2+\cdots. \tag{71}$$

Since the periodicity conditions have been satisfied, ξ is periodic and it follows from its expansibility that each ξ_i separately is periodic. Hence it follows from this property and the initial condition $\xi(0)=1$ that

$$\left.\begin{array}{ll} \xi_i(2\pi+\tau)-\xi_i(\tau)\equiv 0 & (i=0,\,1,\,\ldots\,\infty), \\ \xi_0(0)=1, \quad \xi_i(0)=0 & (i=1,\,2,\,\ldots\,\infty). \end{array}\right\} \tag{72}$$

It will be shown when the solutions are constructed that these properties uniquely define their coefficients.

Case II. In this case we find from (68) for $\mu=0$, $\alpha=\pm a\sqrt{-1}$, that

$$\frac{\partial P}{\partial\alpha}=0, \qquad \frac{1}{2}\frac{\partial^2 P}{\partial\mu^2}=\varphi_1(2\pi)\dot{\psi}_1(2\pi)-\dot{\varphi}_1(2\pi)\psi_1(2\pi),$$

$$\frac{1}{2}\frac{\partial^2 P}{\partial\alpha^2}=4\pi^2, \qquad \frac{\partial^2 P}{\partial\alpha\,\partial\mu}=(-1)^{2a+1}\,2\pi\,[\varphi_1(2\pi)+\dot{\psi}_1(2\pi)].$$

Hence if we let $\alpha=a\sqrt{-1}+\beta$, the expression for D has the form

$$4\pi^2\beta^2+c_{11}\,\beta\,\mu+c_{02}\,\mu^2+\cdots=0, \tag{73}$$

where in general c_{11} and c_{02} are distinct from zero. In order not to multiply cases indefinitely we shall suppose that c_{11} and c_{02} are distinct from zero and that the discriminant of the quadratic terms of (73), viz. $\delta=c_{11}^2-16\pi^2c_{02}$, is distinct from zero. Suppose the quadratic terms factor into

$$4\pi^2(\beta-b_1\mu)\,(\beta-b_2\mu),$$

where now $b_1\neq b_2$ since $\delta\neq 0$.* Then by the theory of implicit functions equation (73) is solvable for β as converging power series of the form

$$\beta_1=b_1\mu+\beta_1^{(2)}\mu^2+\cdots, \qquad \beta_2=b_2\mu+\beta_2^{(2)}\mu^2+\cdots. \tag{74}$$

Hence we get two solutions for β, and consequently for α, as power series in μ starting from the root $\alpha=+a\sqrt{-1}$ for $\mu=0$. There are two similar ones obtained by starting from $\alpha=-a\sqrt{-1}$ for $\mu=0$, but they do not lead to distinct solutions since they differ from the former values by purely imaginary integers. Then, by means of (62) and (61), we obtain the final solutions as before.

There are other sub-cases, for example $b_1=b_2$, all of which can be treated by the theory of implicit functions, but they will be omitted.

*Since $\alpha=\alpha_1$ and $\alpha=-\alpha_1$ are the roots of (63), it follows that in this case $b_1=-b_2$, and that they are therefore distinct unless $b_1=b_2=0$.

Case III. Under the conditions of this case we have for $\mu=0$, instead of equations (66) and (67), the solution

$$x_0=a_1\tau+a_2, \qquad \dot{x}_0=a_1, \qquad \varphi_0=1, \qquad \psi_0=\tau. \tag{75}$$

Hence equation (68) becomes

$$D=e^{-4a\pi}\begin{vmatrix} 1-e^{2a\pi}+\sum_{\lambda=1}^{\infty}\varphi_\lambda(2\pi)\mu^\lambda, & 2\pi+\sum_{\lambda=1}^{\infty}\psi_\lambda(2\pi)\mu^\lambda \\ 0+\sum_{\lambda=1}^{\infty}\dot{\varphi}_\lambda(2\pi)\mu^\lambda, & 1-e^{2a\pi}+\sum_{\lambda=1}^{\infty}\dot{\psi}_\lambda(2\pi)\mu^\lambda \end{vmatrix}=0. \tag{76}$$

As before, D can be expanded into a converging power series in α and μ, and for $\mu=0$ the principal solutions of $D=0$ are $\alpha_1=\alpha_2=0$. We find from (76) for $\mu=\alpha=0$ that

$$\frac{\partial D}{\partial \alpha}=0, \qquad \frac{\partial^2 D}{\partial \alpha^2}=+8\pi^2, \qquad \frac{\partial D}{\partial \mu}=-2\pi\,\dot{\varphi}_1(2\pi).$$

In general $\dot{\varphi}_1(2\pi)$ is distinct from zero, and when it is we know from the theory of implicit functions that (76) can be solved for α in the form

$$\alpha_1=0+\alpha_1^{(1)}\sqrt{\mu}+\alpha_1^{(2)}\mu+\cdots, \qquad \alpha_2=0-\alpha_1^{(1)}\sqrt{\mu}+\alpha_1^{(2)}\mu+\cdots. \tag{77}$$

Since α_1 and α_2 differ only in sign the $\alpha_1^{(2j)}$ are all zero. After α_1 and α_2 have been determined, we obtain the final solutions as before, except now the series proceed in powers of $\sqrt{\mu}$ instead of in powers of μ.

53. Direct Construction of the Solutions in Case I.—On substituting the first of (70) and (71) in (60) and equating the coefficients of the several powers of μ to zero, we obtain

$$\left.\begin{aligned} &\ddot{\xi}_0+2a\sqrt{-1}\,\dot{\xi}_0=0, \\ &\ddot{\xi}_1+2a\sqrt{-1}\,\dot{\xi}_1=-2\alpha_1^{(1)}[\dot{\xi}_0+a\sqrt{-1}\,\xi_0]-\theta_1\xi_0, \\ &\ddot{\xi}_2+2a\sqrt{-1}\,\dot{\xi}_2=-2\alpha_1^{(2)}[\dot{\xi}_0+a\sqrt{-1}\,\xi_0]-2\alpha_1^{(1)}[\dot{\xi}_1+a\sqrt{-1}\,\xi_1] \\ &\qquad\qquad -(\alpha_1^{(1)})^2\xi_0-\theta_1\xi_1-\theta_2\xi_0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{78}$$

the left members of all the equations being the same except for the subscripts, and the first terms on the right being the same except for the superscripts on α_1. There is a similar set of equations defining the other solution, which differ from these only in the sign of $\sqrt{-1}$.

Consider the solutions of (78) subject to the conditions (72). The general solution of the first equation is

$$\xi_0=b_1^{(0)}+b_2^{(0)}e^{2a\sqrt{-1}\,\tau},$$

where $b_1^{(0)}$ and $b_2^{(0)}$ are arbitrary constants of integration. Since in this case $2a$ is not an integer, it follows from (72) that

$$\xi_0=1. \tag{79}$$

The right member of the second equation of (78) now becomes a known function of τ. When the left member of this equation is set equal to zero, its general solution is

$$\xi_1 = b_1^{(1)} + b_2^{(1)} e^{-2a\sqrt{-1}\,\tau}. \tag{80}$$

Now regarding $b_1^{(1)}$ and $b_2^{(1)}$ as variables, according to the method of the variation of parameters, and imposing the conditions $\dot{b}_1^{(1)} + \dot{b}_2^{(1)} e^{-2a\sqrt{-1}\,\tau} = 0$ and that the second equation of (78) shall be satisfied, we obtain

$$\dot{b}_1^{(1)} = -\left[a_1^{(1)} - \frac{\sqrt{-1}\,\theta_1}{2a}\right], \qquad \dot{b}_2^{(1)} = +\left[a_1^{(1)} - \frac{\sqrt{-1}\,\theta_1}{2a}\right] e^{2a\sqrt{-1}\,\tau}. \tag{81}$$

Let the constant part of θ_1 be d_1. Then in order that $b_1^{(1)}$ shall not contain a term proportional to τ, which would make (80) non-periodic, we must impose the condition

$$a_1^{(1)} = \frac{\sqrt{-1}\,d_1}{2a}. \tag{82}$$

The integrals of sines and cosines are cosines and sines respectively, and

$$\int {\sin \atop \cos} j\tau\, e^{2a\sqrt{-1}\,\tau}\, d\tau = \frac{2a\sqrt{-1}}{j^2-4a^2} {\sin \atop \cos} j\tau\, e^{2a\sqrt{-1}\,\tau} {- \atop +} \frac{j}{j^2-4a^2} {\cos \atop \sin} j\tau\, e^{2a\sqrt{-1}\,\tau}.$$

Therefore it follows, when (82) is satisfied, that

$$b_1^{(1)} = P_1(\tau) + B_1^{(1)}, \qquad b_2^{(1)} = Q_1(\tau)\, e^{2a\sqrt{-1}\,\tau} + B_2^{(1)}, \tag{83}$$

where P_1 and Q_1 are periodic functions of τ having the period 2π, and $B_1^{(1)}$ and $B_2^{(1)}$ are the constants of integration. Since $2a$ is not an integer, j^2-4a^2 can not vanish and there are no terms with infinite coefficients. On substituting (83) in (80) and imposing the conditions (72), we get

$$\xi_1 = P_1(\tau) + Q_1(\tau) - P_1(0) - Q_1(0). \tag{84}$$

It is easy to show that all succeeding steps of the integration are entirely similar. The differential equation for the coefficient of μ^i is

$$\ddot{\xi}_i + 2a\sqrt{-1}\,\dot{\xi} = -2a_1^{(i)}[\dot{\xi} + a\sqrt{-1}\,\xi_0] - F_i(\tau),$$

where $F_i(\tau)$ is an entirely known periodic function of τ after the preceding steps have been taken. The general solution of the left member of this equation set equal to zero is the same as (80), except that ξ has the subscript i, and b_1 and b_2 have the superscript i. The equations analogous to (81) are

$$\dot{b}_1^{(i)} = -\left[a_1^{(i)} - \frac{\sqrt{-1}\,F_i}{2a}\right], \qquad \dot{b}_2^{(i)} = +\left[a_1^{(i)} - \frac{\sqrt{-1}\,F_i}{2a}\right] e^{2a\sqrt{-1}\,\tau}.$$

If we represent the constant part of F_i by d_i, we must impose the condition

$$a_1^{(i)} = +\frac{\sqrt{-1}\,d_i}{2a}$$

in order that the solution shall be periodic. Then integrating, substituting in the equation analogous to (80), and imposing the conditions (72), we get

$$\xi_i = P_i(\tau) + Q_i(\tau) - P_i(0) - Q_i(0),$$

where $P_i(\tau)$ and $Q_i(\tau)$ are periodic with the period 2π. Thus the general step in the integration is in all essentials similar to the second step.

54. Direct Construction of the Solutions in Case II.—Since in this case the solutions are also in general developable as power series in μ, we start from equations (78). The general solution of the first equation is

$$\xi_0 = b_1^{(0)} + b_2^{(0)} e^{-2a\sqrt{-1}\,\tau}.$$

Since $2a$ is an integer, ξ_0 is periodic for all values of $b_1^{(0)}$ and $b_2^{(0)}$.

In this case it is convenient in the computation to impose the initial condition $\dot{\xi}(0) = 1$ instead of $\xi(0) = 1$, whence

$$\dot{\xi}_0(0) = 1, \qquad \dot{\xi}_i(0) = 0 \qquad (i = 1, \ldots \infty).$$

Hence we have for the solution at the first step of the integration

$$\xi_0 = b_1^{(0)} + \frac{\sqrt{-1}}{2a} e^{-2a\sqrt{-1}\,\tau}, \tag{85}$$

where $b_1^{(0)}$ is a constant which will be determined at the next step.

The second equation of (78) now becomes

$$\ddot{\xi}_1 + 2a\sqrt{-1}\,\dot{\xi}_1 = -a_1^{(1)}[2a\sqrt{-1}\,b_1^{(0)} + e^{-2a\sqrt{-1}\,\tau}] - b_1^{(0)}\theta_1 - \frac{\sqrt{-1}}{2a}\theta_1 e^{-2a\sqrt{-1}\,\tau}. \tag{86}$$

The equations analogous to (81) are in this case

$$\left.\begin{aligned} \dot{b}_1^{(1)} &= +\frac{\sqrt{-1}}{2a}[2a\sqrt{-1}\,a_1^{(1)} + \theta_1]b_1^{(0)} + \frac{\sqrt{-1}}{2a}\left[a_1^{(1)} + \frac{\sqrt{-1}}{2a}\theta_1\right]e^{-2a\sqrt{-1}\,\tau}, \\ \dot{b}_2^{(1)} &= -\frac{\sqrt{-1}}{2a}[2a\sqrt{-1}\,a_1^{(1)} + \theta_1]b_1^{(0)}e^{2a\sqrt{-1}\,\tau} - \frac{\sqrt{-1}}{2a}\left[a_1^{(1)} + \frac{\sqrt{-1}}{2a}\theta_1\right]. \end{aligned}\right\} \tag{87}$$

In order that ξ_1 shall be periodic the right members of these equations must contain no constant terms. Hence we must impose the conditions

$$[2a\sqrt{-1}\,a_1^{(1)} + d_1]\,b_1^{(0)} + \delta_{21} = 0, \qquad \frac{\sqrt{-1}}{2a}a_1^{(1)} - \frac{1}{4a^2}d_1 + \delta_{22}b_1^{(0)} = 0, \tag{88}$$

where d_1 is the constant part of θ_1, and where δ_{21} and δ_{22} are the constant parts of $\sqrt{-1}\,\theta_1 e^{-2a\sqrt{-1}\,\tau}/2a$ and $\sqrt{-1}\,\theta_1 e^{2a\sqrt{-1}\,\tau}/2a$ respectively. If θ_1 is an even function of τ, then $\delta_{21} = \delta_{22}$, and if it is an odd function, $\delta_{21} = -\delta_{22}$. Equations (88) express the conditions that the right member of (86) shall contain no terms independent of τ, or which involve τ only in the form $e^{-2a\sqrt{-1}\,\tau}$. This is only an expression for the fact that in order that the solutions shall be periodic the right member of the differential equation must not contain terms of the type obtained by integrating the left member set equal to zero.

Upon eliminating $b_1^{(0)}$ from (88), we get

$$[2a\sqrt{-1}\;a_1^{(1)} + d_1]\,[2a\sqrt{-1}\;a_1^{(1)} - d_1] - 4a^2\,\delta_{21}\,\delta_{22} = 0,$$

of which the solutions are

$$a_1^{(1)} = \pm\sqrt{\delta_{21}\,\delta_{22} + \frac{d_1^2}{4a^2}}\;\sqrt{-1}. \tag{89}$$

The two values of $a_1^{(1)}$ are distinct unless $\delta_{21}\,\delta_{22} + d_1^2/4a^2 = 0$. They will not be equal to zero except for special values of the coefficients of the differential

equations, and we shall assume here that they are distinct. This was the case treated in §52, *Case II*, and when the two values of $\alpha_1^{(1)}$ are equal the solutions may be developable in a different form. After $\alpha_1^{(1)}$ has been found, $b_1^{(0)}$ can be obtained at once from either of equations (88). There is a difficulty only if $\delta_{21}=\delta_{22}=0$, when one solution for $b_1^{(0)}$ becomes infinite; but in this case we impose a different initial condition on ξ.

After satisfying equation (88), the integrals of (87) have the form

$$b_1^{(1)}=P_1(\tau)+B_1^{(1)}, \qquad b_2^{(1)}=Q_1(\tau)+B_2^{(1)},$$

where $P_1(\tau)$ and $Q_1(\tau)$ are periodic functions of τ, and $B_1^{(1)}$ and $B_2^{(1)}$ are undetermined constants of integration. These results substituted in equation (80) give, after applying the condition $\xi_1(0)=0$,

$$\xi_1=B_1^{(1)}+P_1(\tau)+Q_1(\tau)e^{-2a\sqrt{-1}\tau}-\left\{Q_1(0)+\frac{\sqrt{-1}}{2a}[\dot{P}_1(0)+\dot{Q}_1(0)]\right\}e^{-2a\sqrt{-1}\tau}, \tag{90}$$

where $B_1^{(1)}$ is so far undetermined.

It is necessary to carry the integration one step further in order to prove that the general term satisfying the periodicity condition and the initial condition can be found. The differential equation for the coefficient of μ^2 is

$$\left.\begin{aligned}\ddot{\xi}_2+2a\sqrt{-1}\,\dot{\xi}_2= &-\alpha_1^{(2)}[2a\sqrt{-1}\,b_1^{(0)}+e^{-2a\sqrt{-1}\tau}]\\ &-2a\sqrt{-1}\,\alpha_1^{(1)}\,B_1^{(1)}-\theta_1\,B_1^{(1)}+F_2(\tau),\end{aligned}\right\} \tag{91}$$

where $\alpha_1^{(2)}$ and $B_1^{(1)}$ are undetermined constants, and where $F_2(\tau)$ is an entirely known periodic function of τ.

In the case under consideration the equations corresponding to (88) are

$$\left.\begin{aligned}-2a\sqrt{-1}\,b_1^{(0)}\,\alpha_1^{(2)}-(2a\sqrt{-1}\,\alpha_1^{(1)}+d_1)B_1^{(1)}+d_2=0,\\ -\alpha_1^{(2)}+2a\sqrt{-1}\,\delta_{22}\,B_1^{(1)}+\delta_2=0,\end{aligned}\right\} \tag{92}$$

where d_2 and δ_2 are known constants depending on F_2. The unknowns $\alpha_1^{(2)}$ and $B_1^{(1)}$ enter (92) linearly, and, by means of (88), the determinant of their coefficients becomes $-4a\sqrt{-1}\ \alpha_1^{(1)}$, which, by hypothesis, is distinct from zero. Therefore $\alpha_1^{(1)}$ and $B_1^{(1)}$ are uniquely determined by these equations. When equations (92) are satisfied, the solution of (91) satisfying the initial condition is

$$\xi_2=B_1^{(2)}+P_2(\tau)+Q_2(\tau)e^{-2a\sqrt{-1}\tau}-\left\{Q_2(0)+\frac{\sqrt{-1}}{2a}[\dot{P}_2(0)+\dot{Q}_2(0)]\right\}e^{-2a\sqrt{-1}\tau}, \tag{93}$$

which has the same form as (90). Therefore the next step can be taken in the same manner. Thus it is seen that the process is unique after the choice of the sign of $\alpha_1^{(1)}$ is made, and in this way two solutions which satisfy the periodicity and initial conditions are obtained.

55. Direct Construction of the Solutions in Case III.—In this case the solutions were proved to have, in general, the form

$$\xi=\xi_0+\xi_1\sqrt{\mu}+\xi_2\,\mu+\cdots, \qquad \alpha=0+\alpha^{(1)}\sqrt{\mu}+\alpha^{(2)}\mu+\cdots. \tag{94}$$

Substituting these equations in (60), we have for the term independent of $\sqrt{\mu}$

$$\ddot{\xi}_{(0)}=0,$$

of which the solution satisfying the conditions (72) is

$$\xi_0=1. \tag{95}$$

The differential equation which the coefficient of $\sqrt{\mu}$ must satisfy is

$$\ddot{\xi}_1+2\alpha^{(1)}\dot{\xi}_0=0,$$

and the solution of this equation which satisfies the conditions (72) is

$$\xi_1=0, \qquad \alpha^{(1)}= \text{ an undetermined constant.} \tag{96}$$

The differential equation which defines the coefficient of μ is then

$$\ddot{\xi}_2+2\alpha^{(2)}\,\dot{\xi}_0=-2\alpha^{(1)}\,\dot{\xi}_1-(\alpha^{(1)})^2\,\xi_0-\theta_1\,\xi_0=-(\alpha^{(1)})^2-\theta_1\,.$$

When the left member of this equation is set equal to zero, its general solution is found to be

$$\xi_2=b_1^{(2)}\,\tau+b_2^{(2)}, \tag{97}$$

where $b_1^{(2)}$ and $b_2^{(2)}$ are the constants of integration. On making use of the variation of parameters and imposing the condition that the differential equation shall be satisfied, we find

$$\dot{b}_1^{(2)}=-[(\alpha^{(1)})^2+\theta_1], \qquad \dot{b}_2^{(2)}=+[(\alpha^{(1)})^2+\theta_1]\,\tau. \tag{98}$$

In order that, when the first of (98) is integrated and the result substituted in (97), there shall be no term proportional to τ^2, the condition

$$(\alpha^{(1)})^2+d_1=0 \tag{99}$$

must be imposed, where d_1 is the constant part of θ_1. This equation determines two values of $\alpha^{(1)}$ which differ only in sign, and they are reals or pure imaginaries when the coefficients of θ_1 are real.

Since there are no more arbitraries available in (98), no more conditions can be satisfied. The first equation of (98) gives rise to integrals of the type

$$-a_j\int {\sin \atop \cos} j\tau\,d\tau = \pm\frac{a_j}{j}{\cos \atop \sin} j\tau.$$

The second of (98) gives rise to the corresponding and associated integrals

$$+a_j\int\tau {\sin \atop \cos} j\tau d\tau=\mp\frac{a_j\tau}{j}{\cos \atop \sin} j\tau+\frac{a_j}{j^2}{\sin \atop \cos} j\tau.$$

Hence, imposing the condition (99), integrating (98), and substituting the results in (97), we have

$$\xi_2=P_2(\tau)-P_2(0), \tag{100}$$

where $P_2(\tau)$ is periodic and $\alpha^{(2)}$ is as yet undetermined.

In determining the coefficient of $\mu^{3/2}$, the equations for $\dot{b}_1^{(3)}$ and $\dot{b}_2^{(3)}$ corresponding to (98) are found to be

$$\dot{b}_1^{(3)} = -[2\alpha^{(1)}\alpha^{(2)} + \varphi_3(\tau)], \qquad \dot{b}_2^{(3)} = +[2\alpha^{(1)}\alpha^{(2)} + \varphi_3(\tau)]\,\tau,$$

where $\varphi_3(\tau)$ is a periodic function of τ. If d_3 is the constant part of φ_3, we must impose the condition

$$2\alpha^{(1)}\alpha^{(2)} + d_3 = 0,$$

which uniquely determines $\alpha^{(2)}$ if $\alpha^{(1)}$ is distinct from zero, as it is in general. If $\alpha^{(1)} = 0$ the expansion may be of another form, for this is an exceptional case in the existence discussion, and it is necessary to go to higher terms of the differential equation to determine the character of the solution. But limiting ourselves here to the case where $\alpha^{(1)}$ is distinct from zero, the solution is carried out as in the preceding step. All the succeeding steps are the same except for the indices and the numerical values of the coefficients.

III. SOLUTION OF THE X AND Y-EQUATIONS FOR THE SPHERICAL PENDULUM.

56. Application to the Spherical Pendulum.—On transforming from t to τ as the independent variable in the second equation of (4), and making use of (50), we get

$$\left.\begin{aligned} &\ddot{x} + [a^2 + \theta_1\,\mu + \theta_2\,\mu^2 + \cdots]\,x = 0, \\ &a^2 = \frac{2(2\alpha_1 + \alpha_3)}{\alpha_1 - \alpha_3}, \qquad \theta_1 = \frac{3}{\alpha_1 - \alpha_3}[\alpha_1 + (\alpha_1 - \alpha_3)\cos 2\tau], \ \ldots\ . \end{aligned}\right\} \quad (101)$$

Obviously a will not in general be an integer. It will be shown that the only integral value it can have in the problem of the spherical pendulum is unity. Suppose a equals the integer n. In this case the second of (101) gives

$$(4 - n^2)\alpha_1 = -(2 + n^2)\alpha_3 .$$

It was shown in §43 that in the problem of the spherical pendulum α_1 is positive and α_3 is negative. Therefore n^2 must be unity or zero. In the former case we have $\alpha_1 = -\alpha_3$, which, because of the inequalities satisfied by α_1 and α_3, can be true only if $\alpha_1 = +l$, $\alpha_3 = -l$. This is the special case of the simple pendulum. If $n = 0$ we have $4\alpha_1 = -2\alpha_3$, which, because of the inequalities to which α_1 and α_3 are subject, can not be satisfied. Therefore a is not an integer and the equations can be integrated by the methods of §53. Upon omitting the subscript on the α in $e^{\alpha\tau}$, so as not to confuse it with α_1, α_2, α_3 defined in §43, it is found by actual computation that

$$\xi_0 = 1, \qquad \alpha = \frac{+3\sqrt{2}\,\alpha_1\sqrt{-1}}{4\sqrt{(\alpha_1 - \alpha_3)(2\alpha_1 + \alpha_3)}},$$

$$\xi^{(1)} = -\frac{(\alpha_1 - \alpha_3)}{4(\alpha_1 + \alpha_3)}(\cos 2\tau - 1) + \frac{\sqrt{2}\sqrt{(\alpha_1 - \alpha_3)(2\alpha_1 + \alpha_3)}}{4(\alpha_1 + \alpha_3)}\sqrt{-1}\,\sin 2\tau,$$

. . . .

The other solution is found from this one simply by changing the sign of $\sqrt{-1}$. Hence the general solution is

$$\left.\begin{aligned}
x &= Ae^{a\tau}\xi^{(1)}(\tau)+Be^{-a\tau}\xi^{(2)}(\tau)\,,\\
a &= \left\{\frac{\sqrt{2}\sqrt{2\alpha_1+\alpha_3}}{\sqrt{\alpha_1-\alpha_3}}+\frac{3\sqrt{2}\,\alpha_1\,\mu}{4\sqrt{(\alpha_1-\alpha_3)(2\alpha_1+\alpha_3)}}+\cdots\right\}\sqrt{-1}=\lambda\sqrt{-1},\\
\xi^{(1)} &= 1-\left[\frac{\alpha_1-\alpha_3}{4(\alpha_1+\alpha_3)}(\cos 2\tau-1)\right.\\
&\qquad\left.-\frac{\sqrt{2}\sqrt{(\alpha_1-\alpha_3)(2\alpha_1+\alpha_3)}}{4(\alpha_1+\alpha_3)}\sqrt{-1}\sin 2\tau\right]\mu+\cdots,\\
\xi^{(2)} &= 1-\left[\frac{\alpha_1-\alpha_3}{4(\alpha_1+\alpha_3)}(\cos 2\tau-1)\right.\\
&\qquad\left.+\frac{\sqrt{2}\sqrt{(\alpha_1-\alpha_3)(2\alpha_1+\alpha_3)}}{4(\alpha_1+\alpha_3)}\sqrt{-1}\sin 2\tau\right]\mu+\cdots.
\end{aligned}\right\}\quad(102)$$

It can be shown from the properties of θ_1, θ_2, . . . and the method of constructing the solutions, that the coefficients of μ_j in $\xi^{(1)}$ and $\xi^{(2)}$ are cosines and sines of even multiples of τ, the highest multiple being $2j$.

It follows from the form of equations (102) that, for real initial conditions, A and B must be conjugate complex quantities, $2A=A_1-\sqrt{-1}\,A_2$, and $2\,B=A_1+\sqrt{-1}\,A_2$. Hence the solution takes the form

$$\left.\begin{aligned}
x &= A_1[x_1\cos\lambda\tau-x_2\sin\lambda\tau]+A_2[x_1\sin\lambda\tau+x_2\cos\lambda\tau]\,,\\
x_1 &= 1-\frac{\alpha_1-\alpha_3}{4(\alpha_1+\alpha_3)}(\cos 2\tau-1)\,\mu+(\text{cosines})\,\mu^2+\cdots,\\
x_2 &= \frac{\sqrt{2}\,\sqrt{(\alpha_1-\alpha_2)\,(2\alpha_1+\alpha_3)}}{4\,(\alpha_1+\alpha_3)}(\sin 2\tau)\mu+(\text{sines})\mu^2+\cdots,
\end{aligned}\right\}\quad(103)$$

where A_1 and A_2 are arbitrary constants.

Since the second and third equations of (4) have the same form, the solution of the latter can differ from that of the former only in the constants of integration. Therefore

$$y=B_1[x_1\cos\lambda\tau-x_2\sin\lambda\tau]+B_2[x_1\sin\lambda\tau+x_2\cos\lambda\tau].$$

The constants A_1, A_2, B_1, and B_2 are subject to the conditions that equations (2), the first equation of (4), and the relation

$$x\,\dot{x}+y\,\dot{y}+z\,\dot{z}=0$$

shall be satisfied. This leaves one arbitrary which may be used to dispose of the orientation of the xy-axes at $\tau=0$. Let the axes be chosen so that $\dot{x}=0$ at $\tau=0$. Then, since $\dot{z}$ also vanishes at $\tau=0$, we have from this equation and the values of x and y above

$$x=A_1[x_1\cos\lambda\tau-x_2\sin\lambda\tau],\qquad y=B_2[x_1\sin\lambda\tau+x_2\cos\lambda\tau].\quad(104)$$

Making use of (104), it is found from (2) and the first of (4), that at $\tau=0$,

$$A_1^2=l^2-\alpha_3^2, \qquad B_2^2=\frac{2l^2(1+\delta)\,(2\alpha_3+c_1)}{(\alpha_1-\alpha_3)\,[\lambda+\dot{x}_2(0)]^2}.$$

Therefore, the solution is completely determined when the positive directions on the x and y-axes are chosen. The well-known properties of the motion can easily be derived from equations (104).

The variables x and y always oscillate around their initial values since they are made up of terms which are the product of two periodic functions that are always finite. Since the period of x_1 and x_2 is π, the solutions are periodic and the curves described by the spherical pendulum are re-entrant provided λ is a rational number. Let the period of x_1 and x_2 be $P_1=\pi$, and that of $\sin\lambda\tau$ and $\cos\lambda\tau$ be $P_2=2\pi/\lambda$. Then, when P_1 and P_2 are commensurable,

$$\frac{P_2}{P_1}=\frac{2}{\lambda}=\frac{2q}{p},$$

p and q being relatively prime integers. Hence the least common multiple of the two periods is

$$P=pP_2=2qP_1=\frac{2p\pi}{\lambda}=2q\pi. \tag{105}$$

In the period P the variables z, x_1, and x_2 make $2q$ complete oscillations, and $\sin\lambda\tau$ and $\cos\lambda\tau$ make p complete oscillations. In the independent variable τ the period of z, x_1, and x_2 is independent of μ, but P_2 is a continuous function of μ. In the original independent variable t both periods are continuous functions of t. But in either variable the period P is a discontinuous function of μ, being finite only when the ratio of P_1 to P_2 is rational. It is seen from the solution expressed in terms of τ, in which P_1 is constant with respect to μ, that this ratio fills a portion of the linear continuum, and therefore that only exceptionally is it rational.

57. Application to the Simple Pendulum.—Since the problem of the simple pendulum is a special case under that of the spherical pendulum, it can be treated by the same methods. Of course, it is not advisable to do so in practice, for x and z must satisfy the relation

$$x^2+z^2=l^2, \tag{106}$$

from which x can be found when z has been determined.

Before discussing the properties of x we must find the expression for z in this case. Since in the simple pendulum z always passes through the

value $-l$, it follows that $a_3 = -l$. From the fact that $z' = 0$ when $z = -l$, we get, by (5), $c_2 = c_1$ and $z' = 0$ for $z = +l$. Hence, in the case of the simple pendulum we have from (50) and (101)

$$\left.\begin{array}{l} a_1 = +l, \qquad a_2 = -l + 2l\mu, \qquad a_3 = -l, \\ z = -l + l(1 - \cos 2\tau)\mu + \frac{1}{8}\, l(1 - \cos 4\tau)\, \mu^2 + \cdots, \qquad a_2 = 1, \\ \theta_1 = \frac{3}{2}(1 + 2\cos 2\tau), \qquad \theta_2 = \frac{3}{32}(5 + 16\cos 2\tau + 4\cos 4\tau), \qquad \ldots; \\ \ddot{x} + \left[1 + \frac{3}{2}(1 + 2\cos 2\tau)\mu + \frac{3}{32}(5 + 16\cos 2\tau + 4\cos 4\tau)\mu^2 + \cdots\right] x = 0. \end{array}\right\} \quad (107)$$

In the expression for z the coefficient of each power of μ separately vanishes at $\tau = 0$ and is a sum of cosines of even multiples of τ. Therefore

$$x^2 = l^2 - z^2$$

contains $\sin^2 \tau$ as a factor. The parameter μ is also a factor. From the relation $\mu(1 - \cos 2\tau) = 2\mu \sin^2 \tau$, it follows that

$$x = \pm \sqrt{l^2 - z^2} \tag{108}$$

is expansible as a power series in $\sqrt{\mu}$, containing only odd powers of $\sqrt{\mu}$. It is easy to show that the coefficient of $(\sqrt{\mu})^{2i+1}$ is a sum of sines of odd multiples of τ, the highest multiple being $2i+1$. We find directly from the second line of (107) and from (108) that

$$x = \pm l \left\{ [2 \sin \tau]\, \mu^{1/2} + \frac{1}{8}\, [-5 \sin \tau + 3 \sin 3\tau]\, \mu^{3/2} + \cdots \right\}, \tag{109}$$

It follows from (109) that the last equation of (107) has a solution of the form

$$x = x_1\, \mu^{1/2} + x_3\, \mu^{3/2} + \cdots, \tag{110}$$

where the x_{2i+1} are periodic with the period 2π instead of π, and where $x_{2i+1}(0) = 0$. It is not possible to determine completely the constants of integration from these conditions, for if (110) is a solution, then, since the last equation of (107) is linear, (110) multiplied by any power series in μ having constant coefficients is also a solution. For example, we have for the determination of x_1 and x_3

$$\ddot{x}_1 + x_1 = 0, \qquad \ddot{x}_3 + x_3 = -\frac{3}{2}(1 + 2\cos 2\tau)\, x_1,$$

the solutions of which, satisfying the conditions $x_1(0) = x_3(0) = 0$, are

$$x_1 = c_1 \sin \tau, \qquad x_3 = c_3 \sin \tau + \frac{3}{16}\, c_1 \sin 3\tau,$$

where c_1 and c_3 are undetermined. This indeterminateness continues as far as the solution is carried, unless additional conditions are imposed.

The value of $\dot{x}$ at $\tau=0$ is an infinite series in $\sqrt{\mu}$ whose general term is not easily obtained; but, from the fact that $z(\pi/2)=a_2=-l+2l\mu$ and from equation (108), we get

$$x\left(\frac{\pi}{2}\right)=\pm 2l\sqrt{\mu}\sqrt{1-\mu}=\pm 2l\sqrt{\mu}\left\{1-\frac{1}{2}\mu-\frac{1}{8}\mu^2\cdots\right\}. \tag{111}$$

On determining c_1 and c_3 by these conditions, we find

$$x=\pm l\left\{[2\sin\tau]\mu^{1/2}+\frac{1}{8}[-5\sin\tau+3\sin 3\tau]\mu^{3/2}+\cdots\right\},$$

agreeing with the direct computation (109).

We may also consider the last equation of (107) from the standpoint of the general theory of linear differential equations having periodic coefficients. From the fact that the part of the coefficient of x which is independent of μ is unity, it follows that the solution of this problem belongs to Case II. Since there is one solution which is periodic with the period 2π, the values of a are independent of μ and are simply $\pm\sqrt{-1}$. We have here the case in which the two values of a not only differ by an imaginary integer for $\mu=0$, but for all values of μ. It follows from §21 that in this case the second solution is either τ times a periodic function or, for special values of the coefficients of the differential equation, a periodic function. In the problem of the simple pendulum the second solution is τ times a periodic function, and is most simply found by integrating the last equation of (107) with the initial conditions

$$x(0)=1, \qquad \dot{x}(0)=0.$$

If we make the transformation $x=e^{a\tau}\xi$, then the last equation of (107) becomes

$$\left.\begin{aligned}\ddot{\xi}+2a\dot{\xi}+a^2\xi+\Big[1&+\frac{3}{2}(1+2\cos 2\tau)\mu\\&+\frac{3}{32}(5+16\cos 2\tau+4\cos 4\tau)\mu^2+\cdots\Big]\xi=0.\end{aligned}\right\} \tag{112}$$

We shall integrate this equation and determine a so that ξ shall be periodic with the period 2π. The chief point of interest will be that a will be independent of μ so far as the work is carried.

The equations corresponding to (85) and (86) are

$$\left.\begin{aligned}\xi_0&=b_1^{(0)}+\frac{\sqrt{-1}}{2}e^{\sqrt{-1}\tau},\\ \ddot{\xi}_1+2\sqrt{-1}\dot{\xi}_1&=-a_1^{(1)}[2\sqrt{-1}b_1^{(0)}+e^{-2\sqrt{-1}\tau}]\\&\quad-\frac{3}{4}(1+2\cos 2\tau)(2b_1^{(0)}+\sqrt{-1}e^{-2\sqrt{-1}\tau}).\end{aligned}\right\} \tag{113}$$

The conditions that the solution of the second of these equations shall be periodic are

$$-2\sqrt{-1}\,\alpha_1^{(1)}b_1^{(0)}-\frac{3}{2}b_1^{(0)}-\frac{3}{4}\sqrt{-1}=0, \qquad -\alpha_1^{(1)}-\frac{3}{2}b_1^{(0)}-\frac{3}{4}\sqrt{-1}=0, \tag{114}$$

of which the solutions are

$$\alpha_1^{(1)}=0, \qquad b_1^{(0)}=-\frac{1}{2}\sqrt{-1}. \tag{115}$$

Hence, we see by direct computation from the differential equations that in this problem α is independent of μ up to μ^2 at least.

58. Application of the Integral Relations.—We now return to the consideration of the problem of the spherical pendulum. Since z and x have been determined the value of y can be found from the first equation of (4). But it will be noticed that in this work no explicit use has been made of the integral (2). Now x, y, and z must satisfy the differential equations, given in the last three equations of (4), and the integral relations

$$\left.\begin{aligned} &x^2+y^2+z^2=l^2, && \dot{x}^2+\dot{y}^2+\dot{z}^2=\frac{4l^2(1+\delta)}{\alpha_1-\alpha_3}(-z+\alpha_1+\alpha_2+\alpha_3),\\ &x\dot{x}+y\dot{y}+z\dot{z}=0, && x\dot{y}-y\dot{x}=c_3, \end{aligned}\right\} \tag{116}$$

where the second equation is determined from (2) by changing the independent variable from t to τ; the third equation expresses the fact that the motion must be along the surface of the sphere; and the fourth equation is obtained from the second and third equations of (4), and expresses the fact that the projection of the areal velocity on the xy-plane is constant.

The solutions of the second and third equations of (4) have been shown to have the form

$$x=a_1\,e^{\alpha\tau}\xi_1+a_2\,e^{-\alpha\tau}\xi_2, \qquad y=b_1\,e^{\alpha\tau}\xi_1+b_2\,e^{-\alpha\tau}\xi_2, \tag{117}$$

where a_1, a_2, b_1, and b_2 are arbitrary constants, ξ_1 and ξ_2 are power series in μ and are periodic with the period π, and α is a pure imaginary which is also a power series in μ, but is not an integer. Moreover, ξ_1 and ξ_2 are conjugate complex quantities.

If we make use of (117), the first equation of (116) becomes

$$(a_1^2+b_1^2)\xi_1^2\,e^{2\alpha\tau}+(a_2^2+b_2^2)\xi_2^2\,e^{-2\alpha\tau}+2(a_1\,a_2+b_1\,b_2)\,\xi_1\xi_2=l^2-z^2. \tag{118}$$

Now it has been shown that z^2 is expansible as a power series in μ and that it is periodic with the periodic π.

Before proceeding further we shall prove a lemma. Suppose there is given

$$F(t)=\sum_{j=1}^{n}a_je^{\sigma_jt}\varphi_j(t)\equiv 0.$$

Suppose the $\varphi_j(t)$ are not identically zero, that they are periodic with the period 2π, and that no two of the σ_j are equal or differ by an imaginary integer.

Then let $e^{2\sigma_j\pi}=k_j$. Suppose for $t=t_1$ that $\varphi_1(t_1), \ldots, \varphi_{n_1}(t_1)$ are distinct from zero. Then we have

$$F(t_1+0\pi)=\sum_{j=1}^{n_1} a_j e^{\sigma_j t_1}\varphi_j(t_1)k_j^0 \quad =0,$$

$$F(t_1+2\pi)=\sum_{j=1}^{n_1} a_j e^{\sigma_j t_1}\varphi_j(t_1)k_j \quad =0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F[t_1+(n_1-1)2\pi]=\sum_{j=1}^{n_1} a_j e^{\sigma_j t_1}\varphi_j(t_1)k^{n_1-1}=0.$$

It follows from these equations that either

$$a_1=a_2=\cdots=a_{n_1}=0$$

or

$$\begin{vmatrix} 1, & 1, & \ldots, & 1 \\ k_1, & k_2, & \ldots, & k_{n_1} \\ \cdots & \cdots & \cdots & \cdots \\ k_1^{n_1-1}, & k_2^{n_1-1}, & \ldots, & k_{n_1}^{n_1-1} \end{vmatrix}=0.$$

Under the hypotheses on the σ_j this determinant is distinct from zero. Therefore $a_1=\cdots=a_{n_1}=0$.

On taking another t for which some of the remaining φ_j do not vanish, we prove similarly that their coefficients are zero. Continuing thus we reach the conclusion that

$$a_j=0 \qquad (j=1, \ldots, n).$$

Upon applying these results to (118), we get the relations

$$a_1^2+b_1^2=0, \qquad a_2^2+b_2^2=0.$$

Therefore

$$b_1=\pm\sqrt{-1}\,a_1, \qquad b_2=\pm\sqrt{-1}\,a_2, \tag{119}$$

from which we get either

$$a_1 a_2+b_1 b_2=0 \quad \text{or} \quad a_1 a_2+b_1 b_2=2a_1 a_2,$$

according as the same sign or the opposite signs are used in front of $\sqrt{-1}$ in (119). It follows from (118) that in the former we have the trivial case $z=\pm l$. Hence we shall take (119) with opposite signs.

The constants c_1 and c_2 of equations (7), upon which μ depends, arise in energy integrals and are independent of the orientation of the x and y-axes. Consequently we may take the axis so that $y(0)=0$ without affecting the work up to this point, and from this it follows that

$$b_1=-b_2=\sqrt{-1}\,a_1=\sqrt{-1}\,a_2. \tag{120}$$

As a consequence of (119) and (120), equation (118) becomes

$$4a_1^2\xi_1\xi_2=l^2-z^2. \tag{121}$$

Let us suppose that z has been computed and that we wish the coefficients of ξ_1 and ξ_2. These quantities have the form

$$
\left.\begin{aligned}
\xi_1 = 1 &+ [a_0^{(1)} + a_2^{(1)} \cos 2\tau + \sqrt{-1}\, b_2^{(1)} \sin 2\tau]\,\mu + \cdots \\
&+ [a_0^{(j)} + a_2^{(j)} \cos 2\tau + \cdots + a_{2j}^{(j)} \cos 2j\tau \\
&+ \sqrt{-1}\, b_2^{(j)} \sin 2\tau + \cdots + \sqrt{-1}\, b_{2j}^{(j)} \sin 2j\tau]\,\mu^j + \cdots, \\
\xi_2 = 1 &+ [a_0^{(1)} + a_2^{(1)} \cos 2\tau - \sqrt{-1}\, b_2^{(1)} \sin 2\tau]\,\mu + \cdots \\
&+ [a_0^{(j)} + a_2^{(j)} \cos 2\tau + \cdots + a_{2j}^{(j)} \cos 2j\tau \\
&- \sqrt{-1}\, b_2^{(j)} \sin 2\tau - \cdots - \sqrt{-1}\, b_{2j}^{(j)} \sin 2j\tau]\,\mu^j + \cdots, \\
z = \alpha_3 &+ [c_0^{(1)} + c_2^{(1)} \cos 2\tau]\,\mu + \cdots \\
&+ [c_0^{(j)} + c_2^{(j)} \cos 2\tau + \cdots + c_{2j}^{(j)} \cos 2j\tau]\,\mu^j + \cdots,
\end{aligned}\right\} \qquad (122)
$$

where the $a_{2i}^{(j)}$ and $c_{2i}^{(j)}$ satisfy the relations

$$
\left.\begin{aligned}
&a_0^{(j)} + a_2^{(j)} + \cdots + a_{2j}^{(j)} = 0, \\
&c_0^{(j)} + c_2^{(j)} + \cdots + c_{2j}^{(j)} = 0 \qquad (j = 1, \ldots \infty).
\end{aligned}\right\} \qquad (123)
$$

The constant coefficients of these solutions, as determined from the differential equations, are expressed in terms of α_1, α_3, and μ, and therefore we must express (116) in terms of the same parameters in order that it may be possible to compare these results with those obtained from the integrals. We find from (5), (7), and the first of (10) that

$$
l^2 = -\alpha_3 (2\alpha_1 + \alpha_3) - (\alpha_1^2 - \alpha_3^2)\,\mu. \qquad (124)
$$

While the constant a_1 is arbitrary in the solution of the differential equations, it must be subjected to the condition that the pendulum shall move on the sphere whose radius is l. This condition may very well make it a power series in μ; hence it must be expressible in the form

$$
a_1 = a_1^{(0)} + a_1^{(1)} \mu + a_1^{(2)} \mu^2 + \cdots.
$$

In fact, we find from (121) at $\tau = 0$, upon making use of (50) and (124), that

$$
a_1 = \sum_{j=0}^{\infty} a_1^{(j)} \mu^j = \pm \tfrac{1}{2} \sqrt{-2\alpha_3(\alpha_1 + \alpha_3)} \left[1 + \frac{\alpha_1 - \alpha_3}{4\alpha_3}\mu + \cdots\right]. \qquad (125)
$$

If we substitute (122), (124), and (125) in (121), we get results of the form

$$
F_0 + F_1 \mu + F_2 \mu^2 + \cdots = G_0 + G_1 \mu + G_2 \mu^2 + \cdots. \qquad (126)
$$

Since this equation is an identity in μ, we have

$$
F_j = G_j \qquad (j = 0, \ldots \infty). \qquad (127)
$$

By hypothesis z has been determined; therefore the G_j are fully known. It follows from (122) that the F_j and the G_j have the form

$$
\left.\begin{aligned}
F_j &= A_0^{(j)} + A_2^{(j)} \cos 2\tau + \cdots + A_{2j}^{(j)} \cos 2j\tau, \\
G_j &= B_0^{(j)} + B_2^{(j)} \cos 2\tau + \cdots + B_{2j}^{(j)} \cos 2j\tau,
\end{aligned}\right\} \qquad (128)
$$

where the $B_0^{(j)}, \ldots, B_{2j}^{(j)}$ are known constants. Since equations (127) are identities in τ, we have

$$A_{2i}^{(j)} = B_{2i}^{(j)} \qquad (i=0, \ldots, j;\ j=0, \ldots \infty). \qquad (129)$$

On substituting (122) in the second of (116), we get from this integral

$$[a_1^2+b_1^2][\alpha\xi_1+\dot{\xi}]^2 e^{2\alpha\tau}+[a_2^2+b_2^2][\alpha\xi_2-\dot{\xi}]^2 e^{-2\alpha\tau}-2(a_1a_2+b_1b_2)[\alpha^2\xi_1\xi_2$$
$$+\alpha(\dot{\xi}_1\xi_2-\xi_1\dot{\xi}_2)-\dot{\xi}_1\dot{\xi}_2] = -\dot{z}^2+\frac{4l^2(1+\delta)}{\alpha_1-\alpha_3}[-z+\alpha_1+\alpha_2+\alpha_3].$$

When we reduce this equation by (120) and (121), we obtain

$$4a_1^2[\alpha(\xi_1\dot{\xi}_2-\dot{\xi}_1\xi_2)+\dot{\xi}_1\dot{\xi}_2] = \alpha^2(l^2-z^2)-\dot{z}^2+\frac{4l^2(1+\delta)}{\alpha_1-\alpha_3}[-z+\alpha_1+\alpha_2+\alpha_2]. \quad (130)$$

Now, on substituting (122) and the series for z in this equation, we get an expression of the form

$$H_0+H_1\mu+H_2\mu^2+\cdots = K_0+K_1\mu+K_2\mu^2+\cdots.$$

From the fact that this expression is an identity in μ, it follows that

$$H_j=K_j \qquad (j=0, \ldots \infty). \qquad (131)$$

The K_j are known except for the expansions of α. The constant K_0 involves $(\alpha^{(0)})^2$, and the $K_j\ (j=1, \ldots \infty)$ involve the $\alpha^{(j)}$ linearly.

On referring to (122), we see that the H_j and the K_j have the form

$$\left.\begin{aligned} H_j &= C_0^{(j)}+C_2^{(j)}\cos 2\tau+\cdots+C_{2j}^{(j)}\cos 2j\tau, \\ K_j &= D_0^{(j)}+D_2^{(j)}\cos 2\tau+\cdots+D_{2j}^{(j)}\cos 2j\tau. \end{aligned}\right\} \qquad (132)$$

Since (131) are identities in τ, it follows that

$$C_{2i}^{(j)} = D_{2i}^{(j)} \qquad (i=0, \ldots, j;\ j=0, \ldots \infty). \qquad (133)$$

It will now be shown that equations (123), (129), and (133) determine uniquely the $a_{2i}^{(j)}, b_{2i}^{(j)}, a_1^{(j)}, \alpha^{(j)}\ (j>0)$, in the order of increasing values of j when z and δ are known. To do this it is necessary to develop the explicit forms of (129) and (133) by reference to equations (121) and (130). It is necessary to eliminate α_2 and l^2 from their right members by equations (124) and the first of (10). When $j=0$, we get from (121) and (130)

$$\left.\begin{aligned} 4(a_1^{(0)})^2 &= -2\alpha_3(\alpha_1+\alpha_3), \\ 0 &= -(\alpha^{(0)})^2[2\alpha_3(\alpha_1+\alpha_3)]-\frac{4\alpha_3(2\alpha_1+\alpha_3)(\alpha_1+\alpha_3)}{\alpha_1-\alpha_3}. \end{aligned}\right\} \qquad (134)$$

The first of these equations determines $a_1^{(0)}$ except as to sign. The sign of $a_1^{(0)}$ depends upon which is taken as the positive end of the x-axis. The second equation gives

$$(\alpha^{(0)})^2 = -a^2 = -\frac{2(2\alpha_1+\alpha_3)}{\alpha_1-\alpha_3}, \qquad (135)$$

agreeing with the result in (101).

When $j=1$, we find from (121) and (130) that

$$\left.\begin{aligned}
A_0^{(1)} &= 8(a_1^{(0)})^2 a_0^{(1)} + 8a_1^{(0)} a_1^{(1)} = -(\alpha_1^2-\alpha_3^2) - 2\alpha_3 c_0^{(1)} = B_0^{(1)},\\
A_2^{(1)} &= 8(a_1^{(0)})^2 a_2^{(1)} + 0 \qquad = \qquad 0 - 2\alpha_3 c_2^{(1)} = B_2^{(1)},\\
C_0^{(1)} &= 0 = -4\alpha^{(0)}\alpha_3(\alpha_1+\alpha_3)\alpha^{(1)} - (\alpha^{(0)})^2(\alpha_1^2-\alpha_3^2) - 2(\alpha^{(0)})^2\alpha_3 c_0^{(1)}\\
&\qquad + 4[\alpha_1^2 - 2(\alpha_1+\alpha_3)^2] + \frac{4\alpha_3(2\alpha_1+\alpha_3)}{\alpha_1-\alpha_3}[c_0^{(1)} - (\alpha_1+\alpha_3)\delta_1] \quad = D_0^{(1)},\\
C_2^{(1)} &= -16(a_1^{(0)})^2\alpha^{(0)}\sqrt{-1}\, b_2^{(1)} = -2(\alpha^{(0)})^2\alpha_3 c_2^{(1)} + \frac{4\alpha_3(2\alpha_1+\alpha_3)}{\alpha_1-\alpha_3} c_2^{(1)} = D_2^{(1)},
\end{aligned}\right\} \quad (136)$$

to which we must add the first equation of (123) for $j=1$. The unknowns in these five equations are $a_0^{(1)}$, $a_1^{(1)}$, $a_2^{(1)}$, $\alpha^{(1)}$, and $b_2^{(1)}$, which enter linearly. The second equation determines $a_2^{(1)}$; then $a_0^{(1)}$ is found from the first of (123); then the first of (136) defines $a_1^{(1)}$, while $\alpha^{(1)}$ and $b_2^{(1)}$ are given by the last two equations of (136).

We shall apply (136) in computing the first terms of the solutions. We find from (49) and (50) that

$$\delta_1 = \frac{1}{2}, \qquad c_0^{(1)} = -c_2^{(1)} = \frac{1}{2}(\alpha_1 - \alpha_3). \quad (137)$$

Upon substituting in (136) and solving these equations and the first of (123) for $a_2^{(1)}$, $a_0^{(1)}$, $a_1^{(1)}$, $\alpha^{(1)}$, and $b_2^{(1)}$ in order, we get

$$\left.\begin{aligned}
a_2^{(1)} &= -a_0^{(1)} = \frac{-(\alpha_1-\alpha_3)}{4(\alpha_1+\alpha_3)}, & a_1^{(1)} &= \frac{-(\alpha_1-\alpha_3)\sqrt{\alpha_1+\alpha_3}}{4\sqrt{-2\alpha_3}},\\
\alpha^{(1)} &= \frac{+3\sqrt{2}\,\alpha_1\sqrt{-1}}{4\sqrt{(\alpha_1-\alpha_3)(2\alpha_1+\alpha_3)}}, & b_2^{(1)} &= \frac{\sqrt{2}(\alpha_1-\alpha_3)(2\alpha_1+\alpha_3)}{4(\alpha_1+\alpha_3)},
\end{aligned}\right\} \quad (138)$$

agreeing exactly with the results obtained in (102).

For a general value of j the equations corresponding to (136) are

$$\left.\begin{aligned}
A_0^{(j)} &= 8(a_1^{(0)})^2 a_0^{(j)} + 8a_1^{(0)} a_1^{(j)} + \overline{A}_0^{(j)} = B_0^{(j)},\\
A_{2i}^{(j)} &= 8(a_1^{(0)})^2 a_{2i}^{(j)} + 0 \qquad + \overline{A}_{2i}^{(j)} = B_{2i}^{(j)} \qquad (i=1, \ldots, j),\\
C_0^{(j)} &= \overline{C}_0^{(j)} = -4\alpha^{(0)}\alpha_3(\alpha_1+\alpha_3)\alpha^{(j)} + \overline{D}_0^{(j)},\\
C_{2i}^{(j)} &= -16(a_1^{(0)})^2\alpha^{(0)}\sqrt{-1}\, i\, b_{2i}^{(j)} + \overline{C}_{2i}^{(j)} = D_{2i}^{(j)} \qquad (i=1, \ldots, j),\\
0 &= a_0^{(j)} + a_2^{(j)} + \cdots + a_{2j}^{(j)}.
\end{aligned}\right\} \quad (139)$$

The unknowns in these equations, after $a_{2i}^{(k)}$, $a_1^{(k)}$, $\alpha_1^{(k)}$, $b_{2i}^{(k)}$ $(k=0, \ldots, j-1)$ have been determined, are $a_0^{(j)}$, $a_1^{(j)}$, $a_{2i}^{(j)}$, $\alpha^{(j)}$, $b_{2i}^{(j)}$ $(i=1, \ldots, j)$. The $\overline{A}_{2i}^{(j)}$, $\overline{C}_{2i}^{(j)}$, and $\overline{D}_{2i}^{(j)}$ $(i=1, \ldots, j)$ are known quantities depending upon the coefficients having smaller numbers for the superscripts. The $j+2$ equations of the first two and the last lines define uniquely the $j+2$ quantities $a_1^{(j)}$ and $a_{2i}^{(j)}$ $(i=0, \ldots, j)$. The equation of the third line determines $\alpha^{(j)}$, and the equations of the fourth line, the coefficients $b_{2i}^{(j)}$ $(i=1, \ldots, j)$.

Therefore we have the interesting result that in this problem the coefficients of the general solution can all be determined from the integral relations alone, the solution of the z-equation having been previously obtained from another integral in § 49.

The last two equations of (116) are unused integrals. Let us consider the last equation, which is the more complicated. By means of (117), we get

$$4\, a_1^2\, \alpha\, \sqrt{-1}\, \xi_1\, \xi_2 - 2\, a_1^2\, \sqrt{-1}\, (\xi_1\, \dot{\xi}_2 - \dot{\xi}_1\, \xi_2) = c_3\,.$$

Upon reducing by (121), this equation becomes

$$2\, a_1^2\, (\xi_1\, \dot{\xi}_2 - \dot{\xi}_1\, \xi_2) = \alpha\, (l^2 - z^2) + \sqrt{-1}\, c_3\,. \tag{140}$$

The constant c_3 will be a power series in μ having constant coefficients. Hence, on expanding (140) as a power series in μ, we have

$$L_1\, \mu + L_2\, \mu^2 + \cdots = M_0 + M_1\, \mu + M_2\, \mu^2 + \cdots\,, \tag{141}$$

where the M_i involve linearly in the terms independent of τ the unknown coefficients of the expansion of c_3. Since this equation is an identity in μ, we have

$$L_j = M_j \qquad (j=0, \ldots \infty).$$

It follows from (122) and (140) that

$$L_j = E_0^{(j)} + E_2^{(j)} \cos 2\tau + \cdots + E_{2j}^{(j)} \cos 2j\tau,$$
$$M_j = F_0^{(j)} + F_2^{(j)} \cos 2\tau + \cdots + F_{2j}^{(j)} \cos 2j\tau,$$

from which it follows that

$$E_{2i}^{(j)} = F_{2i}^{(j)} \qquad (i=0, \ldots, j;\ j=0, \ldots \infty). \tag{142}$$

The $E_{2i}^{(j)}$ and the $F_{2i}^{(j)}$ $(i=1, \ldots, j)$ are known functions of the coefficients already computed, while the $F_0^{(j)}$ involves the unknown coefficient of μ^j in the expansion of c_3. Consequently equations (142) determine this constant for $i=0$, and also furnish a check on the earlier computation of the coefficients for $i=1, \ldots, j$.

CHAPTER IV.

PERIODIC ORBITS ABOUT AN OBLATE SPHEROID.

BY WILLIAM DUNCAN MACMILLAN.

59. Introduction.—The orbit of a particle about an oblate spheroid is not, in general, closed geometrically. The motion of the particle is not, therefore, in general, periodic from a geometric point of view. But if we consider the orbit as described by the particle in a revolving meridian plane which passes constantly through the particle, several classes of closed orbits can be found in which the motion is periodic. The failure of these orbits to close in space arises from the incommensurability of the period of rotation of the line of nodes with the period of motion in the revolving plane. When these periods happen to be commensurable the orbits are closed in space and the motion is therefore periodic, though the period may be very great. Indeed, it seems that much of the difficulty in giving mathematical expressions to the orbits about an oblate spheroid rests upon the incommensurability of periods. The difficulty arising from the node can be overcome in the manner just described, but elsewhere it is more troublesome.

Orbits closed in the revolving plane are considered most conveniently in two general classes: I, Those which re-enter after one revolution; II, those which re-enter after many revolutions. The existence of both classes is established in this chapter and convenient methods for constructing the solutions are given. Orbits which re-enter after the first revolution are naturally the simpler and will be considered in the first part of the chapter. Those lying in the equatorial plane of the spheroid become straight lines in the revolving plane, and within the realm of convergence of the series employed all orbits in the equatorial plane are periodic. When the motion is not in the equatorial plane there exists one, and only one, orbit for assigned values of the inclination and the mean distance. These orbits reduce to circles with the vanishing of the oblateness of the spheroid.

In considering orbits which re-enter only after many revolutions the differential equations are found to be very complex, and one would despair of ever finding any of these orbits by direct computation. However, a proof of their existence and a method for the constructions of the solutions are given by the aid of theorems on the character of the solutions of non-homogeneous linear differential equations with periodic coefficients.

These periodic orbits of many revolutions involve five *arbitrary* constants. One, only, is lacking for a complete integration of the differential equations. The orbits are all symmetric with respect to the equatorial plane.

60. The Differential Equations.—The differential equations of motion of a particle about an oblate spheroid are*

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=-\frac{k^2Mx}{R^3}\left[1+\tfrac{3}{10}b^2\mu^2\,\frac{x^2+y^2-4z^2}{R^4}+\cdots\right]=\frac{\partial V}{\partial x},\\ \frac{d^2y}{dt^2}&=-\frac{k^2My}{R^3}\left[1+\tfrac{3}{10}b^2\mu^2\,\frac{x^2+y^2-4z^2}{R^4}+\cdots\right]=\frac{\partial V}{\partial y},\\ \frac{d^2z}{dt^2}&=-\frac{k^2Mz}{R^3}\left[1+\tfrac{3}{10}b^2\mu^2\,\frac{3(x^2+y^2)-2z^2}{R^4}+\cdots\right]=\frac{\partial V}{\partial z}.\end{aligned}\right\}\qquad(1)$$

The symbols employed are defined as follows:

The x, y, z are rectangular coördinates, the origin being at the center of the spheroid and the xy-plane being the plane of the equator,
k is the Gaussian constant, b is the polar radius of the spheroid,
M is the mass of the spheroid, μ is the eccentricity of the spheroid,

$$R=\sqrt{x^2+y^2+z^2},\qquad V=\frac{k^2M}{R}\left[1+\frac{b^2\mu^2}{10}\,\frac{x^2+y^2-2z^2}{R^4}+\cdots\right].$$

Since $\dfrac{1}{x}\dfrac{\partial V}{\partial x}=\dfrac{1}{y}\dfrac{\partial V}{\partial y}$, we obtain one integral of areas, namely

$$x\frac{dy}{dt}-y\frac{dx}{dt}=c_1.\qquad(2)$$

That is, the projection of the area described by the radius vector upon the equatorial plane is proportional to the time. We have also the vis viva integral

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=2V+c_2.\qquad(3)$$

There are no other integrals which can be expressed in a finite number of terms, and for further integration we are compelled to resort to the use of infinite series.

It will be advantageous to transform the differential equations to cylindrical coördinates by the substitutions

$$\left.\begin{aligned}x&=ar\cos v, & y&=ar\sin v, & z&=aq,\\ k^2M&=n^2a^3, & c_1&=ck\sqrt{Ma}, & nt&=\tau,\\ R&=a\sqrt{r^2+q^2}, & \tfrac{3}{10}\frac{b^2}{a^2}&=\theta_1^2. & &\end{aligned}\right\}\qquad(4)$$

*Moulton's *Introduction to Celestial Mechanics*, p. 113.

After these substitutions equations (1) become

$$\left.\begin{array}{lrl}(a) & r''-r(v')^2 = & \dfrac{-r}{(r^2+q^2)^{\frac{3}{2}}} - \dfrac{r^3-4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\,\theta_1^2\,\mu^2+\cdots, \\ (b) & rv''+2r'v' = & 0, \\ (c) & q'' = & \dfrac{-q}{(r^2+q^2)^{\frac{3}{2}}} - \dfrac{3r^2q-2q^3}{(r^2+q^2)^{\frac{7}{2}}}\,\theta_1^2\,\mu^2+\cdots,\end{array}\right\} \qquad (5)$$

where the accents denote derivatives with respect to τ.

The integral of (b) is $r^2v'=c$, by means of which v' can be eliminated from equation (a), and the equations then take the form

$$\left.\begin{array}{ll}(a) & r''=\dfrac{c^2}{r^3} - \dfrac{r}{(r^2+q^2)^{\frac{3}{2}}} - \dfrac{r^3-4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\,\theta_1^2\,\mu^2+\cdots, \\ (b) & q''= \quad - \dfrac{q}{(r^2+q^2)^{\frac{3}{2}}} - \dfrac{3r^2q-2q^3}{(r^2+q^2)^{\frac{7}{2}}}\,\theta_1^2\,\mu^2+\cdots, \\ (c) & v'=\dfrac{c}{r^2}.\end{array}\right\} \qquad (6)$$

The first two of these equations are independent of the third, so that r and q may be considered as being rectangular coördinates in a revolving plane which passes always through the polar axis of the spheroid and through the particle itself. The problem is thus reduced to the consideration of the motion in this plane, for, when r is known, v is obtained from the last equation by a simple quadrature.

61. Surfaces of Zero Velocity.—The velocity integral in the revolving plane is

$$r'^2+q'^2=\frac{2}{(r^2+q^2)^{\frac{1}{2}}}+\frac{2}{3}\,\frac{r^2-2q^2}{(r^2+q^2)^{\frac{5}{2}}}\,\theta_1^2\,\mu^2+\cdots-\frac{c^2}{r^2}+c_2. \qquad (7)$$

If we put the velocity equal to zero, the resulting equation represents a two-parameter family of curves. For assigned values of the parameters c and c_2, there is defined a curve in the revolving plane. On one side of this curve the motion is real and on the other side it is imaginary. For values of $c_2<0$, this curve is closed and the motion is real on the inside. As the plane revolves this curve generates a surface of the general form of an anchor ring.

For $\mu^2=0$, this curve belongs to the ordinary two-body problem and the motion is elliptic, parabolic, or hyperbolic according as c_2 is negative, zero, or positive. Its equation is

$$\frac{2}{(r^2+q^2)^{\frac{1}{2}}}-\frac{c^2}{r^2}+c_2=0.$$

On putting

$$r=\rho\cos\varphi, \qquad q=\rho\sin\varphi,$$

we find, by solving for ρ, that

$$\rho=\frac{1}{c_2}\left[-1\pm\sqrt{1+\frac{c^2c_2}{\cos^2\varphi}}\right].$$

For negative values of c_2 this equation represents two closed ovals which do not inclose the origin. If $c^2 c_2 = -1$ the oval shrinks upon the points $\rho = -1/c_2$, $\varphi = 0$ and π. The corresponding orbit is therefore a circle in the equatorial plane. As c_2 approaches zero the ovals open out rapidly and approach the limiting curves

$$\rho = \frac{c^2}{2\cos^2\varphi}.$$

For values of $c_2 > 0$, there is but one positive value for ρ, which is

$$\rho = \frac{1}{c_2}\left[-1 + \sqrt{1 + \frac{c^2 c_2}{\cos^2\varphi}}\right].$$

If $c^2 \neq 0$, none of these curves cross the axis $\varphi = \pi/2$. But if $c^2 = 0$, we have the circle $\rho = -2/c_2$ inside of which the motion is real when c_2 is negative.

For values of $\mu^2 \neq 0$, but sufficiently small, we can put

$$r = (\rho + \bar{\rho})\cos\varphi, \qquad q = (\rho + \bar{\rho})\sin\varphi,$$

and solve for $\bar{\rho}$ as a power series in μ^2. We find in this manner

$$\bar{\rho} = \frac{1}{3}\,\frac{2 - 3\cos^2\varphi}{\rho(1 + c_2\rho)}\,\theta_1^2\,\mu^2 + \cdots,$$

which is the correction to be applied to the corresponding surface in the two-body problem.

I. ORBITS RE-ENTRANT AFTER ONE REVOLUTION.

62. Symmetry.—On returning to the differential equations (*a*) and (*b*) of (6), we observe that if we change

$$r \text{ into} + r, \qquad q \text{ into} - q, \qquad \tau \text{ into} - \tau,$$

the differential equations remain unchanged. Hence, if at some epoch $\tau = \tau_0$

$$r = \alpha, \qquad r' = 0, \qquad q = 0, \qquad q' = \beta,$$

that is, if at the epoch $\tau = \tau_0$, the particle crosses the r-axis perpendicularly, it follows from the form of the differential equations that the orbit is symmetrical with respect to the r-axis and with respect to the epoch $\tau = \tau_0$. In other words, r is an even function of $\tau - \tau_0$, and q is an odd function of $\tau - \tau_0$. If now at some other epoch, $\tau = \tau_0 + T$, the particle again crosses the r-axis perpendicularly, the orbit is symmetrical with respect to this epoch also. It is clear, therefore, that the orbit is a closed one, and that the motion in it is periodic, for, at $\tau = \tau_0 + T$ and at $\tau = \tau_0 - T$ it must have been at the same point and moving with the same velocity in the same direction. Hence sufficient conditions for periodicity, with the period $2T$, are

$$r'(\tau_0) = q(\tau_0) = 0, \qquad r'(\tau_0 + T) = q(\tau_0 + T) = 0.$$

From the areas integral, $v' = c/r^2$, it follows that if r is periodic v will have the form $v = A(\tau - \tau_0) +$ periodic terms, where A is a constant.

63. Existence of Periodic Orbits in the Equatorial Plane.—In the case where $q=0$ equations (6) reduce to

$$\left.\begin{aligned} &(a) & r'' &= \frac{c^2}{r^3}-\frac{1}{r^2}-\frac{\theta_1^2\mu^2}{r^4}-\frac{\theta_2^2\mu^4}{r^6}+\cdots, \\ &(b) & v' &= \frac{c}{r^2}. \end{aligned}\right\} \quad (8)$$

The first of these equations is independent of the second and can be integrated separately. It represents motion in a straight line in the revolving plane. It admits the constant solution

$$r=r_0=1, \qquad c^2=c_0^2=1+\theta_1^2\mu^2+\theta_2^2\mu^4+\cdots,$$

which represents a point in the revolving plane, or a circle in the equatorial plane.

In order to investigate the oscillations about this point let us put

$$r=1+\rho e, \qquad c^2=c_0^2+\epsilon e,$$

where ρ is a variable whose initial value can be arbitrarily assigned, e is a parameter corresponding to the eccentricity in the two-body problem, and ϵ is a parameter to be determined so that ρ shall be periodic.

On substituting these values in (8a) and expanding as power series in e, the terms independent of e cancel out, and it is possible to divide through by e. The equation becomes

$$\left.\begin{aligned} \rho''+[1-\theta_1^2\mu^2-3\theta_2^2\mu^4+\cdots]\rho = &+[1-3\rho e+6\rho^2e^2-10\rho^3e^3+\cdots]\epsilon \\ &+[3-6\theta_1^2\mu^2-15\theta_2^2\mu^4+\cdots]\rho^2 e \\ &+[-6+10\theta_1^2\mu^2+46\theta_2^2\mu^4+\cdots]\rho^3e^2 \\ &+[+10-20\theta_1^2\mu^2-111\theta_2^2\mu^4+\cdots]\rho^4e^3 \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (9)$$

We can simplify this equation somewhat by dividing through by the coefficient of ρ in the left member and then substituting

$$\mathsf{T}=\tau\sqrt{1-\theta_1^2\mu^2-3\theta_2^2\mu^4+\cdots}, \qquad \delta=\frac{\epsilon}{1-\theta_1^2\mu^2-3\theta_2^2\mu^4+\cdots}.$$

The equation then becomes

$$\left.\begin{aligned} \frac{d^2\rho}{d\mathsf{T}^2}+\rho = &[1-3\rho e+6\rho^2e^2-10\rho^3e^3+\cdots]\delta+[3+a_1]\rho^2e \\ &+[-6+a_2]\rho^3e^2+[10+a_3]\rho^4e^3+\cdots, \end{aligned}\right\} \quad (10)$$

where

$$\begin{aligned} a_1 &= -\ 3\theta_1^2\mu^2-(\ 3\theta_1^4+\ 6\theta_2^2)\mu^4+\cdots, \\ a_2 &= +\ 4\theta_1^2\mu^2+(\ 4\theta_1^4+28\theta_2^2)\mu^4+\cdots, \\ a_3 &= -10\theta_1^2\mu^2-(10\theta_1^4+81\theta_2^2)\mu^4+\cdots. \end{aligned}$$

Equation (10) can be integrated as a power series in δ and e with the initial values

$$\rho = -1, \qquad \rho' = 0.$$

By Poincaré's extension of Cauchy's theorem, §§ 14–16, this solution converges for values of δ and e sufficiently small, and for all values of T in the interval $0 \leqq \mathsf{T} \gtreqless T$, where T is finite, but otherwise arbitrary.

The condition for periodicity is simply

$$\rho' = 0 \text{ at } \mathsf{T} = T. \tag{11}$$

If we choose $T = \pi$, an inspection of equation (10) shows that for $e = 0$ the solution for ρ is periodic with the period 2π, whatever may be the value of δ. Consequently equation (11) must carry e as a factor. After integrating equation (10), we find that the condition (11) is, explicitly,

$$0 = -\left[\tfrac{3}{2} + a_1\right]\pi\delta e - \left[\tfrac{3}{2} + \tfrac{5}{2}a_1 + \tfrac{5}{12}a_1^2 + \tfrac{3}{8}a_2\right]\pi e^2 + \text{higher degree terms}. \tag{12}$$

Upon dividing out the factor e, there remains an equation in which the linear terms in δ and e are present, and this equation can be solved for δ as a power series in e. We find

$$\delta = \left[-1 + 2\theta_1^2\mu^2 + \left(\tfrac{3}{2}\theta_1^4 - \theta_2^2\right)\mu^4 + \cdots\right]e + \cdots . \tag{13}$$

If this value of δ be substituted in equation (10), it will then admit periodic solutions for ρ having the period 2π for all values of e sufficiently small. Furthermore the solution as a power series in e is unique.

64. Existence of Periodic Orbits which are Inclined to the Equatorial Plane.—For $\mu^2 = 0$ the differential equations (6) admit the circular solution

$$c^2 = 1, \qquad r = 1, \qquad v = \tau, \qquad q = 0. \tag{14}$$

In order to investigate the existence of orbits not lying in the equatorial plane, but having the period 2π for $\mu^2 \neq 0$, let us put

$$r = 1 + \rho, \qquad q = 0 + \sigma, \qquad c^2 = 1 + \epsilon, \tag{15}$$

and take the initial conditions

$$\rho = a, \qquad \rho' = 0, \qquad \sigma = 0, \qquad \sigma' = \beta\mu.$$

The conditions for periodicity are then

$$\rho' = \sigma = 0 \quad \text{at } \tau = \pi.$$

We have three arbitrary constants at our disposal, a, β, and ϵ, and two conditions to be satisfied. We will therefore let β remain arbitrary and determine a and ϵ so as to satisfy the two conditions.

After making the substitutions (15) and expanding, equations (6) become

$$\left.\begin{aligned}
&(a) \quad \rho''+\rho=\epsilon-3\rho\epsilon+3\rho^2+\frac{3}{2}\sigma^2-\theta_1^2\mu^2+6\rho^2\epsilon-6\rho^3-6\rho\sigma^2\\
&\qquad\qquad +4\rho\theta_1^2\mu^2+\text{ higher degree terms,}\\
&(b) \quad \sigma''+\sigma=3\rho\sigma-6\rho^2\sigma+\frac{3}{2}\sigma^3-\sigma\theta_1^2\mu^2+\text{ higher degree terms.}
\end{aligned}\right\} \tag{16}$$

In order to integrate these equations let us put

$$\rho=\sum_{i,j,k=0}^{\infty}\rho_{ijk}\,\epsilon^i\alpha^j\mu^k, \qquad \sigma=\sum_{i,j,k=0}^{\infty}\sigma_{ijk}\,\epsilon^i\alpha^j\mu^k. \tag{17}$$

The ρ_{ijk} and σ_{ijk} can be found by successive integrations, the constants of integration being determined so as to satisfy the initial conditions. In the series thus obtained put $\tau=\pi$. The two conditions for periodicity give the two equations

$$\left.\begin{aligned}
&(a) \quad \rho'(\pi)=0=a_1\epsilon^2+a_2\epsilon\alpha+a_3\epsilon^3+a_4\epsilon^2\alpha+a_5\epsilon\alpha^2+a_6\alpha^3+a_7\epsilon\mu^2\\
&\qquad\qquad +a_8\alpha\mu^2+a_9\mu^4+\cdots,\\
&(b) \quad \sigma(\pi)=0=\beta\mu\,[b_1\epsilon+b_2\epsilon^2+b_3\alpha^2+b_4\epsilon\alpha+b_5\mu^2+\cdots],
\end{aligned}\right\} \tag{18}$$

Equation (18a) involves only the even powers of μ, while (18b) involves only the odd powers. After dividing (18b) by $\beta\mu$, we can solve it for ϵ as a power series in α and μ^2 of the form

$$\epsilon=c_1\alpha^2+c_2\mu^2+c_3\alpha^3+c_4\alpha\mu^2+c_5\mu^4+\cdots. \tag{19}$$

On substituting (19) in (18a), we obtain a series of the form

$$(a) \qquad 0=d_1\alpha\mu^2+d_2\alpha^3+d_3\mu^4+d_4\alpha^2\mu^2+d_5\alpha^4+\cdots. \tag{20}$$

If in this equation we make the substitution

$$(b) \qquad \alpha=\left(\gamma-\frac{d_3}{d_1}\right)\mu^2,$$

we obtain

$$(c) \qquad 0=\mu^4\,[f_1\gamma+f_2\mu^2+f_3\gamma\mu^2+f_4\gamma^2\mu^2+\cdots],$$

which can be solved uniquely for γ as a power series in μ^2. This solution substituted in (20b) gives α as a power series in μ^2. This value of α substituted in (19) gives ϵ as a power series in μ^2. We thus have a solution

$$\alpha=\mu^2P_1(\mu^2), \qquad \epsilon=\mu^2P_2(\mu^2), \qquad \beta=\text{arbitrary},$$

where P_1 and P_2 are power series in μ^2. Newton's parallelogram shows that equation (20a) has two additional solutions, but as they are imaginary we we shall not develop them.

65. Existence of Orbits in a Meridian Plane.—If in equations (6) we put the area constant c equal to zero, the motion of the particle is in a meridian plane; that is, the plane has ceased to revolve, and the orbit in this plane is the true orbit. After changing to polar coördinates by the substitution

$$r=p\cos\varphi, \qquad q=p\sin\varphi,$$

the differential equations are

$$\left.\begin{aligned} p''-p(\varphi')^2+\frac{1}{p^2}&=-\frac{-\frac{3}{4}+\frac{3}{2}\cos 2\varphi+\frac{1}{4}\cos 4\varphi}{p^4}\,\theta_1^2\,\mu^2+\cdots, \\ p\,\varphi''+2p'\,\varphi'&=-\frac{\frac{1}{2}\sin 2\varphi-\frac{1}{4}\sin 4\varphi}{p^4}\,\theta_1^2\,\mu^2+\cdots. \end{aligned}\right\} \qquad (21)$$

For $\mu^2=0$, equations (21) have the periodic solution

$$p=1, \qquad \varphi=\tau;$$

that is, a circle. For $\mu^2\neq 0$, we will put

$$p=1+\rho, \qquad \varphi=\tau+\sigma,$$

with the initial values

$$\rho=\alpha, \quad \rho'=0, \quad \sigma=0, \quad \sigma'=\beta,$$

where α and β are two new arbitraries. By §§14–16, ρ, ρ', σ, and σ' are expansible as power series in α, β, and μ^2 with τ entering the coefficients. The conditions for periodicity are that at $\tau=\pi$

$$\rho'=\sigma=0.$$

If we integrate equations (21) and then put $\tau=\pi$, we obtain from the periodicity conditions two equations of the form

$$\left.\begin{aligned} &(a) \quad \sigma(\pi)=0=a_1\,\alpha+a_2\beta+a_3\,\alpha^2+a_4\,\alpha\,\beta+a_5\,\beta^2+a_6\,\mu^2+\,a_7\mu^4+\cdots, \\ &(b) \quad \rho'(\pi)=0=\qquad\qquad\quad +b_3\,\alpha^2+b_4\,\alpha\,\beta+b_5\,\beta^2+0.\mu^2+\,b_7\mu^4+\cdots. \end{aligned}\right\} \qquad (22)$$

The first of equations (22) can be solved for α as a power series in β and μ^2. This expression for α substituted in (b) gives rise to an equation of the form

$$(c) \qquad 0=c_1\,\beta\,\mu^2+c_2\,\beta^3+c_3\,\beta^2\,\mu^2+c_4\,\mu^4+\cdots.$$

This equation has the same form as (20) and can be solved in the same way, giving a solution for β as a power series in μ^2, vanishing with μ^2. This expression for β substituted in the equation for α gives a unique value for α as a power series in μ^2, vanishing with μ^2. Therefore periodic orbits exist for $\mu^2\neq 0$, which are analytic continuations of circular orbits for $\mu=0$.

We have thus proved the existence of the following three classes of periodic orbits which have the period 2π:

I. Orbits lying in the equatorial plane whose generating orbit is a circle.
II. Orbits inclined to the equatorial plane whose generating orbit is a circle.
III. Orbits in a meridian plane whose generating orbit is a circle.

66. Construction of Periodic Solutions in the Equatorial Plane.—We consider first orbits in the equatorial plane. We take the differential equations (8), and by means of the transformations there given we proceed at once to the integration of equation (10). It was shown in equation (13) that δ can be expanded uniquely as a power series in e in such a manner that the solution for ρ as a power series in e shall be periodic with the period 2π. Since the series is periodic with the same period for all values of e sufficiently small, it follows that the coefficient of each power of e is itself periodic. Since the solution exists and is unique, it must be possible to determine the δ uniquely by the condition that the solution shall be periodic. In the existence proof it was shown that δ vanishes with e. Therefore ρ and δ have the form

$$\rho=\rho_0+\rho_1 e+\rho_2 e^2+\rho_3 e^3+\cdots, \qquad \delta=\delta_1 e+\delta_2 e^2+\delta_3 e^3+\cdots. \tag{23}$$

The ρ_j are to be determined by the integration of equation (10) and by the initial values

$$\rho(0)=-1, \quad \frac{d\rho(0)}{d\tau}=0. \tag{24}$$

The δ_j are to be determined in such a manner that the ρ_j shall be periodic.

Upon substituting (23) in (10) and equating the coefficients, we find

$$\left.\begin{aligned}
&(a) \quad \frac{d^2\rho_0}{d\tau^2}+\rho_0=0,\\
&(b) \quad \frac{d^2\rho_1}{d\tau^2}+\rho_1=\delta_1+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]\rho_0^2,\\
&(c) \quad \frac{d^2\rho_2}{d\tau^2}+\rho_2=\delta_2-3\rho_0\delta_1+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]2\rho_0\rho_1\\
&\qquad\qquad +[-6+4\theta_1^2\mu^2+(4\theta_1^4+28\theta_2^2)\mu^4+\cdots]\rho_0^3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(d) \quad \frac{d^2\rho_n}{d\tau^2}+\rho_n=\delta_n-3\rho_0\delta_{n-1}+[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^4+\cdots]2\rho_0\rho_{n-1}\\
&\qquad\qquad +f_n(\rho_0,\ldots,\rho_{n-2}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{25}$$

These equations can be integrated in succession. The solution of (a) which satisfies the initial conditions is

$$\rho_0=-\cos\tau. \tag{26}$$

Since the initial conditions are independent of e, every ρ_j except ρ_0 must vanish at $\tau=0$. On substituting (26) in (25b) and integrating, we have

$$\rho_1=\delta_1(1-\cos\tau)+\left[3-3\theta_1^2\mu^2-(\theta_1^4+6\theta_2^2)\mu^2+\cdots\right]\left[\tfrac{1}{2}-\tfrac{1}{3}\cos\tau-\tfrac{1}{6}\cos 2\tau\right]. \tag{27}$$

The constants of integration in equation (27) have been determined so as to satisfy the initial conditions, but the constant δ_1 is, as yet, undetermined.

On substituting the values of ρ_0 and ρ_1 in (25c), we find

$$\left.\begin{aligned}\frac{d^2\rho_2}{d\mathsf{T}^2}+\rho_2=&+\Big[\delta_2+\delta_1(3-3\theta_1^2\mu^2\cdots)+(3-6\theta_1^2\mu^2\cdots)\Big]\\&+\Big[\delta_1\left(-3+6\theta_1^2\mu^2+(2\theta_1^4+12\theta_2^2)\mu^4\cdots\right)\\&\quad+\left(-3+12\theta_1^2\mu^2+(-\tfrac{11}{2}\theta_1^4+9\theta_2^2)\mu^4\cdots\right)\Big]\cos\mathsf{T}\\&+\Big[\delta_1(3-3\theta_1^2\mu^2\cdots)+(3-18\theta_1^2\mu^2\cdots)\Big]\cos 2\mathsf{T}\\&+\Big[3-4\theta_1^2\mu^2\cdots\Big]\cos 3\mathsf{T}.\end{aligned}\right\}\quad(28)$$

In order that the solution of this equation shall be periodic the coefficient of $\cos\mathsf{T}$ must be zero. This is the condition that determines δ_1, and consequently

$$\delta_1=-1+2\theta_1^2\mu^2+\left(\tfrac{3}{2}\theta_1^4-\theta_2^2\right)\mu^4+\cdots,$$

which agrees with (13) of the existence proof. With this value of δ_1 equation (28) becomes

$$\frac{d^2\rho_2}{d\mathsf{T}^2}+\rho_2=[\delta_2+3\theta_1^2\mu^2\cdots]+[-9\theta_1^2\mu^2\cdots]\cos 2\mathsf{T}+[3-4\theta_1^2\mu^2\cdots]\cos 3\mathsf{T}.$$

The solution of this equation which satisfies the initial conditions is

$$\left.\begin{aligned}\rho_2=\delta_2(1-\cos\mathsf{T})+[3\theta_1^2\mu^2+\cdots]+\left[\tfrac{3}{8}-\tfrac{13}{2}\theta_1^2\mu^2+\cdots\right]\cos\mathsf{T}\\+[3\theta_1^2\mu^2+\cdots]\cos 2\mathsf{T}+\left[-\tfrac{3}{8}+\tfrac{1}{2}\theta_1^2\mu^2+\cdots\right]\cos 3\mathsf{T}.\end{aligned}\right\}\quad(29)$$

The constant δ_2 is as yet entirely arbitrary. It is determined by the periodicity condition on ρ_3 in the same manner that δ_1 was determined by the periodicity condition on ρ_2. Without giving the details of the computation, its value is found to be

$$\delta_2=-6\theta_1^2\mu^2+\cdots.$$

This method of integration can be carried as far as is desired. In order to show this, let us suppose that $\rho_0,\ldots,\rho_{n-1}$ have been computed and that all the constants are known except δ_{n-1}. From (25d) we have

$$\frac{d^2\rho_n}{d\mathsf{T}^2}+\rho_n=\delta_n-3\rho_0\delta_{n-1}+[3-3\theta_1^2\mu^2\cdots]\,2\rho_0\rho_{n-1}+f_n(\rho_0,\ldots,\rho_{n-2}),\qquad(30)$$

where $f_n(\rho_0,\ldots,\rho_{n-2})$ is a polynomial in the ρ_j and contains only known terms. It is easy to see that ρ_{n-1} depends upon δ_{n-1} in the following way,

$$\rho_{n-1}=\delta_{n-1}(1-\cos\mathsf{T})+\text{ known terms.}$$

Equation (30) may therefore be written

$$\begin{aligned}\frac{d^2\rho_n}{d\mathsf{T}^2}+\rho_n=\delta_n+[3-3\theta_1^2\mu^2\cdots]\,\delta_{n-1}+[-3+6\theta_1^2\mu^2\cdots]\,\delta_{n-1}\cos\mathsf{T}\\+[3-3\theta_1^2\mu^2+\cdots]\cos 2\mathsf{T}+\text{ known terms.}\end{aligned}$$

In order that the solution of this equation shall be periodic the coefficient of $\cos \mathsf{T}$ must be zero. This condition determines δ_{n-1}. The equation can then be integrated, and the constants of integration will be determined by the conditions that, at $\mathsf{T}=0$,

$$\rho_n = \frac{d\rho_n}{d\mathsf{T}} = 0.$$

Everything is then determined with the exception of δ_n, and we have

$$\rho_n = (1-\cos\mathsf{T})\delta_n + \text{known terms}.$$

On substituting the values of δ_1 and δ_2 in the solution as far as it has been computed, we find

$$\rho_0 = -\cos\mathsf{T},$$

$$\begin{aligned}\rho_1 = &\left[\tfrac{1}{2}+\tfrac{1}{2}\theta_1^2\mu^2+(\theta_1^4-4\theta_2^2)\mu^4\cdots\right]+\left[-\theta_1^2\mu^2+\left(-\tfrac{7}{6}\theta_1^4+3\theta_2^2\right)\mu^4\cdots\right]\cos\mathsf{T}\\ &+\left[-\tfrac{1}{2}+\tfrac{1}{2}\theta_1^2\mu^2+\left(\tfrac{1}{6}\theta_1^4+\theta_2^2\right)\mu^4\cdots\right]\cos 2\mathsf{T},\end{aligned}$$

$$\begin{aligned}\rho_2 = &\left[-3\theta_1^2\mu^2+\cdots\right]+\left[\tfrac{3}{8}-\tfrac{1}{2}\theta_1^2\mu^2+\cdots\right]\cos\mathsf{T}+\left[3\theta_1^2\mu^2\cdots\right]\cos 2\mathsf{T}\\ &+\left[-\tfrac{3}{8}+\tfrac{1}{2}\theta_1^2\mu^2\cdots\right]\cos 3\mathsf{T},\end{aligned}$$

$$\delta_1 = -1+2\theta_1^2\mu^2+\left(\tfrac{3}{2}\theta_1^4-\theta_2^2\right)\mu^4\cdots, \qquad \delta_2 = 0-6\theta_1^2\mu^2+\cdots.$$

From these expressions the series for r becomes

$$\left.\begin{aligned}(a)\quad r = 1 - e\cos\mathsf{T} + \Big\{&\left[\tfrac{1}{2}+\tfrac{1}{2}\theta_1^2\mu^2+(\theta_1^4-4\theta_2^2)\mu^4+\cdots\right]\\ &+\left[-\theta_1^2\mu^2+\left(-\tfrac{7}{6}\theta_1^4+3\theta_2^2\right)\mu^4+\cdots\right]\cos\mathsf{T}\\ &+\left[-\tfrac{1}{2}+\tfrac{1}{2}\theta_1^2\mu^2+\left(\tfrac{1}{6}\theta_1^4+\theta_2^2\right)\mu^4\cdots\right]\cos 2\mathsf{T}\Big\}e^2\\ &+\Big\{\left[-3\theta_1^2\mu^2+\cdots\right]+\left[+\tfrac{3}{8}-\tfrac{1}{2}\theta_1^2\mu^2\cdots\right]\cos\mathsf{T}\\ &+\left[3\theta_1^2\mu^2\cdots\right]\cos 2\mathsf{T}+\left[-\tfrac{3}{8}+\tfrac{1}{2}\theta_1^2\mu^2\cdots\right]\cos 3\mathsf{T}\Big\}e^3+\cdots\end{aligned}\right\} \quad (31)$$

On substituting this value of r in the equation (8b), transforming to the independent variable T, and integrating, we find

$$\begin{aligned}(b)\quad v - v_0 = +\Big\{&\left[1+\theta_1^2\mu^2+\left(\tfrac{1}{2}\theta_1^4+2\theta_2^2\right)\mu^4+\cdots\right]\\ &+\left[\theta_1^2\mu^2+\left(-\tfrac{9}{4}\theta_1^4+\tfrac{19}{2}\theta_2^2\right)\mu^4+\cdots\right]e^2+\cdots\Big\}\mathsf{T}\\ +\Big\{&\left[2+2\theta_1^2\mu^2+(\theta_1^4+4\theta_2^2)\mu^4+\cdots\right]\sin\mathsf{T}\Big\}e\\ +\Big\{&\left[2\theta_1^2\mu^2+\left(\tfrac{13}{3}\theta_1^4-6\theta_2^2\right)\mu^4+\cdots\right]\sin\mathsf{T}\\ &+\left[\tfrac{5}{4}+\tfrac{3}{4}\theta_1^2\mu^2+\left(-\tfrac{1}{24}\theta_1^4+\tfrac{3}{2}\theta_2^2\right)\mu^4+\cdots\right]\sin 2\mathsf{T}\Big\}e^2\end{aligned}$$

. . . .

Equations (31a) and (31b) are the periodic solutions sought. If we return to the symbols defined in the original differential equations (1) by means of equations (5), with the additional notation

$$n\sqrt{1-\theta_1^2\mu^2-3\,\theta_2^2\mu^4\cdots}=\nu,$$

we have the following expressions for the polar coördinates:

$$\left.\begin{aligned}R=a\Big\{&1-e\cos\nu t+\Big[\tfrac{1}{2}-\tfrac{1}{2}\cos 2\nu t\Big]e^2+\Big[\tfrac{3}{8}\cos\nu t-\tfrac{3}{8}\cos 3\nu t\Big]e^3+\cdots\\&+\frac{b^2}{a^2}\Big[\Big(\tfrac{3}{20}-\tfrac{3}{10}\cos\nu t+\tfrac{3}{20}\cos 2\nu t\Big)e^2\\&+\Big(-\tfrac{9}{10}-\tfrac{3}{20}\cos\nu t+\tfrac{9}{10}\cos 2\nu t+\tfrac{3}{20}\cos 3\nu t\Big)e^3+\cdots\Big]\mu^2+\cdots\Big\},\end{aligned}\right\}\quad(32)$$

$$\left.\begin{aligned}v-v_0=\nu t&+2\sin\nu t\cdot e+\tfrac{5}{4}\sin 2\nu t\cdot e^2+\cdots\\&+\frac{b^2}{a^2}\Big[\Big(\tfrac{3}{10}+\tfrac{3}{10}e^2+\cdots\Big)\nu t+\Big(\tfrac{3}{5}\sin\nu t\Big)e^2\\&+\Big(\tfrac{3}{5}\sin\nu t+\tfrac{9}{40}\sin 2\nu t\Big)e^2+\cdots\Big]\mu^2+\frac{b^4}{a^4}\Big[\cdots\Big]\mu^4+\cdots.\end{aligned}\right\}\quad(33)$$

Equations (32) and (33) contain four arbitrary constants,* a, e, v_0, and t_0. Since the differential equations of motion in the equatorial plane are of the fourth order, these series, within the realm of their convergence, represent the general solution. The expression for the radius vector, R, is always periodic with the period $2\pi/\nu$. At the expiration of this period v has increased by the quantity

$$2\pi\Big[\frac{b^2}{a^2}\mu^2\Big(\tfrac{3}{10}+\tfrac{3}{10}e^2+\cdots\Big)+\cdots\Big]=2\pi\Theta \qquad (34)$$

in excess of 2π; that is, the line of apsides has rotated forward by this amount. If Θ is commensurable with unity the orbit is eventually closed geometrically. If $\Theta=I/J$, where I and J are relatively prime integers, then $v=2(I+J)\pi$ at $t=2J\pi/\nu$, and the particle is at its initial position with its initial components of velocity. The particle has completed $I+J$ revolutions, and the line of apsides has completed I revolutions. The mean sidereal period is

$$P=\frac{2\,\pi}{\nu\,(1+\Theta)}\cdot \qquad (35)$$

Equation (34) for the rotation of the line of apsides has an interesting application in the case of Jupiter's fifth satellite. On the hypothesis that Jupiter is a homogeneous spheroid whose equatorial diameter is 90,190 miles and whose polar diameter is 84,570 miles, that the mean distance of the satellite is 112,500 miles, that the eccentricity of its orbit is .006, and that

*The constant a is also contained implicitly in ν through the constant n, and t can obviously be replaced by $(t-t_0)$ since t does not occur explicitly in the differential equations (1).

its sidereal period is $11^h\ 0^m\ 22^s.7$, equation (34) gives for the rotation of the line of apsides 1440° per year. The values derived from observations are somewhat discordant, but are in the neighborhood of 883° per year. If we still keep the hypothesis that Jupiter is homogeneous in density and of the same oblateness as before, we are compelled to suppose that the value adopted for its polar radius was about 9,000 miles too great. In reality Jupiter must be much more dense at the center than at its surface, and therefore it is not necessary to suppose so large a reduction in making an allowance for its atmosphere.

67. Construction of Periodic Solutions for Orbits Inclined to the Equatorial Plane.—By means of the area integral the problem has been reduced to the three equations (6), the first two of which are

$$(a)\qquad r'' = \frac{c^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - \frac{r^3-4rq^2}{(r^2+q^2)^{\frac{7}{2}}}\,\theta_1^2\,\mu^2 \cdots,$$

$$(b)\qquad q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - \frac{3r^2q-2q^3}{(r^2+q^2)^{\frac{7}{2}}}\,\theta_1^2\,\mu^2 \cdots.$$

After the solution of these equations has been obtained, the third coördinate is found from the equation

$$(c)\qquad v' = \frac{c}{r^2}.$$

We have already proved, equations (14) to (20), the existence of periodic solutions of these equations of the following type:

$$\left.\begin{aligned} c^2 &= 1+c_2\,\mu^2+c_4\,\mu^4+\cdots, \\ r &= 1+\rho_2\,\mu^2+\rho_4\,\mu^4+\cdots, \\ q &= q_1\,\mu+q_3\,\mu^3+q_5\,\mu^5+\cdots, \end{aligned}\right\}\qquad (36)$$

with the initial conditions

$$r'(0) = q(0) = 0, \qquad q'(0) = \beta\,\mu,$$

the constant β being arbitrary. We can therefore integrate the equations so as to satisfy these initial conditions, and determine the c_j in such a manner as to render the solution periodic.

On substituting (36) in (6) there results

$$\left.\begin{aligned} 0 = \left[\rho_2''+\rho_2-\tfrac{3}{2}q_1^2-c_2+\theta_1^2\right]\mu^2+\Big[\rho_4''+\rho_4-3\,\rho_2^2-3\,q_1\,q_3+6\,\rho_2\,q_1^2 \\ +\tfrac{15}{8}q_1^4+(3\,c_2-4\,\theta_1^2)\,\rho_2-\tfrac{15}{2}\theta_1^2\,q_1^2-c_4\Big]\mu^4+\cdots, \end{aligned}\right\}\ (37)$$

$$\left.\begin{aligned} 0 = \left[q_1''+q_1\right]\mu+\left[q_3''+q_3-3\rho_2q_1-\tfrac{3}{2}q_1^3+3\theta_1^2\,q_1\right]\mu^3+\Big[q_5''+q_5-3\,\rho_2q_3-\tfrac{9}{2}\,q_1^2q_3 \\ +6\,\rho_2^2\,q_1-3\rho_4q_1+\tfrac{15}{2}\,\rho_2\,q_1^3+\tfrac{15}{8}q_1^5+3\,\theta_1^2\,q_3-15\,\theta_1^2\,\rho_2\,q_1-\tfrac{25}{2}\theta_1^2\,q_1^3\Big]\mu^5+\cdots \end{aligned}\right\}\ (38)$$

Equation (37) contains only the even powers of μ, and (38) only the odd powers. For the integration we have:

Coefficient of μ. The coefficient of μ is defined by $q_1''+q_1=0$, and the solution of this equation satisfying the initial conditions is

$$q_1=\beta \sin \tau. \tag{39}$$

Coefficient of μ^2. The coefficient of μ^2, from (37), is defined by

$$\rho_2''+\rho_2=\frac{3}{2}q_1^2+c_2-\theta_1^2=\left(\frac{3}{4}\beta^2+c_2-\theta_1^2\right)-\frac{3}{4}\beta^2 \cos 2\tau.$$

The solution of this equation which satisfies the assigned initial conditions is

$$\rho_2=\left(\frac{3}{4}\beta^2+c_2-\theta_1^2\right)+\alpha_2 \cos \tau+\frac{1}{4}\beta^2 \cos 2\tau.$$

The constant c_2 is determined by the periodicity condition on q_3, where it is found that it must have the value $c_2=2\theta_1^2-\beta^2$; and α_2, which is determined by the periodicity condition on ρ_4, is found to be zero. If we anticipate these determinations, we have

$$\rho_2=\left(\theta_1^2-\frac{1}{4}\beta^2\right)+\frac{1}{4}\beta^2 \cos 2\tau. \tag{40}$$

Coefficient of μ^3. The coefficient of μ^3, from (38), is defined by

$$q_3''+q_3=q_1\left(3\rho_2+\frac{3}{2}q_1^2-3\theta_1^2\right)=(3\beta^2+3c_2-6\theta_1^2)\beta \sin\tau+\frac{3}{4}\alpha_2\beta\sin 2\tau.$$

In order that the solution shall be periodic it is necessary that the coefficient of $\sin\tau$ be zero. Therefore $c_2=2\theta_1^2-\beta^2$. On substituting this value and integrating, we find

$$q_3=\beta_3 \sin \tau-\frac{1}{4}\alpha_2\beta \sin 2\tau.$$

From the initial conditions we must have $q_3'(0)=0$, and therefore

$$\beta_3=\frac{1}{2}\alpha_2\beta.$$

But it will be shown in the next step that $\alpha_2=0$, and consequently that

$$q_3\equiv 0. \tag{41}$$

Coefficient of μ^4. It follows from (37) that the coefficient of μ^4 is defined by

$$\rho_4''+\rho_4=3\rho_2^2+3q_1q_3-6\rho_2q_1^2-\frac{15}{8}q_1^4+(4\theta_1^2-3c_2)\rho_2+\frac{15}{2}\theta_1^2q_1^2+c_4. \tag{42}$$

Before expanding the right member of this equation we will examine the coefficient of $\cos\tau$, which we know must be zero from the periodicity condition. It is noticed in the first place that terms in $\cos\tau$ can arise only through terms involving ρ_2 and q_3, and secondly that all such terms carry α_2 as a factor. No other arbitrary enters the coefficient; therefore we must take $\alpha_2=0$. It can be shown by induction that the arbitrary constant α_i(the coefficient of $\cos\tau$), which arises in the integration of ρ_i, is determined by the periodicity condition on ρ_{i+2}, and further that its value is zero. The proof is omitted for the sake of brevity.

Upon substituting the value $\alpha_2=0$ in ρ_2 and q_3 and expanding the right member of (42), we find

$$\rho_4''+\rho_4=\left[c_4+\theta_1^2+\tfrac{11}{4}\theta_1^2\beta^2-\tfrac{3}{64}\beta^4\right]+\left[\tfrac{1}{4}\theta_1^2\beta^2-\tfrac{3}{16}\beta^4\right]\cos 2\tau+\tfrac{15}{64}\beta^4\cos 4\tau.$$

Since the constants of integration must both be zero, the solution is

$$\rho_4=\left[c_4+\theta_1^2+\tfrac{11}{4}\theta_1^2\beta^2-\tfrac{3}{64}\beta^4\right]+\left[-\tfrac{1}{12}\theta_1^2\beta^2+\tfrac{1}{16}\beta^4\right]\cos 2\tau-\tfrac{1}{64}\beta^4\cos 4\tau.$$

If we anticipate the value of c_4 which is found below, we have

$$\rho_4=\left[-3\theta_1^4-\tfrac{11}{12}\theta_1^2\beta^2-\tfrac{3}{64}\beta^4\right]+\left[-\tfrac{1}{12}\theta_1^2\beta^2+\tfrac{1}{16}\beta^4\right]\cos 2\tau-\tfrac{1}{64}\beta^4\cos 4\tau.$$

Coefficient of μ^5. We find from (38) that the coefficient of μ^5 is defined by

$$\begin{aligned} q_5''+q_5&=3\rho_2 q_3+\tfrac{9}{2}q_1^2 q_3-6\rho_2^2 q_1+3\rho_4 q_1-\tfrac{15}{2}\rho_2 q_1^2-\tfrac{15}{8}q_1^5-3\theta_1^2 q_3+15\theta_1^2\rho_2 q_1+\tfrac{25}{2}\theta_1^2 q_1^3\\ &=[3c_4+12\theta_1^4+11\theta_1^2\beta^2]\,\beta\sin\tau-\theta_1^2\beta^3\sin 3\tau. \end{aligned}$$

From the periodicity condition we have

$$c_4=-4\theta_1^4-\tfrac{11}{3}\theta_1^2\beta^2.$$

On integrating and imposing the initial conditions, we find at this step

$$q_5=-\tfrac{3}{8}\theta_1^2\beta^3\sin\tau+\tfrac{1}{8}\theta_1^2\beta^3\sin 3\tau. \qquad (43)$$

This is sufficient to make evident the general character of the series. The r-equation contains only even multiples of τ and the q-equation contains only odd multiples. The r-equation contains only even powers of μ and of τ, while the q-equation is odd in both these respects. The series are therefore triply even and odd.

On collecting the various coefficients, we have the following series:

$$
\left.
\begin{aligned}
&(a) && r=1+\left[\left(\theta_1^2-\tfrac{1}{4}\beta^2\right)+\tfrac{1}{4}\beta^2\cos 2\tau\right]\mu^2+\left[\left(-3\theta_1^4-\tfrac{11}{12}\theta_1^2\beta^2-\tfrac{3}{64}\beta^4\right)\right.\\
&&&\qquad \left.+\left(-\tfrac{1}{12}\theta_1^2\beta^2+\tfrac{1}{16}\beta^4\right)\cos 2\tau-\tfrac{1}{64}\beta^4\cos 4\tau\right]\mu^4+\cdots,\\
&(b) && q=\left[\beta\sin\tau\right]\mu+\left[0\right]\mu^3+\left[-\tfrac{3}{8}\theta_1^2\beta^3\sin\tau+\tfrac{1}{8}\theta_1^2\beta^3\sin 3\tau\right]\mu^5+\cdots,\\
&(c) && v-v_0=\left[1-\theta_1^2\mu^2-\left(\tfrac{3}{2}\theta_1^4+\tfrac{1}{6}\theta_1^2\beta^2\right)\mu^4+\cdots\right]\tau+\left[-\tfrac{1}{4}\beta^2\sin 2\tau\right]\mu^2\\
&&&\qquad +\left[\left(\tfrac{11}{12}\theta_1^2\beta^2-\tfrac{1}{8}\beta^4\right)\sin 2\tau+\tfrac{1}{64}\beta^4\sin 4\tau\right]\mu^4+\cdots,\\
&(d) && c^2=1+\left[2\theta_1^2-\beta^2\right]\mu^2+\left[-4\theta_1^4-\tfrac{11}{3}\theta_1^2\beta^2\right]\mu^4+\cdots.
\end{aligned}
\right\}\quad(44)
$$

In this solution the constants* a, β, v_0, and τ_0 are arbitrary. As is shown by equation (44c) the nodes regress, the measure of regression being

$$2\pi\left[\theta_1^2\mu^2+\left(\tfrac{3}{2}\theta_1^4+\tfrac{1}{6}\theta_1^2\beta^2\right)\mu^4+\cdots\right].$$

The generating orbit of these solutions is a circle in the equatorial plane. A circle having any assigned inclination might have been used, *e. g.*,

$$r=\sqrt{1-s^2\sin^2\tau},\qquad q=s\sin\tau,\qquad v=\tan^{-1}\left[\sqrt{1-s^2}\tan\tau\right],\qquad(45)$$

where s is the sine of the inclination. The solution thus obtained would have been identical with (44). If we should expand (45) as power series in s and put $s=\beta\mu$, we should find that the terms thus obtained are identical with the terms independent of θ_1^2 in the solution which has been worked out. It might therefore be of assistance in the physical interpretation of the constants to put $\beta\mu=s$ in the series (44).

68. Construction of Periodic Solutions in a Meridian Plane.—When the constant c is zero the motion is in a meridian plane. The equations of motion (21) are

$$
\left.
\begin{aligned}
p''-p(\varphi')^2+\frac{1}{p^2}&=-\frac{-\frac{3}{4}+\frac{3}{2}\cos 2\varphi+\frac{1}{4}\cos 4\varphi}{p^4}\,\theta_1^2\mu^2+\cdots,\\
p\varphi''+2p'(\varphi')^2&=-\frac{\frac{1}{2}\sin 2\varphi-\frac{1}{4}\sin 4\varphi}{p^4}\,\theta_1^2\mu^2+\cdots.
\end{aligned}
\right\}\quad(46)
$$

We have proved in §65 the existence of periodic solutions of these equations as power series in μ^2, which, for $\mu^2=0$, reduce to the circle $p=1$, $\varphi=\tau$. Let us therefore put

$$p=1+p_2\mu^2+p_4\mu^4+\cdots,\qquad \varphi=\tau+\varphi_2\mu^2+\varphi_4\mu^4+\cdots.$$

*The constant a is contained implicitly through τ and θ_1^2; see equations (4).

Upon substituting these expressions in (46), expanding, and collecting the coefficients of the various powers of μ^2, we find

$$\left.\begin{aligned}
0=&\left[p_2''-3\,p_2-2\,\varphi_2'-\tfrac{3}{4}\theta_1^2+\tfrac{3}{2}\theta_1^2\cos 2\tau+\tfrac{1}{4}\theta_1^2\cos 4\tau\right]\mu^2\\
&+\left[p_4''-3p_4-2\varphi_4'-2p_2\varphi_2'-\varphi_2'^2+3p_2^2+(3-6\cos 2\tau-\cos 4\tau)p_2\theta_1^2\right.\\
&\left.+(-3\sin 2\tau-\sin 4\tau)\,\varphi_2\,\theta_1^2\right]\mu^4+\cdots,\\
0=&\left[\varphi_2''+2\,p_2'+\tfrac{1}{2}\theta_1^2\sin 2\tau-\tfrac{1}{4}\theta_1^2\sin 4\tau\right]\mu^2\\
&+\left[\varphi_4''+2p_4'+\varphi_2''\,p_2+2p_2'\,\varphi_2'+(-2\sin 2\tau+\sin 4\tau)\,\theta_1^2\,p_2\right.\\
&\left.+(\cos 2\tau-\cos 4\tau)\,\theta_1^2\,\varphi_2\right]\mu^4+\cdots.
\end{aligned}\right\}\tag{47}$$

The initial conditions are $p'(0)=\varphi(0)=0$. On proceeding to the integration, we have:

Coefficients of μ^2. The coefficients of μ^2 are defined by

$$\left.\begin{aligned}
&(a)\qquad p_2''-3p_2-2\varphi_2'=\tfrac{3}{4}-\tfrac{3}{2}\,\theta_1^2\cos 2\tau-\tfrac{1}{4}\,\theta_1^2\cos 4\tau,\\
&(b)\qquad \varphi_2''+2p_2'=\ -\tfrac{1}{2}\,\theta_1^2\sin 2\tau+\tfrac{1}{4}\,\theta_1^2\sin 4\tau.
\end{aligned}\right\}\tag{48}$$

On integrating (*b*) once, we have

$$(c)\qquad \varphi_2'=-2p_2+\tfrac{1}{4}\theta_1^2\cos 2\tau-\tfrac{1}{16}\theta_1^2\cos 4\tau+c_1.$$

If we substitute this value of φ_2' in (*a*), the latter becomes

$$(d)\qquad p_2''+p_2=\left(2c_1+\tfrac{3}{4}\theta_1^2\right)-\theta_1^2\cos 2\tau-\tfrac{3}{8}\theta_1^2\cos 4\tau.$$

The integration of this equation gives for the general solution

$$p_2=\left(2c_1+\tfrac{3}{4}\theta_1^2\right)+c_2\sin\tau+c_3\cos\tau+\tfrac{1}{3}\theta_1^2\cos 2\tau+\tfrac{1}{40}\theta_1^2\cos 4\tau.$$

Since $p_2'(0)=0$ we must take $c_2=0$. On substituting this value of p_2 in (*c*) and integrating, we get

$$\varphi_2=\left(-3c_1-\tfrac{3}{2}\theta_1^2\right)\tau-2c_3\sin\tau-\tfrac{5}{24}\,\theta_1^2\sin 2\tau-\tfrac{9}{320}\theta_1^2\sin 4\tau+c_4.$$

From the initial conditions, φ_2 must be zero when $\tau=0$; therefore $c_4=0$. It must also be periodic; therefore $2c_1=-\theta_1^2$. All of the constants of integration are now determined except c_3, which will be determined by the periodicity condition on p_4.

The differential equations for p_4 and φ_4 are just the same as for p_2 and φ_2 except in the right members. The process of integration is just the same. In the right members only even multiples of τ occur except in terms carrying the undetermined constant c_3 as a factor. In the equation corresponding to (48d) there will be a term in $\cos\tau$ carrying c_3 as a factor. But the integral from this term will be non-periodic unless $c_3=0$. Upon putting $c_3=0$, the integration proceeds just as before and the constants are determined in the same manner. The same steps are repeated in the coefficients of μ^6, and so on for all higher powers. Therefore no odd multiples of τ occur in the solution. We have, therefore,

$$\left.\begin{aligned} p&=1+\left[-\tfrac{1}{4}+\tfrac{1}{3}\cos 2\tau+\tfrac{1}{40}\cos 4\tau\right]\theta_1^2\mu^2+\cdots,\\ \varphi&=\tau+\left[-\tfrac{5}{24}\sin 2\tau-\tfrac{9}{320}\sin 4\tau\right]\theta_1^2\mu^2+\cdots.\end{aligned}\right\} \tag{49}$$

Since the series involve only even multiples of τ, the orbits are symmetrical with respect to both the r-axis and the q-axis.

This completes the formal construction of the solutions of which the existence was proved in §§63, 64, and 65.

II. ORBITS RE-ENTRANT AFTER MANY REVOLUTIONS.

69. The Differential Equations.—The orbits which we have previously considered have had the common property of involving only the period 2π. Since this period is independent of the oblateness of the spheroid, the derivation of these orbits has been relatively simple. We shall proceed now to investigate a class of orbits which involves, beside the period 2π, another period $2\pi/\lambda$, where λ is a function of the oblateness of the spheroid, the inclination of the orbit to the equator, and the mean distance of the particle. We will start out from the solution which involved an arbitrary inclination (§67). Into this solution four arbitrary constants were introduced, viz., inclination, mean distance, longitude of node, and the epoch. Two more arbitraries are necessary for a complete solution, viz., constants corresponding to the eccentricity and to the longitude of perihelion. In what follows we shall introduce the constant corresponding to eccentricity.

We have found for the differential equations a certain solution, given in (44), which we may write

$$r=\varphi(\beta,\,\mu;\,\tau),\qquad q=\psi(\beta,\,\mu;\,\tau),\qquad c^2=c_0^2,$$

which is symmetric with respect to the equatorial plane. That is to say, at $\tau=0$ the particle is in the equatorial plane and its motion is perpendicular to the radius vector. Its initial distance is φ (0).

Suppose now we change the initial distance slightly and also the initial velocity so that

$$r(0)=\varphi(0)+\alpha,\qquad q(0)=0,\qquad r'(0)=0,\qquad q'(0)=\psi'(0)+\gamma,$$

and give an increment to the constant of areas so that $c^2=c_0^2+\epsilon$. Can we determine these three constants α, γ, and ϵ, as functions of β and μ, in such a manner that the series for r and q shall be periodic? The solutions can be expressed in the form

$$r=\varphi(\beta,\ \mu;\ \tau)+\rho, \qquad q=\psi(\beta,\ \mu;\ \tau)+\sigma, \qquad c^2=c_0^2+\epsilon,$$

where ρ and σ are the necessary additions to φ and ψ. If now we substitute these expressions in the differential equations (6), all the terms independent of ρ, σ, and ϵ will drop out, and there will remain the following differential equations for ρ and σ:

$$\left.\begin{aligned}
(a)\quad & \rho''+\Big\{1+\Big[\Big(-4\theta_1^2+\tfrac{3}{2}\beta^2\Big)-\tfrac{9}{2}\beta^2\cos 2\tau\Big]\mu^2+\Big[\Big(20\theta_1^4+6\theta_1^2\beta^2+\tfrac{3}{8}\beta^4\Big)\\
&+\Big(\theta_1^2\beta^2-\tfrac{3}{2}\beta^4\Big)\cos 2\tau+\tfrac{9}{8}\beta^4\cos 4\tau\Big]\mu^4+\cdots\Big\}\rho+\Big\{\Big[-3\beta\sin\tau\Big]\mu\\
&+\Big[\Big(-3\theta_1^2\beta+\tfrac{9}{8}\beta^3\Big)\sin\tau-\tfrac{3}{8}\beta^3\sin 3\tau\Big]\mu^3+\cdots\Big\}\sigma\\
&=\Big\{3+\Big[\Big(-16\theta_1^2+6\beta^2\Big)-12\beta^2\cos 2\tau\Big]\mu^2+\Big[\Big(90\theta_1^4+\tfrac{59}{4}\theta_1^2\beta^2+\tfrac{69}{64}\beta^4\Big)\\
&+\Big(\tfrac{1}{4}\theta_1^2\beta^2-\tfrac{63}{16}\beta^4\Big)\cos 2\tau+\tfrac{183}{64}\beta^4\cos 4\tau\Big]\mu^4+\cdots\Big\}\rho^2+\Big\{\Big[-12\beta\sin\tau\Big]\mu\\
&+\Big[\Big(-90\theta_1^2\beta+\tfrac{45}{4}\beta^3\Big)\sin\tau-\tfrac{15}{4}\beta^3\sin 3\tau\Big]\mu^3+\cdots\Big\}\rho\sigma\\
&+\Big\{\tfrac{3}{2}+\Big[\Big(\tfrac{3}{2}\theta_1^2-\tfrac{33}{8}\beta^2\Big)+\tfrac{33}{8}\beta^2\cos 2\tau\Big]\mu^2+\cdots\Big\}\sigma^2+\cdots\\
&+\Big[\Big\{1+\Big[\Big(-3\theta_1^2+\tfrac{3}{4}\beta^2\Big)-\tfrac{3}{4}\beta^2\cos 2\tau\Big]\mu^2+\Big[\Big(15\theta_1^4-\tfrac{1}{4}\theta_1^2\beta^2+\tfrac{45}{64}\beta^4\Big)\\
&+\Big(\tfrac{13}{4}\theta_1^2\beta^2-\tfrac{15}{16}\beta^4\Big)\cos 2\tau+\tfrac{15}{4}\beta^4\cos 4\tau\Big]\mu^4+\cdots\Big\}\\
&+\Big\{-3+\Big[\Big(12\theta_1^2-3\beta^2\Big)+3\beta^2\cos 2\tau\Big]\mu^2+\cdots\Big\}\rho+\cdots\Big]\epsilon,\\
(b)\quad & \sigma''+\Big\{1+\Big[-\tfrac{3}{2}\beta^2+\tfrac{3}{2}\beta^2\cos 2\tau\Big]\mu^2+\Big[-4\theta_1^2\beta^2+7\theta_1^2\beta^2\cos 2\tau\Big]\mu^4+\cdots\Big\}\sigma\\
&+\Big\{\Big[-3\beta\sin\tau\Big]\mu+\Big[\Big(-3\theta_1^2\beta+\tfrac{9}{8}\beta^3\Big)\sin\tau-\tfrac{3}{8}\beta^3\sin 3\tau\Big]\mu^3+\cdots\Big\}\rho\\
&=\Big\{\Big[-6\beta\sin\tau\Big]\mu+\Big[\Big(-21\theta_1^2\beta+\tfrac{63}{8}\beta^3\Big)\sin\tau-\tfrac{21}{8}\beta^3\sin 3\tau\Big]\mu^3+\cdots\Big\}\rho^2\\
&+\Big\{3+\Big[\Big(3\theta_1^2-\tfrac{33}{4}\beta^2\Big)+\tfrac{33}{4}\beta^2\cos 2\tau\Big]\mu^2+\cdots\Big\}\rho\sigma\\
&+\Big\{\Big[\tfrac{9}{2}\beta\sin\tau\Big]\mu+\cdots\Big\}\sigma^2+\cdots.
\end{aligned}\right\}\tag{50}$$

In the first of these equations the coefficients of all the terms containing odd powers of σ involve only sines of odd multiples of τ and odd powers of μ; all other coefficients involve only even powers of μ and cosines of even

multiples of τ. In the second equation the coefficient of every odd power of σ involves only even powers of μ and cosines of even multiples of τ; all other coefficients involve only odd powers of μ and sines of odd multiples of τ. These properties play an important rôle throughout the entire discussion.

70. The Equations of Variation.—Considering merely the terms of the differential equations (50) which are linear in ρ and σ, we have

$$
\left.\begin{aligned}
(a)\quad & \rho''+\Big\{1+\Big[\big(-4\theta_1^2+\tfrac{3}{2}\beta^2\big)-\tfrac{9}{2}\beta^2\cos 2\tau\Big]\mu^2+\Big[\big(20\theta_1^4+6\theta_1^2\beta^2+\tfrac{3}{8}\beta^4\big) \\
& +\big(\theta_1^2\beta^2-\tfrac{3}{2}\beta^4\big)\cos 2\tau+\tfrac{9}{8}\beta^4\cos 4\tau\Big]\mu^4+\cdots\Big\}\rho+\Big\{\Big[-3\beta\sin\tau\Big]\mu \\
& +\Big[\big(-3\theta_1^2\beta+\tfrac{9}{8}\beta^3\big)\sin\tau-\tfrac{3}{8}\beta^3\sin 3\tau\Big]\mu^3+\cdots\Big\}\sigma=0, \\
(b)\quad & \sigma''+\Big\{1+\Big[-\tfrac{3}{2}\beta^2+\tfrac{3}{2}\beta^2\cos 2\tau\Big]\mu^2+\cdots\Big\}\sigma+\Big\{\Big[-3\beta\sin\tau\Big]\mu \\
& +\Big[\big(-3\theta_1^2\beta+\tfrac{9}{8}\beta^3\big)\sin\tau-\tfrac{3}{8}\beta^3\sin 3\tau\Big]\mu^3+\cdots\Big\}\rho=0.
\end{aligned}\right\} \quad (51)
$$

The solutions of these equations have the general form

$$
\rho=\sum_{j=1}^{4} c_j e^{\lambda_j\tau}\varphi_j(\tau), \qquad \sigma=\sum_{j=1}^{4} c_j e^{\lambda_j\tau}\psi_j(\tau),
$$

where c_j and λ_j are constants, and $\varphi_j(\tau)$ and $\psi_j(\tau)$ are periodic functions of τ with the period 2π. The four values of the λ_j (real or imaginary) are associated in pairs, equal in value but of opposite sign (§33); and since the solution (44) contains two arbitrary constants, *i. e.*, the origin of time, τ_0, and the mean distance, a, it is known *a priori* that one pair of the λ_j has the value 0 (§33). If we suppose that $\lambda_3=\lambda_4=0$, the two corresponding solutions have the form

$$
\rho=c_3\varphi_3(\tau)+c_4[\varphi_4(\tau)+\tau\varphi_3(\tau)], \qquad \sigma=c_3\psi_3(\tau)+c_4[\psi_4(\tau)+\tau\psi_3(\tau)].
$$

The values of $\varphi_3(\tau)$ and $\psi_3(\tau)$ can be obtained at once by differentiating the solutions for r and q [equations (44)] with respect to τ; and the values of $[\varphi_4(\tau)+\tau\,\varphi_3(\tau)]$ and $[\psi_4(\tau)+\tau\,\psi_3(\tau)]$ are obtained by differentiating ar and aq with respect to a. Thus two of the solutions of the fundamental set can be found without integration.

We will consider first the two solutions in which the λ_j are not zero. Let us substitute in (51) the expressions

$$
\rho=e^{i\lambda\tau}\varphi(\tau), \qquad \sigma=e^{i\lambda\tau}\psi(\tau) \qquad (i=\sqrt{-1}).
$$

After dividing out the exponential, there remains

$$\left.\begin{array}{ll}(a) & \varphi''+2i\lambda\varphi'+[1-\lambda^2+a_2\mu^2+a_4\mu^4+\cdots]\varphi+[a_1\mu+a_3\mu^3+\cdots]\psi=0,\\ (b) & \psi''+2i\lambda\psi'+[1-\lambda^2+b_2\mu^2+b_4\mu^4+\cdots]\psi+[a_1\mu+a_3\mu^3+\cdots]\varphi=0,\end{array}\right\} \quad (52)$$

where

$$a_1=-3\beta\sin\tau,$$
$$a_3=\left(-3\theta_1^2\beta+\tfrac{9}{8}\beta^3\right)\sin\tau-\tfrac{3}{8}\beta^3\sin 3\tau,$$
$$a_2=\left(-4\theta_1^2+\tfrac{3}{2}\beta^2\right)-\tfrac{9}{2}\beta^2\cos 2\tau,$$
$$a_4=\left((20\theta_1^4+6\theta_1^2\beta^2+\tfrac{3}{8}\beta^4\right)+\left(\theta_1^2\beta^2-\tfrac{3}{2}\beta^4\right)\cos 2\tau+\tfrac{9}{8}\beta^4\cos 4\tau,$$
$$b_2=-\tfrac{3}{2}\beta^2+\tfrac{3}{2}\beta^2\cos 2\tau,$$

b_4 = a sum of cosines of even multiples of τ.

With respect to equations (52), it is known that φ and ψ are periodic with the period 2π and that λ vanishes with μ, since the problem then reduces to the two-body problem, in which the characteristic exponents are all zero. Since φ, ψ, and λ are analytic in μ, we may put

$$\varphi=\sum_{j=0}^{\infty}\varphi_j\mu^j,\qquad \psi=\sum_{j=0}^{\infty}\psi_j\mu^j,\qquad \lambda=\sum_{j=0}^{\infty}\lambda_j\mu^j.$$

The expressions for φ_j, ψ_j, and λ are determined from (52) as follows:

Coefficients of μ^0. The terms independent of μ are found to be

$$\left.\begin{array}{ll}\varphi_0''+\varphi_0=0, & \varphi_0=a_1^{(0)}\cos\tau+a_2^{(0)}\sin\tau,\\ \psi_0''+\psi_0=0, & \psi_0=\gamma_1^{(0)}\cos\tau+\gamma_2^{(0)}\sin\tau.\end{array}\right\} \quad (53)$$

Coefficients of μ. The differential equations for the terms in μ are

$$\varphi_1''+\varphi_1=-2i\lambda_1\varphi_0'-a_1\psi_0,\qquad \psi_1''+\psi_1=-2i\lambda_1\psi_0'-a_1\varphi_0. \quad (54)$$

Since the periodicity conditions demand that the coefficients of $\cos\tau$ and $\sin\tau$ in the right members shall be zero, we must take $\lambda_1=0$, after which we get, on making use of (53),

$$\left.\begin{array}{l}\varphi_1''+\varphi_1=\tfrac{3}{2}\beta\gamma_2^{(0)}-\tfrac{3}{2}\beta\gamma_2^{(0)}\cos 2\tau+\tfrac{3}{2}\beta\gamma_1^{(0)}\sin 2\tau,\\ \psi_1''+\psi_1=\tfrac{3}{2}\beta a_2^{(0)}-\tfrac{3}{2}\beta a_2^{(0)}\cos 2\tau+\tfrac{3}{2}\beta a_1^{(0)}\sin 2\tau.\end{array}\right\} \quad (55)$$

Upon integrating, we have

$$\left.\begin{array}{l}\varphi_1=a_1^{(1)}\cos\tau+a_2^{(1)}\sin\tau+\tfrac{3}{2}\beta\gamma_2^{(0)}+\tfrac{1}{2}\beta\gamma_2^{(0)}\cos 2\tau-\tfrac{1}{2}\beta\gamma_1^{(0)}\sin 2\tau,\\ \psi_1=\gamma_1^{(1)}\cos\tau+\gamma_2^{(1)}\sin\tau+\tfrac{3}{2}\beta a_2^{(0)}+\tfrac{1}{2}\beta a_2^{(0)}\cos 2\tau-\tfrac{1}{2}\beta a_1^{(0)}\sin 2\tau.\end{array}\right\} \quad (56)$$

Coefficients of μ^2. The coefficients of μ^2 are defined by

$$\varphi_2''+\varphi_2=-2i\lambda_2\varphi_0'-a_2\varphi_0-a_1\psi_1, \qquad \psi_2''+\psi_2=-2i\lambda_2\psi_0'-b_2\psi_0-a_1\varphi_1; \quad (57)$$

or, expanding by (53) and (56),

$$\left.\begin{aligned}\varphi_2''+\varphi_2&=\tfrac{3}{2}\beta\gamma_2^{(1)}-\tfrac{3}{2}\beta\gamma_2^{(1)}\cos 2\tau+\tfrac{3}{2}\beta\gamma_1^{(1)}\sin 2\tau+[4\theta_1^2\alpha_2^{(0)}+2i\lambda_2\alpha_1^{(0)}]\sin\tau\\&\quad+[4\theta_1^2\alpha_1^{(0)}-2i\lambda_2\alpha_2^{(0)}]\cos\tau+3\beta^2\alpha_2^{(0)}\sin 3\tau+3\beta^2\alpha_1^{(0)}\cos 3\tau,\\ \psi_2''+\psi_2&=\tfrac{3}{2}\beta\alpha_2^{(1)}-\tfrac{3}{2}\beta\alpha_2^{(1)}\cos 2\tau+\tfrac{3}{2}\beta\alpha_1^{(1)}\sin 2\tau+\left[\tfrac{15}{2}\beta^2\gamma_2^{(0)}+2i\lambda_2\gamma_1^{(0)}\right]\sin\tau\\&\quad+2i\lambda_2\gamma_2^{(0)}\cos\tau.\end{aligned}\right\}\quad(58)$$

In order to satisfy the periodicity conditions we must have

$$\left.\begin{aligned}2i\lambda_2\alpha_1^{(0)}+4\theta_1^2\alpha_2^{(0)}&=0, &\qquad 2i\lambda_2\gamma_1^{(0)}+\tfrac{15}{2}\beta^2\gamma_2^{(0)}&=0,\\ 4\theta_1^2\alpha_1^{(0)}-2i\lambda_2\alpha_2^{(0)}&=0, & +2i\lambda_2\gamma_2^{(0)}&=0.\end{aligned}\right\}\quad(59)$$

The last two of these equations are satisfied by taking

$$\gamma_1^{(0)}=\gamma_2^{(0)}=0.$$

On solving the first two, we find

$$\lambda_2=\pm 2\theta_1^2, \qquad \alpha_1^{(0)}-i\alpha_2^{(0)}=0.$$

Equations (59) can also be satisfied by

$$\lambda_2=\alpha_1^{(0)}=\alpha_2^{(0)}=\gamma_2^{(0)}=0, \qquad \gamma_1^{(0)} \text{ arbitrary.}$$

This leads to the development of the solution in which the characteristic exponent is zero, which will be discussed, beginning with equations (80), by using the integral relations.

It was known at the outset that the two values of λ are equal but of opposite sign. We will choose the one with the positive sign. The solution for the negative λ can be derived from it. The condition $\alpha_1^{(0)}-i\alpha_2^{(0)}=0$ still leaves us with an arbitrary constant. Since the equations are linear, this constant will enter the solution linearly, and may therefore be taken equal to unity, inasmuch as the solution after development is multiplied by an arbitrary constant. We will take then $\alpha_1^{(0)}=1$, which therefore makes $\alpha_2^{(0)}=-i$. Consequently

$$\phi_0=\cos\tau-i\sin\tau, \qquad \psi_0=0. \quad (60)$$

On integrating equations (58) with these values of $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, we get

$$\left.\begin{aligned}\varphi_2&=\tfrac{3}{2}\beta\gamma_2^{(1)}+\alpha_1^{(2)}\cos\tau+\alpha_2^{(2)}\sin\tau+\tfrac{1}{2}\beta\gamma_2^{(1)}\cos 2\tau-\tfrac{1}{2}\beta\gamma_1^{(1)}\sin 2\tau\\&\quad+3\beta^2\cos 3\tau-3\beta^2 i\sin 3\tau,\\ \psi_2&=\tfrac{3}{2}\beta\alpha_2^{(1)}+\gamma_1^{(2)}\cos\tau+\gamma_2^{(2)}\sin\tau+\tfrac{1}{2}\beta\alpha_2^{(1)}\cos 2\tau-\tfrac{1}{2}\beta\alpha_1^{(1)}\sin 2\tau.\end{aligned}\right\}\quad(61)$$

Coefficients of μ^3. The terms of the third degree in μ are defined by

$$\left.\begin{aligned}
\varphi_3''+\varphi_3=&+\tfrac{3}{2}\beta\gamma_2^{(2)}+\left[-2\lambda_3+4\theta_1^2(\alpha_1^{(1)}-i\alpha_2^{(1)})\right]\cos\tau\\
&+\left[2i\lambda_3+4\theta_1^2(\alpha_2^{(1)}+i\alpha_1^{(1)})\right]\sin\tau+\tfrac{3}{2}\beta\gamma_1^{(2)}\sin 2\tau-\tfrac{3}{2}\beta\gamma_2^{(2)}\cos 2\tau,\\
\psi_3''+\psi_3=&+\left[\tfrac{3}{2}\alpha_2^{(2)}\beta-\tfrac{3}{2}\theta_1^2 i\beta-\tfrac{21}{16}i\beta^3\right]+\left[-2i\lambda_2\gamma_2^{(1)}+\tfrac{3}{4}\beta^2\gamma_1^{(1)}\right]\cos\tau\\
&+\left[2i\lambda_2\gamma_1^{(1)}+\tfrac{9}{4}\beta^2\gamma_2^{(1)}\right]\sin\tau+\left[-\tfrac{3}{2}\alpha_2^{(2)}\beta+\tfrac{11}{2}\theta_1^2 i\beta+\tfrac{21}{16}i\beta^3\right]\cos 2\tau\\
&+\left[\tfrac{3}{2}\alpha_1^{(2)}\beta+\tfrac{11}{2}\theta_1^2\beta-\tfrac{9}{16}\beta^3\right]\sin 2\tau-\tfrac{3}{4}\beta^2\gamma_1^{(1)}\cos 3\tau-\tfrac{3}{4}\beta^2\gamma_2^{(1)}\sin 3\tau.
\end{aligned}\right\}\quad(62)$$

From the periodicity conditions we must have

$$-2\lambda_3+4\theta_1^2(\alpha_1^{(1)}-i\alpha_2^{(1)})=0, \qquad \tfrac{3}{4}\beta^2\gamma_1^{(1)}-2i\lambda_2\gamma_2^{(1)}=0,$$

$$2i\lambda_3+4\theta_1^2(\alpha_2^{(1)}+i\alpha_1^{(1)})=0, \qquad 2i\lambda_2\gamma_1^{(1)}+\tfrac{9}{4}\beta^2\gamma_2^{(1)}=0.$$

The last two equations can be satisfied only if $\gamma_1^{(1)}=\gamma_2^{(1)}=0$. The first two can be satisfied only if $\lambda_3=(\alpha_1^{(1)}-i\,\alpha_2^{(1)})=0$. The condition $\alpha_1^{(1)}-i\,\alpha_2^{(1)}=0$ again leaves us an arbitrary constant; it gives us $\varphi_1=c(\cos\tau-i\,\sin\tau)$, but this is the same as φ_0 multiplied by $c\mu$. That is, the solution is repeating itself one degree higher in μ, and this, of course, should be expected since the equations are linear, and φ_0 multiplied by any power of μ must satisfy them. We may then choose the arbitrary multiplier equal to zero, which is the same as choosing $\alpha_1^{(1)}=\alpha_2^{(1)}=0$. On integrating (62) with these values, we find

$$\left.\begin{aligned}
\varphi_3=&+\tfrac{3}{2}\beta\gamma_2^{(2)}+\alpha_1^{(3)}\cos\tau+\alpha_2^{(3)}\sin\tau+\tfrac{1}{2}\beta\gamma_2^{(2)}\cos 2\tau-\tfrac{1}{2}\beta\gamma_1^{(2)}\sin 2\tau,\\
\psi_3=&+\left[\tfrac{3}{2}\alpha_2^{(2)}\beta-\tfrac{3}{2}i\theta_1^2\beta-\tfrac{21}{16}i\beta^3\right]+\gamma_1^{(3)}\cos\tau+\gamma_2^{(3)}\sin\tau\\
&+\left[\tfrac{1}{2}\alpha_2^{(2)}\beta-\tfrac{11}{6}i\theta_1^2\beta-\tfrac{7}{16}i\beta^3\right]\cos 2\tau+\left[-\tfrac{1}{2}\alpha_1^{(2)}\beta-\tfrac{11}{6}\theta_1^2\beta+\tfrac{3}{16}\beta^3\right]\sin 2\tau.
\end{aligned}\right\}\quad(63)$$

It can be shown by induction at this point that φ and λ are even series in μ, and that ψ is an odd series in μ. Furthermore, φ contains only odd multiples of τ, and ψ only even multiples. Consequently

$$\gamma_1^{(2)}=\gamma_2^{(2)}=\alpha_1^{(3)}=\alpha_2^{(3)}=\gamma_1^{(3)}=\gamma_2^{(3)}=0,$$

and all

$$\varphi_{2j+1}=\psi_{2j}=0.$$

Coefficient of μ^4. The term of the fourth degree in μ is defined by

$$\left.\begin{aligned}
\varphi_4''+\varphi_4=&+\left[-2\lambda_4+4\theta_1^2(\alpha_1^{(2)}-i\,\alpha_2^{(2)})-16\theta_1^4-10\theta_1^2\beta^2\right]\cos\tau\\
&+\left[2i\lambda_4+4\theta_1^2(i\,\alpha_1^{(2)}+\alpha_2^{(2)})+16i\theta_1^4\right]\sin\tau\\
&+\left[6\theta_1^2\beta^2+3\beta^2\alpha_1^{(2)}+\tfrac{3}{16}\beta^4\right]\cos 3\tau+\left[-6i\theta_1^2\beta+3\beta^2\alpha_2^{(2)}-\tfrac{45}{16}i\beta^4\right]\sin 3\tau\\
&-\tfrac{21}{16}\beta^4\cos 5\tau+\tfrac{21}{16}i\beta^4\sin 5\tau.
\end{aligned}\right\}\quad(64)$$

The conditions that φ_4 shall be periodic are

$$\begin{aligned} -2\ \lambda_4+4\,\theta_1^2(a_1^{(2)}-i\,a_2^{(2)})-16\ \ \theta_1^4-10\,\theta_1^2\,\beta^2&=0,\\ +2i\,\lambda_4+4\,\theta_1^2(i\,a_1^{(2)}+a_2^{(2)})+16\,i\,\theta_1^4\qquad\qquad&=0. \end{aligned}$$

On solving these equations, we find

$$\lambda_4=-8\,\theta_1^4-\tfrac{5}{2}\,\theta_1^2\,\beta^2,\qquad\qquad a_1^{(2)}-i\,a_2^{(2)}=\tfrac{5}{4}\beta^2.$$

In the last equation we can choose $a_1^{(2)}=5/4\,\beta^2$ and $a_2^{(2)}=0$. This choice will make the coefficient of $\sin\tau$ in φ_2 equal to zero, and since the same thing occurs for each φ_j it is evident that the corresponding choice will simplify the solution by making the coefficient of $\sin\tau$ equal to zero for all powers of μ. We have, then, on integrating and collecting results

$$\left.\begin{aligned} \varphi_4&=a_4\cos\tau+\left[-\tfrac{3}{4}\theta_1^2\,\beta^2-\tfrac{63}{128}\beta^4\right]\cos 3\tau+\left[\tfrac{3}{4}i\,\theta_1^2\,\beta^2+\tfrac{45}{128}i\,\beta^4\right]\sin 3\tau\\ &\qquad\qquad+\tfrac{7}{128}\beta^4\cos 5\tau-\tfrac{7}{128}i\,\beta^4\ \sin 5\tau,\\ \psi_3&=\left[-\tfrac{3}{2}i\,\theta_1^2\,\beta-\tfrac{21}{16}i\,\beta^3\right]+\left[-\tfrac{11}{6}i\,\theta_1^2\,\beta-\tfrac{7}{13}i\,\beta^3\right]\cos 2\tau\\ &\qquad\qquad+\left[-\tfrac{11}{6}\theta_1^2\,\beta-\tfrac{7}{16}\beta^3\right]\sin 2\tau,\\ \varphi_2&=\tfrac{5}{4}\beta^2\cos\tau-\tfrac{3}{8}\,\beta^2\cos 3\tau+\tfrac{3}{8}\,i\,\beta^2\sin 3\tau,\\ \psi_1&=-\tfrac{3}{2}i\,\beta-\tfrac{1}{2}\,i\,\beta\cos 2\tau-\tfrac{1}{2}\beta\sin 2\tau,\\ \varphi_0&=\cos\tau-i\,\sin\tau,\\ \lambda\ &=2\theta_1^2\,\mu^2+\left[-8\theta_1^4-\tfrac{5}{2}\theta_1^2\,\beta^2\right]\mu^4+\cdots\,. \end{aligned}\right\}\quad(65)$$

The coefficient, a_4, of $\cos\tau$ in φ_4 is determined by the periodicity condition of φ_6. That this process of determining the values of the λ_j and the constants of integration arising at each step is general can be shown as follows: Let us suppose that we have computed all terms of φ up to and including φ_j with the exception of the constants of integration in φ_j. We have then

$$\varphi_j=a_1^{(j)}\cos\tau+a_2^{(j)}\sin\tau+\text{known terms}.$$

It follows from equations (52) that the $a_1^{(j)}$ and $a_2^{(j)}$ enter ψ_{j+1} only as shown explicitly in

$$\psi_{j+1}''+\psi_{j+1}=\tfrac{3}{2}\,\beta\,a_2^{(j)}-\tfrac{3}{2}\,\beta a_2^{(j)}\cos 2\tau+\tfrac{3}{2}\,\beta a_1^{(j)}\sin 2\tau+\cdots\,.$$

Consequently, in so far as ψ_{j+1} depends upon these terms, it is

$$\psi_{j+1}=\tfrac{3}{2}\beta a^{(j)}+\tfrac{1}{2}\,\beta a_2^{(j)}-\tfrac{1}{2}\beta a_1^{(j)}\sin 2\tau.$$

Similarly, in so far as φ_{j+2} depends upon constants as yet undetermined, it is found from equations (52) that

$$\left.\begin{aligned}\varphi''_{j+2}+\varphi_{j+2}=&-2i\lambda_2\varphi'_j-2i\lambda_{j+2}\varphi'_0+\left[(4\theta_1^2-\tfrac{3}{2}\beta^2)+\tfrac{9}{2}\beta^2\cos 2\tau\right]\varphi_j\\&+\left[3\beta\sin\tau\right]\psi_{j+1}\\=&+\left[-2\lambda_{j+2}+4\theta_1^2(a_1^{(j)}-ia_2^{(j)})+A_{j+2}\right]\cos\tau\\&+\left[2i\lambda_{j+2}+4\theta_1^2(ia_1^{(j)}+a_2^{(j)})+B_{j+2}\right]\sin\tau\\&+3\beta^2a_1^{(j)}\cos 3\tau+3\beta^2a_2^{(j)}\sin 3\tau,\end{aligned}\right\}\quad(66)$$

where A_{j+2} and B_{j+2} are the known terms in the coefficients of $\cos\tau$ and $\sin\tau$ respectively. From the periodicity condition we must have

$$-2\lambda_{j+2}+4\theta_1^2(a_1^{(j)}-ia_2^{(j)})+A_{j+2}=0,\qquad 2i\lambda_{j+2}+4\theta_1^2(ia_1^{(j)}+a_2^{(j)})+B_{j+2}=0.\quad(67)$$

The solution of these equations is

$$\lambda_{j+2}=\tfrac{1}{4}\left[A_{j+2}+iB_{j+2}\right],\qquad a_1^{(j)}-ia_2^{(j)}=-\frac{A_{j+2}-iB_{j+2}}{8\theta_1^2}.\quad(68)$$

As has already been pointed out, we can choose $a_2^{(j)}=0$, and we have then

$$a_1^{(j)}=-\frac{A_{j+2}-iB_{j+2}}{8\theta_1^2}.\quad(69)$$

In order to show that λ_{j+2} and $a_1^{(j)}$ are real, it will be sufficient to show that A_{j+2} is real and that B_{j+2} is a pure imaginary. This is readily proved by induction, for, up to $j=4$ inclusive, we have

$$\varphi_j=\sum_{\kappa=1}^{j+1}m_\kappa\cos\kappa\tau+i\sum_{\kappa=1}^{j+1}n_\kappa\sin\kappa\tau,\qquad \psi_j=i\sum_{\kappa=0}^{j-1}f_\kappa\cos\kappa\tau+\sum_{\kappa=0}^{j-1}g_\kappa\sin\kappa\tau,$$

where m_κ, n_κ, f_κ, and g_κ are all real. From the form of the differential equations it follows at once that the same forms hold for $j=5$, then 6, and so on. That is, A_{j+2} is real while B_{j+2} is purely imaginary.

Furthermore, it is to be noticed that A_{j+2} and B_{j+2} do not contain any terms in β independent of θ_1^2, and consequently the θ_1^2, which appears in the denominator of $a_1^{(j)}$, will divide out. This is proved as follows: If θ_1^2 be put equal to zero in the differential equations, then equations (52) become the equations of variation of a circular orbit in the ordinary two-body problem, the plane of the circle being inclined to the plane of reference by an angle whose sine is $\beta\mu=s$. Equations (6) can then be written

$$r''=\frac{(1-s^2)(1-e^2)}{r^3}-\frac{r}{(r^2+q^2)^{\frac{3}{2}}}=R,\qquad q''=-\frac{q}{(r^2+q^2)^{\frac{3}{2}}}=Q,\quad(70)$$

where the constant c^2 is given the form $(1-s^2)(1-e^2)$. For these equations we have the solution

$$r=\frac{(1-e^2)\sqrt{1-s^2\sin^2(\theta-\Omega)}}{1+e\cos(\theta-\theta_0)},\qquad q=\frac{(1-e^2)s\sin(\theta-\Omega)}{1+e\cos(\theta-\theta_0)},\quad(71)$$

where

$$\theta-\theta_0=(\tau-\tau_0)+2e\sin(\tau-\tau_0)+\cdots$$

If now we form the equations of variation by varying r, q, and e, that is by putting

$$r=r_0+\rho, \qquad q=q_0+\sigma, \qquad e=e_0+\epsilon,$$

where r_0, q_0, and e_0 are the values in (71), we find

$$\rho''=\frac{\partial R}{\partial r}\rho+\frac{\partial R}{\partial q}\sigma+\frac{\partial R}{\partial e}\epsilon, \qquad \sigma''=\frac{\partial Q}{\partial r}\rho+\frac{\partial Q}{\partial q}\sigma+\frac{\partial Q}{\partial e}\epsilon. \tag{72}$$

Three solutions of equations (72) are given by

$$\left.\begin{array}{lll} (1) & (2) & (3) \\ \rho=c_1\dfrac{\partial r_0}{\partial \Omega}, & \rho=c_2\dfrac{\partial r_0}{\partial \tau_0}, & \rho=c_3\dfrac{\partial r_0}{\partial e_0}, \\ \sigma=c_1\dfrac{\partial q_0}{\partial \Omega}, & \sigma=c_2\dfrac{\partial q_0}{\partial \tau_0}, & \sigma=c_3\dfrac{\partial q_0}{\partial e_0}, \\ \epsilon=c_1\dfrac{\partial e_0}{\partial \Omega}=0; & \epsilon=c_2\dfrac{\partial e_0}{\partial \tau_0}=0; & \epsilon=c_3\dfrac{\partial e_0}{\partial e_0}. \end{array}\right\} \tag{73}$$

If $e_0\neq 0$ these three solutions are distinct. The case in which we are interested is that for which $e_0=0$, but then these three solutions are not distinct, for the first two coincide, as is readily seen by putting $e=0$ in (71). Since the equations are linear, it follows that

$$\rho=c_4\left[\frac{\partial r_0}{\partial \Omega}-\frac{\partial r_0}{\partial \tau_0}\right], \qquad \sigma=c_4\left[\frac{\partial q_0}{\partial \Omega}-\frac{\partial q_0}{\partial \tau_0}\right], \qquad \epsilon=c_4\left[\frac{\partial e_0}{\partial \Omega}-\frac{\partial e_0}{\partial \tau_0}\right], \tag{74}$$

is also a solution; but, since it vanishes with e_0, it carries e_0 as a factor which we can divide out and absorb in the arbitrary constant c_4. For $e_0=0$ this solution does not now vanish, and it is moreover distinct from the first solution. Thus we have three distinct solutions even when $e_0=0$, but since $\partial Q/\partial e_0\equiv 0$ and $\partial R/\partial e_0$ carries e_0 as a factor, equations (72) pass over to the equations of variation of a circle when $e_0=0$. For these equations we have therefore three solutions which are periodic with the period 2π. The fourth solution is not periodic for it involves a term of the form τ times a periodic function.

Let us return now to the solution which we have developed, (65), and consider only the terms which belong to the two-body problem, viz., the terms which are independent of θ_1^2. This solution can be separated into two solutions, one of which is real, the other purely imaginary. The real solution is the third one of (73), and the purely imaginary solution is the second. Since both of these solutions are certainly periodic with the period 2π, it follows that no terms in β alone can occur in the A_{j+2} and B_{j+2}, (67), because the presence of such terms would give rise to non-periodic terms in the two-body problem. Hence A_{j+2} and B_{j+2} carry θ_1^2 as a factor which can be divided out of equation (69). Furthermore, λ_{j+2}, equation (68), carries θ_1^2 as a factor, and therefore λ vanishes with the oblateness of the spheroid.

The solution which we have obtained may be written

$$\rho^{(1)}=e^{i\lambda\tau}[\varphi^{(1)}+i\,\varphi^{(2)}], \qquad \sigma^{(1)}=e^{i\lambda\tau}[\psi^{(1)}+i\,\psi^{(2)}], \tag{75}$$

where

$$\left.\begin{aligned}
\varphi^{(1)}&=\cos\tau+\left[\tfrac{5}{4}\beta^2\cos\tau-\tfrac{3}{8}\beta^2\cos 3\tau\right]\mu^2+\left[a_4\cos\tau+\left(-\tfrac{3}{4}\theta_1^2\beta^2-\tfrac{63}{128}\beta^4\right)\cos 3\tau\right.\\
&\qquad\left.+\tfrac{7}{128}\beta^4\cos 5\tau\right]\mu^4+\cdots,\\
\varphi^{(2)}&=-\sin\tau+\left[\tfrac{3}{8}\beta^2\sin 3\tau\right]\mu^2+\left[\left(\tfrac{3}{4}\theta_1^2\beta^2+\tfrac{45}{128}\beta^4\right)\sin 3\tau-\tfrac{7}{128}\beta^4\sin 5\tau\right]\mu^4+\cdots\\
\psi^{(1)}&=\left[-\tfrac{1}{2}\beta\sin 2\tau\right]\mu+\left[\left(-\tfrac{11}{6}\theta_1^2\beta-\tfrac{7}{16}\beta^3\right)\sin 2\tau\right]\mu^3+\cdots,\\
\psi^{(2)}&=\left[-\tfrac{3}{2}\beta-\tfrac{1}{2}\beta\cos 2\tau\right]\mu+\left[\left(-\tfrac{3}{2}\theta_1^2\beta-\tfrac{21}{16}\beta^3\right)\right.\\
&\qquad\left.+\left(-\tfrac{11}{6}\theta_1^2\beta^2-\tfrac{7}{16}\beta^3\right)\cos 2\tau+\cdots\right]\mu^3+\cdots.
\end{aligned}\right\}\tag{76}$$

By putting $e^{i\lambda\tau}=\cos\lambda\tau+i\,\sin\lambda\tau$, the solution takes the form

$$\left.\begin{aligned}
\rho^{(1)}&=[\varphi^{(1)}\cos\lambda\tau-\varphi^{(2)}\sin\lambda\tau]+i[\varphi^{(2)}\cos\lambda\tau+\varphi^{(1)}\sin\lambda\tau],\\
\sigma^{(1)}&=[\psi^{(1)}\cos\lambda\tau-\psi^{(2)}\sin\lambda\tau]+i[\psi^{(2)}\cos\lambda\tau+\psi^{(1)}\sin\lambda\tau].
\end{aligned}\right\}\tag{77}$$

We have thus one solution of the differential equations. A second solution can be derived from it by merely changing the sign of i, or

$$\left.\begin{aligned}
\rho^{(2)}&=[\varphi^{(1)}\cos\lambda\tau-\varphi^{(2)}\sin\lambda\tau]-i[\varphi^{(2)}\cos\lambda\tau+\varphi^{(1)}\sin\lambda\tau],\\
\sigma^{(2)}&=[\psi^{(1)}\cos\lambda\tau-\psi^{(2)}\sin\lambda\tau]-i[\psi^{(2)}\cos\lambda\tau+\psi^{(1)}\sin\lambda\tau].
\end{aligned}\right\}\tag{78}$$

By adding and subtracting these two solutions, we have finally

$$\rho=A\,[\rho^{(1)}+\rho^{(2)}]+B\,[\rho^{(1)}-\rho^{(2)}], \qquad \sigma=A\,[\sigma^{(1)}+\sigma^{(2)}]+B\,[\sigma^{(1)}-\sigma^{(2)}],$$

A and B being arbitrary constants.

As above developed, there is a certain arbitrariness in these solutions, owing to the manner in which the constants of integration were determined. They may be reduced to a normal form by multiplying each solution by the proper series in μ^2 with constant coefficients. By this process we can make

$$\rho^{(1)}(0)+\rho^{(2)}(0)=1, \qquad \sigma^{(1)}(0)-\sigma^{(2)}(0)=\beta\mu. \tag{79}$$

Since $[\rho^{(1)}-\rho^{(2)}]$ and $[\sigma^{(1)}+\sigma^{(2)}]$ are sine series, they vanish for $\tau=0$.

The third and fourth solutions of the equations of variation are

$$\rho_3=C'\frac{\partial r}{\partial\tau}, \quad \sigma_3=C'\frac{\partial q}{\partial\tau}; \qquad \rho_4=D'\,\frac{\partial\,(ar)}{\partial a}, \quad \sigma_4=D'\,\frac{\partial\,(aq)}{\partial a}, \tag{80}$$

the r and q being defined by equations (44). In performing the differentiation in this last solution it will be remembered that τ and θ_1^2 are functions of a. The third solution also can be normalized by giving the arbitrary constant such a form that

$$\rho_3(0)=0, \qquad \sigma_3(0)=C\beta\mu.$$

As already stated, the fourth solution is non-periodic and has the form

$$\rho_4=D\left[\tau\frac{\partial r_0}{\partial\tau}+\varphi_4\right],\qquad\qquad \sigma_4=D\left[\tau\frac{\partial q_0}{\partial\tau}+\psi_4\right],\tag{81}$$

where φ_4 and ψ_4 are periodic functions of τ with the period 2π. As in the previous solutions, this can be normalized so that at $\tau=0$

$$\varphi_4=\beta^2\mu^2,\qquad\qquad \psi_4=0.$$

The functions φ_4 and ψ_4 are also easily found by substituting (81) in the equations of variation and solving for these variables (which must be periodic).

Upon carrying out the foregoing operations, we find the following fundamental set of solutions:

$$\left.\begin{aligned}
\rho=&+A\Big\{\cos(1-\lambda)\tau+\Big[-\tfrac{1}{4}\beta^2\cos(1-\lambda)\tau+\tfrac{5}{8}\beta^2\cos(1+\lambda)\tau\\
&\quad-\tfrac{3}{8}\beta^2\cos(3-\lambda)\tau\Big]\mu^2+\Big[\Big(-\tfrac{1}{2}\alpha_4+\tfrac{3}{4}\theta_1^2\beta^2+\tfrac{21}{32}\beta^4\Big)\cos(1-\lambda)\tau\\
&\quad+\Big(\tfrac{1}{2}\alpha_4-\tfrac{35}{64}\beta^4\Big)\cos(1+\lambda)\tau+\Big(-\tfrac{3}{4}\theta_1^2\beta^2-\tfrac{3}{32}\beta^4\Big)\cos(3-\lambda)\tau\\
&\qquad\qquad-\tfrac{9}{128}\beta^4\cos(3+\lambda)\tau+\tfrac{7}{128}\beta^4\cos(5-\lambda)\tau\Big]\mu^4+\cdots\Big\}\\
&+B\Big\{\tfrac{1}{2}\sin(1-\lambda)\tau+\Big[\Big(-\tfrac{1}{8}\beta^2-\tfrac{5}{6}\theta_1^2\Big)\sin(1-\lambda)\tau-\tfrac{5}{16}\beta^2\sin(1+\lambda)\tau\\
&\qquad\qquad-\tfrac{3}{16}\beta^2\sin(3-\lambda)\tau\Big]\mu^2+\cdots\Big\}\\
&+C\Big\{\Big[-\tfrac{1}{2}\beta^2\sin2\tau\Big]\mu^2+\Big[\Big(\tfrac{1}{16}\theta_1^2\beta^2-\tfrac{1}{8}\beta^4\Big)\sin2\tau+\tfrac{1}{16}\beta^4\sin4\tau\Big]\mu^4+\cdots\Big\}\\
&+D\Big\{\Big\{\Big[\tfrac{3}{4}\beta^2+\tfrac{1}{4}\beta^2\cos2\tau\Big]\mu^2+\Big[\Big(\tfrac{9}{8}\theta_1^2\beta^2-\tfrac{3}{32}\beta^4\Big)\\
&\quad+\Big(-\tfrac{9}{8}\theta_1^2\beta^2+\tfrac{1}{8}\beta^4\Big)\cos2\tau-\tfrac{1}{32}\beta^4\cos4\tau\Big]\mu^4+\cdots\Big\}\\
&\quad+\tau\Big\{\Big[\tfrac{3}{4}\beta^4\sin2\tau\Big]\mu^4+\Big[\Big(\tfrac{91}{16}\theta_1^2\beta^4+\tfrac{15}{64}\beta^6\Big)\sin2\tau-\tfrac{3}{32}\beta^6\sin4\tau\Big]\mu^6+\cdots\Big\}\Big\},
\end{aligned}\right\}\tag{82}$$

$$\left.\begin{aligned}
\sigma=&+A\Big\{\Big[\tfrac{3}{2}\beta\sin\lambda\tau-\tfrac{1}{2}\beta\sin(2-\lambda)\tau\Big]\mu+\Big[\tfrac{3}{2}\theta_1^2\beta\sin\lambda\tau\\
&\qquad\qquad-\tfrac{11}{6}\theta_1^2\beta\sin(2-\lambda)\tau\Big]\mu^3+\cdots\Big\}\\
&+B\Big\{\Big[\tfrac{3}{4}\beta\cos\lambda\tau+\tfrac{1}{4}\beta\cos(2-\lambda)\tau\Big]\mu+\Big[-\tfrac{1}{2}\theta_1^2\beta\cos\lambda\tau\\
&\qquad\qquad+\tfrac{1}{2}\theta_1^2\beta\cos(2-\lambda)\tau\Big]\mu^3+\cdots\Big\}\\
&+C\Big\{\Big[\beta\cos\tau\Big]\mu+\Big[0\Big]\mu^3+\cdots\Big\}\\
&+D\Big\{\Big\{\Big[\tfrac{1}{2}\beta\sin\tau\Big]\mu+\Big[\Big(-\tfrac{7}{4}\theta_1^2\beta+\tfrac{1}{2}\beta^3\Big)\sin\tau\Big]\mu^3+\cdots\Big\}\\
&\quad+\tau\Big\{\Big[-\tfrac{3}{2}\beta^3\cos\tau\Big]\mu^3+\Big[\Big(-\tfrac{53}{16}\theta_1^2\beta^3+\tfrac{21}{16}\beta^5\Big)\cos\tau\Big]\mu^5+\cdots\Big\}\Big\}.
\end{aligned}\right\}\tag{83}$$

71. Special Theorems for the Non-Homogeneous Equations.—The general theorems, proved in Chapter I, Section IV, on the character of the solutions of non-homogeneous linear differential equations with periodic coefficients, presuppose merely the conditions that the coefficients are periodic with the period 2π. Additional facts with regard to the solutions can be established when additional facts are specified with regard to the coefficients of the differential equations. The equations of variation, (51), may be written

$$\frac{d\rho_1}{d\tau}=\rho_2\,, \qquad \frac{d\rho_2}{dt}=\bar{\bar{\theta}}_2\rho_1+\bar{\theta}_3\sigma_1\,, \qquad \frac{d\sigma_1}{d\tau}=\sigma_2\,, \qquad \frac{d\sigma_2}{d\tau}=\bar{\theta}_3\rho_1+\bar{\bar{\theta}}_4\sigma_1\,, \tag{84}$$

where the notation with respect to the θ's has the following significance: Even subscripts denote functions even in τ, and odd subscripts denote functions odd in τ; one dash indicates that only odd multiples of τ are involved, and two dashes indicate that only even multiples of τ are involved. The solution of equations (82) and (83) may be characterized in the same manner, and are then

$$\left.\begin{aligned}
\rho_1&=A\bar{\alpha}_2(\tau)+B\bar{\alpha}_1(\tau)+C\bar{\bar{\alpha}}_3(\tau)+D[\bar{\bar{\alpha}}_4(\tau)+\tau\,\bar{\bar{\alpha}}_3(\tau)],\\
\rho_2&=A\bar{\beta}_1(\tau)+B\bar{\beta}_2(\tau)+C\bar{\bar{\beta}}_4(\tau)+D[\bar{\bar{\beta}}_3(\tau)+\tau\,\bar{\bar{\beta}}_4(\tau)],\\
\sigma_1&=A\bar{\bar{\gamma}}_1(\tau)+B\bar{\bar{\gamma}}_2(\tau)+C\bar{\gamma}_4(\tau)+D[\bar{\gamma}_3(\tau)+\tau\,\bar{\gamma}_4(\tau)],\\
\sigma_2&=A\bar{\bar{\delta}}_2(\tau)+B\bar{\bar{\delta}}_1(\tau)+C\bar{\delta}_3(\tau)+D[\bar{\delta}_4(\tau)+\tau\,\bar{\delta}_3(\tau)],
\end{aligned}\right\} \tag{85}$$

where the notation is the same as for the θ's with the exception that in the first two solutions every integral multiple of τ is increased by $\pm\lambda\tau$, *e. g.*, $\cos(3+\lambda)\tau$. On these terms the dashes refer only to the integral part.

Suppose now we have the non-homogeneous differential equations

$$\left.\begin{aligned}
\frac{d\rho_1}{d\tau}&=\rho_2\,, &\qquad \frac{d\rho_2}{d\tau}&=\bar{\bar{\theta}}_2\rho_1+\bar{\theta}_3\sigma_1+g(\tau),\\
\frac{d\sigma_1}{d\tau}&=\sigma_2\,, &\qquad \frac{d\sigma_2}{d\tau}&=\bar{\theta}_3\rho_1+\bar{\bar{\theta}}_4\sigma_1+f(\tau),
\end{aligned}\right\} \tag{86}$$

where $g(\tau)$ and $f(\tau)$ are periodic with the period 2π. Since the characteristic exponents are 0, 0, $\pm\sqrt{-1}\lambda$, by §31, the general solution has the form

$$\left.\begin{aligned}
\rho_1&=(\rho_1)+\xi_1=(\rho_1)+\omega_1(\tau)+a\tau\,\bar{\bar{\alpha}}_3+b\left[\tfrac{1}{2}\tau^2\bar{\bar{\alpha}}_3+\tau\,\bar{\bar{\alpha}}_4\right],\\
\rho_2&=(\rho_2)+\xi_2=(\rho_2)+\omega_2(\tau)+a\tau\,\bar{\bar{\beta}}_4+b\left[\tfrac{1}{2}\tau^2\bar{\bar{\beta}}_4+\tau\,\bar{\bar{\beta}}_3\right],\\
\sigma_1&=(\sigma_1)+\eta_1=(\sigma_1)+\omega_3(\tau)+a\tau\,\bar{\gamma}_4+b\left[\tfrac{1}{2}\tau^2\bar{\gamma}_4+\tau\,\bar{\gamma}_3\right],\\
\sigma_2&=(\sigma_2)+\eta_2=(\sigma_2)+\omega_4(\tau)+a\tau\,\bar{\delta}_3+b\left[\tfrac{1}{2}\tau^2\bar{\delta}_3+\tau\,\bar{\delta}_4\right],
\end{aligned}\right\} \tag{87}$$

where the (ρ_i) and (σ_i) are the complementary functions, and the ξ_i and η_i are the particular integrals of which the ω_i are the periodic parts, and where a and b are constants which depend upon the differential equations.

Let us suppose that $g(\tau)$ is an even function of τ and that $f(\tau)$ is an odd function of τ, and let us seek the character of the solutions which satisfy the initial conditions

$$\rho_2(0)=\sigma_1(0)=0.$$

On changing τ into $-\tau$ in equations (86), we get

$$\left.\begin{aligned}
&\frac{d}{d\tau}\rho_1(-\tau)=-\rho_2(-\tau), \quad && \frac{d}{d\tau}\rho_2(-\tau)=-\bar{\bar{\theta}}_2\rho_1(-\tau)+\bar{\theta}_3\sigma_1(-\tau)-g(\tau),\\
&\frac{d}{d\tau}\sigma_2(-\tau)=-\sigma_2(-\tau), \quad && \frac{d}{d\tau}\sigma_2(-\tau)=+\bar{\theta}_3\rho_1(-\tau)-\bar{\bar{\theta}}_4\sigma_1(-\tau)+f(\tau).
\end{aligned}\right\} \quad (88)$$

From equations (86) and (88) we obtain, by eliminating $g(\tau)$ and $f(\tau)$, the differential equations

$$\left.\begin{aligned}
&\frac{d}{d\tau}[\rho_1(\tau)-\rho_1(-\tau)]= [\rho_2(\tau)+\rho_2(-\tau)],\\
&\frac{d}{d\tau}[\rho_2(\tau)+\rho_2(-\tau)]=\bar{\bar{\theta}}_2[\rho_1(\tau)-\rho_1(-\tau)]+\bar{\theta}_3[\sigma_1(\tau)+\sigma_1(-\tau)],\\
&\frac{d}{d\tau}[\sigma_1(\tau)+\sigma_1(-\tau)]= [\sigma_2(\tau)-\sigma_2(-\tau)],\\
&\frac{d}{d\tau}[\sigma_2(\tau)-\sigma_2(-\tau)]=\bar{\theta}_3[\rho_1(\tau)-\rho_1(-\rho)]+\bar{\bar{\theta}}_4[\rho_1(\tau)+\rho_1(-\tau)].
\end{aligned}\right\} \quad (89)$$

These equations are the same as the original homogeneous set (84). Hence their general solutions have the same form, viz.,

$$\left.\begin{aligned}
\rho_1(\tau)-\rho_1(-\tau)&=A\,\bar{\alpha}_2(\tau)+B\,\bar{\alpha}_1(\tau)+C\,\bar{\bar{\alpha}}_3(\tau)+D[\bar{\bar{\alpha}}_4(\tau)+\tau\bar{\bar{\alpha}}_3(\tau)],\\
\rho_2(\tau)+\rho_2(-\tau)&=A\,\bar{\beta}_1(\tau)+B\,\bar{\beta}_2(\tau)+C\,\bar{\bar{\beta}}_4(\tau)+D[\bar{\bar{\beta}}_3(\tau)+\tau\bar{\bar{\beta}}_4(\tau)],\\
\sigma_1(\tau)+\sigma_1(-\tau)&=A\,\bar{\bar{\gamma}}_1(\tau)+B\,\bar{\bar{\gamma}}_2(\tau)+C\,\bar{\gamma}_4(\tau)+D[\bar{\gamma}_3(\tau)+\tau\bar{\gamma}_4(\tau)],\\
\sigma_2(\tau)-\sigma_2(-\tau)&=A\,\bar{\bar{\delta}}_2(\tau)+B\,\bar{\bar{\delta}}_1(\tau)+C\,\bar{\delta}_3(\tau)+D[\bar{\delta}_4(\tau)+\tau\bar{\delta}_3(\tau)].
\end{aligned}\right\} \quad (90)$$

Upon putting $\tau=0$, we find from the first and the fourth of these equations that

$$0=A\bar{\alpha}_2(0)+D\bar{\bar{\alpha}}_4(0), \qquad 0=A\bar{\bar{\delta}}_2(0)+D\bar{\delta}_4(0).$$

Either $A=D=0$, or the determinant $\bar{\alpha}_2(0)\bar{\bar{\delta}}_4(0)-\bar{\alpha}_4(0)\bar{\bar{\delta}}_2(0)=0$. But it is readily verified that this determinant is not zero. Therefore $A=D=0$. By virtue of the hypotheses made on the initial values, it follows from the second and third equations that

$$0=B\bar{\beta}_2(0)+C\bar{\bar{\beta}}_4(0), \qquad 0=B\bar{\bar{\gamma}}_2(0)+C\bar{\gamma}_4(0),$$

and hence $B=C=0$; consequently

$$\left.\begin{aligned} \rho_1(\tau)-\rho_1(-\tau)&=0, & \rho_2(\tau)+\rho_2(-\tau)&=0,\\ \sigma_1(\tau)+\sigma_1(-\tau)&=0, & \sigma_2(\tau)-\sigma_2(-\tau)&=0. \end{aligned}\right\} \quad (91)$$

Since these equations are identities in τ, we have the following theorem:

Theorem I. *If $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function of τ, and if $\rho_2(0)=\sigma_1(0)=0$, then $\rho_1(\tau)$ and $\sigma_2(\tau)$ are even functions of τ, and $\rho_2(\tau)$ and $\sigma_1(\tau)$ are odd functions of τ.*

In the same way it can be shown that if $g(\tau)$ is odd and $f(\tau)$ is even and if $\rho_1(0)=\sigma_2(0)=0$, then ρ_1 and σ_2 are odd, and ρ_2 and σ_1 are even.

Let us suppose now that $g(\tau)$ contains only even multiples of τ, and that $f(\tau)$ contains only odd multiples of τ. The general form of the solution will be the same as (87), and ξ_1, ξ_2, η_1, and η_2 satisfy the differential equations

$$\left.\begin{aligned} \frac{d}{d\tau}\xi_1(\tau)&=\xi_2(\tau), & \frac{d}{d\tau}\xi_2(\tau)&=\bar{\bar{\theta}}_2\,\xi_1(\tau)+\bar{\theta}_3\,\eta_1(\tau)+\bar{\bar{g}}(\tau),\\ \frac{d}{d\tau}\eta_1(\tau)&=\eta_2(\tau), & \frac{d}{d\tau}\eta_2(\tau)&=\bar{\theta}_3\,\xi_1(\tau)+\bar{\bar{\theta}}_4\,\eta_1(\tau)+\bar{f}(\tau). \end{aligned}\right\} \quad (92)$$

Let us denote $\xi_i(\tau+\pi)$ by $\xi_i'(\tau)$ and $\eta_i(\tau+\pi)$ by $\eta_i'(\tau)$. Then by changing τ into $\tau+\pi$ in (92), we have

$$\left.\begin{aligned} \frac{d}{d\tau}\xi_1'(\tau)&=\xi_2'(\tau), & \frac{d}{d\tau}\xi_2'(\tau)&=+\bar{\bar{\theta}}_2\,\xi_1'(\tau)-\bar{\theta}_3\,\eta_1'(\tau)+\bar{\bar{g}}(\tau),\\ \frac{d}{d\tau}\eta_1'(\tau)&=\eta_2'(\tau), & \frac{d}{d\tau}\eta_2'(\tau)&=-\bar{\theta}_3\,\xi_1'(\tau)+\bar{\bar{\theta}}_4\,\eta_1'(\tau)-\bar{f}(\tau). \end{aligned}\right\} \quad (93)$$

From equations (92) and (93) it follows that

$$\left.\begin{aligned} \frac{d}{d\tau}[\xi_1-\xi_1']&=[\xi_2-\xi_2'], & \frac{d}{d\tau}[\xi_2-\xi_2']&=\bar{\bar{\theta}}_2[\xi_1-\xi_1']+\bar{\theta}_3[\eta_1+\eta_1'],\\ \frac{d}{d\tau}[\eta_1+\eta_1']&=[\eta_2+\eta_2'], & \frac{d}{d\tau}[\eta_2+\eta_2']&=\bar{\theta}_3[\xi_1-\xi_1']+\bar{\bar{\theta}}_4[\eta_1+\eta_1']. \end{aligned}\right\} \quad (94)$$

The solutions of these equations, which have the same form as (84), are

$$
\left.\begin{aligned}
\xi_1-\xi_1' &= A\overline{\alpha}_2(\tau)+B\overline{\alpha}_1(\tau)+C\overline{\overline{\alpha}}_3(\tau)+D[\overline{\overline{\alpha}}_4(\tau)+\tau\overline{\overline{\alpha}}_3(\tau)],\\
\xi_2-\xi_2' &= A\overline{\beta}_1(\tau)+B\overline{\beta}_2(\tau)+C\overline{\overline{\beta}}_4(\tau)+D[\overline{\overline{\beta}}_3(\tau)+\tau\overline{\overline{\beta}}_4(\tau)],\\
\eta_1+\eta_1' &= A\overline{\overline{\gamma}}_1(\tau)+B\overline{\overline{\gamma}}_2(\tau)+C\overline{\gamma}_4(\tau)+D[\overline{\gamma}_3(\tau)+\tau\overline{\gamma}_4(\tau)],\\
\eta_2+\eta_2' &= A\overline{\overline{\delta}}_2(\tau)+B\overline{\overline{\delta}}_1(\tau)+C\overline{\delta}_3(\tau)+D[\overline{\delta}_4(\tau)+\tau\overline{\delta}_3(\tau)].
\end{aligned}\right\}\tag{95}
$$

On forming these expressions directly from (87), we get

$$
\left.\begin{aligned}
\xi_1-\xi_1' &= \omega_1(\tau)-\omega_1(\tau+\pi)-\left(a\pi+\tfrac{1}{2}b\pi^2\right)\overline{\overline{\alpha}}_3-b\pi(\tau\overline{\overline{\alpha}}_3+\overline{\overline{\alpha}}_4),\\
\xi_2-\xi_2' &= \omega_2(\tau)-\omega_2(\tau+\pi)-\left(a\pi+\tfrac{1}{2}b\pi^2\right)\overline{\overline{\beta}}_4-b\pi(\tau\overline{\overline{\beta}}_4+\overline{\overline{\beta}}_3),\\
\eta_1+\eta_1' &= \omega_3(\tau)+\omega_3(\tau+\pi)-\left(a\pi+\tfrac{1}{2}b\pi^2\right)\overline{\gamma}_4-b\pi(\tau\overline{\gamma}_4+\overline{\gamma}_3),\\
\eta_2+\eta_2' &= \omega_4(\tau)+\omega_4(\tau+\pi)-\left(a\pi+\tfrac{1}{2}b\pi^2\right)\overline{\delta}_3-b\pi(\tau\overline{\delta}_3+\overline{\delta}_4).
\end{aligned}\right\}\tag{96}
$$

A comparison of equations (95) and (96) shows that

$$
A=B=0, \qquad C=-\left(a\pi+\tfrac{1}{2}b\pi^2\right), \qquad D=-b\pi,
$$

$$
\begin{aligned}
\omega_1(\tau)-\omega_1(\tau+\pi)&=0, \qquad & \omega_3(\tau)+\omega_3(\tau+\pi)&=0,\\
\omega_2(\tau)-\omega_2(\tau+\pi)&=0, \qquad & \omega_4(\tau)+\omega_4(\tau+\pi)&=0.
\end{aligned}
$$

Therefore $\omega_1(\tau)$ and $\omega_2(\tau)$ contain only even multiples of τ, while $\omega_3(\tau)$ and $\omega_4(\tau)$ contain only odd multiples of τ, and by carrying this result into equation (87), we have

$$
\left.\begin{aligned}
\xi_1&=\overline{\overline{\omega}}_1(\tau)+a\tau\overline{\overline{\alpha}}_3+b\left[\tfrac{1}{2}\tau^2\,\overline{\overline{\alpha}}_3+\tau\,\overline{\overline{\alpha}}_4\right], &\quad \eta_1&=\overline{\omega}_3(\tau)+a\tau\overline{\gamma}_4+b\left[\tfrac{1}{2}\tau^2\,\overline{\gamma}_4+\tau\,\overline{\gamma}_3\right],\\
\xi_2&=\overline{\overline{\omega}}_2(\tau)+a\tau\overline{\overline{\beta}}_4+b\left[\tfrac{1}{2}\tau^2\,\overline{\overline{\beta}}_4+\tau\,\overline{\overline{\beta}}_3\right], &\quad \eta_2&=\overline{\omega}_4(\tau)+a\tau\overline{\delta}_3+b\left[\tfrac{1}{2}\tau^2\,\overline{\delta}_3+\tau\,\overline{\delta}_4\right].
\end{aligned}\right\}\tag{97}
$$

These results may be expressed in

Theorem II. *If $g(\tau)$ contains only even multiples of τ and $f(\tau)$ contains only odd multiples of τ, then ξ_1 and ξ_2 contain only even multiples of τ, and η_1 and η_2 contain only odd multiples of τ.*

If in addition to the above hypotheses we suppose that $g(\tau)$ is an even function of τ and $f(\tau)$ is an odd function of τ, then ξ_1 and η_2 are even functions and ξ_2 and η_1 are odd functions; therefore $b=0$. But if $g(\tau)$ is an odd function and $f(\tau)$ is an even function of τ, then ξ_1 and η_2 are odd functions and ξ_2 and η_1 are even functions, and in this case $a=0$.

In the same manner we can prove

Theorem III. *If $g(\tau)$ contains only odd multiples of τ and $f(\tau)$ contains only even multiples of τ, then ξ_1 and ξ_2 contain only odd multiples of τ, and η_1 and η_2 contain only even multiples of τ. Furthermore ξ_1, ξ_2, η_1, and η_2 are periodic with the period 2π.*

If $g(\tau)$ is of the form $\Sigma\, m_j \cos(j \pm \lambda)\tau$ and also if $f(\tau)$ has the form $\Sigma\, n_j \sin(j \pm \lambda)\tau$, then, since $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, the form of the solution is, by §§30 and 31,

$$\left.\begin{aligned}
\xi_1 &= \sum_\kappa p_\kappa^{(1)} \cos(\kappa \pm \lambda)\tau + \sum_\kappa p_\kappa^{(2)} \sin(\kappa \pm \lambda)\tau + a^{(1)}\tau\,\overline{\alpha}_1(\tau) + a^{(2)}\tau\,\overline{\alpha}_2(\tau),\\
\xi_2 &= \sum_\kappa q_\kappa^{(2)} \cos(\kappa \pm \lambda)\tau + \sum_\kappa q_\kappa^{(1)} \sin(\kappa \pm \lambda)\tau + a^{(1)}\tau\,\overline{\beta}_2(\tau) + a^{(2)}\tau\,\overline{\beta}_1(\tau),\\
\eta_1 &= \sum_\kappa r_\kappa^{(2)} \cos(\kappa \pm \lambda)\tau + \sum_\kappa r_\kappa^{(1)} \sin(\kappa \pm \lambda)\tau + a^{(1)}\tau\,\overline{\overline{\gamma}}_2(\tau) + a^{(2)}\tau\,\overline{\overline{\gamma}}_1(\tau),\\
\eta_2 &= \sum_\kappa s_\kappa^{(1)} \cos(\kappa \pm \lambda)\tau + \sum_\kappa s_\kappa^{(2)} \sin(\kappa \pm \lambda)\tau + a^{(1)}\tau\,\overline{\overline{\delta}}_1(\tau) + a^{(2)}\tau\,\overline{\overline{\delta}}_2(\tau);
\end{aligned}\right\} \quad (98)$$

but, since $g(\tau)$ is an even function and $f(\tau)$ is an odd function, ξ_1 and η_2 are even functions, and ξ_2 and η_1 are odd functions. Therefore all the coefficients in (98) which have the superfix (2) are zero. But if $g(\tau)$ were an odd function of τ and $f(\tau)$ an even function, then all the coefficients in (98) which have the superfix (1) would be zero. Therefore

Theorem IV. *If $g(\tau)$ has the form $\Sigma m_j \cos(j \pm \lambda)\tau$, and if $f(\tau)$ has the form $\Sigma n_j \sin(j \pm \lambda)\tau$, where $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equation, then the particular solution has the form*

$$\left.\begin{aligned}
\xi_1 &= \sum_a p_a \cos(a \pm \lambda)\tau + A\tau\,\overline{\alpha}_1(\tau), & \eta_1 &= \sum_c p_c \sin(c \pm \lambda)\tau + A\tau\,\overline{\overline{\gamma}}_2(\tau),\\
\xi_2 &= \sum_b p_b \sin(b \pm \lambda)\tau + A\tau\,\overline{\beta}_2(\tau), & \eta_2 &= \sum_d p_d \cos(d \pm \lambda)\tau + A\tau\,\overline{\overline{\delta}}_1(\tau).
\end{aligned}\right\} \quad (99)$$

From similar reasoning we have

Theorem V. *If $g(\tau)$ has the form $\Sigma m_j \sin(j \pm \lambda)\tau$ and if $f(\tau)$ has the form $\Sigma n_j \cos(j \pm \lambda)\tau$, where $\pm\sqrt{-1}\,\lambda$ are the characteristic exponents of the homogeneous equations, then the particular solution has the form*

$$\left.\begin{aligned}
\xi_1 &= \sum_a r_a \sin(a \pm \lambda)\tau + B\tau\,\overline{\alpha}_2(\tau), & \eta_1 &= \sum_c r_c \cos(c \pm \lambda)\tau + B\tau\,\overline{\overline{\gamma}}_1(\tau),\\
\xi_2 &= \sum_b r_b \cos(b \pm \lambda)\tau + B\tau\,\overline{\beta}_1(\tau), & \eta_2 &= \sum_d r_d \sin(d \pm \lambda)\tau + B\tau\,\overline{\overline{\delta}}_2(\tau).
\end{aligned}\right\} \quad (100)$$

It is understood that a, b, c, d, and j are integers in Theorems IV and V.

72. Integration of the Differential Equations.—It will be convenient hereafter to use as notation for the fundamental set of solutions, (82) and (83), of the equations of variation

$$\left.\begin{aligned}\rho&=A\,\alpha_2(\tau)+B\,\alpha_1(\tau)+C\,\alpha_3(\tau)+D\,[\alpha_4(\tau)+\tau\,\alpha_3(\tau)],\\ \sigma&=A\,\gamma_1(\tau)+B\,\gamma_2(\tau)+C\,\gamma_4(\tau)+D\,[\gamma_3(\tau)+\tau\,\gamma_4(\tau)],\end{aligned}\right\}\tag{101}$$

where the α and γ-functions are characterized as follows:

$\alpha_2(\tau)$	involves only terms of the form	$\cos\,[(2n+1)\pm\lambda]\,\tau$,
$\gamma_1(\tau)$	" " " " "	$\sin\,[\,2n\qquad\pm\lambda]\,\tau$,
$\alpha_1(\tau)$	" " " " "	$\sin\,[(2n+1)\pm\lambda]\,\tau$,
$\gamma_2(\tau)$	" " " " "	$\cos\,[\,2n\qquad\pm\lambda]\,\tau$,
$\alpha_3(\tau)$	" " " " "	$\sin\,[\,2n\qquad+0]\,\tau$,
$\gamma_4(\tau)$	" " " " "	$\cos\,[(2n+1)+0]\,\tau$,
$\alpha_4(\tau)$	" " " " "	$\cos\,[\,2n\qquad+0]\,\tau$,
$\gamma_3(\tau)$	" " " " "	$\sin\,[(2n+1)+0]\,\tau$.

It will be also convenient to write the differential equations (50) for ρ and σ in the form

$$\left.\begin{aligned}\rho''+\theta_2\,\rho+\theta_3\,\sigma&=\theta_{001}\,\epsilon+\theta_{101}\,\rho\,\epsilon+\theta_{200}\,\rho^2+\theta_{110}\,\rho\,\sigma+\theta_{020}\,\sigma^2+\cdots,\\ \sigma''+\theta_4\,\sigma+\theta_3\,\rho&=\qquad\qquad\quad\;\overline{\theta}_{200}\,\rho^2+\overline{\theta}_{110}\,\rho\,\sigma+\overline{\theta}_{020}\,\sigma^2+\cdots,\end{aligned}\right\}\tag{102}$$

where all the θ's are periodic with the period 2π; θ_2 and θ_4 contain only cosines of even multiples of τ; and θ_3 contains only sines of odd multiples of τ. On the right side of the first equation the coefficients of terms carrying odd powers of σ contain only sines of odd multiples of τ, while all the other coefficients contain only cosines of even multiples of τ. In the second equation odd powers of σ have coefficients involving only cosines of even multiples of τ, while all other coefficients contain only sines of odd multiples of τ.

The initial conditions are

$$\rho(0)=\alpha,\qquad \rho'(0)=0,\qquad \sigma(0)=0,\qquad \sigma'(0)=\delta.$$

We will integrate equations (102) as power series in α, δ, and ϵ, with τ entering in the coefficients. We know that these series are convergent for any arbitrarily chosen interval for τ, $0\gtreqqless\tau\leqq T$, provided $|\alpha|$, $|\delta|$, and $|\epsilon|$ are sufficiently small. The equations of variation involve the period $2\pi/\lambda$. The solutions are not periodic unless λ is rational. Hence the constants upon which λ depends must be chosen in advance, so that λ shall be rational. We will suppose then that $\lambda=j/\kappa$, where j and κ are relatively prime integers. Then the first two solutions of the equations of variation are periodic with the period $2\kappa\pi$.

Since ρ and σ are expansible in powers of α, δ, and ϵ, we may write

$$\rho=\rho_{100}\,\alpha+\rho_{010}\,\delta+\rho_{001}\,\epsilon+\rho_{200}\,\alpha^2+\rho_{110}\,\alpha\delta+\rho_{020}\,\delta^2+\rho_{101}\,\alpha\epsilon+\rho_{011}\,\delta\epsilon+\rho_{002}\,\epsilon^2+\cdots,$$

$$\sigma=\sigma_{100}\,\alpha+\sigma_{010}\,\delta+\sigma_{001}\,\epsilon+\sigma_{200}\,\alpha^2+\sigma_{110}\,\alpha\delta+\sigma_{020}\,\delta^2+\sigma_{101}\,\alpha\epsilon+\sigma_{011}\,\delta\epsilon+\sigma_{002}\,\epsilon^2+\cdots.$$

The differential equations for the ρ_{ijk} and σ_{ijk} are obtained by substituting these expressions in (102) and equating the coefficients of similar powers of the parameters.

Coefficients of α. The coefficients of α are defined by

$$\rho_{100}''+\theta_2\,\rho_{100}+\theta_3\,\sigma_{100}=0, \qquad \sigma_{100}''+\theta_4\,\sigma_{100}+\theta_3\,\rho_{100}=0. \tag{103}$$

The solution of these equations, which are the same as the equations of variation, is

$$\rho_{100}=A\,\alpha_2(\tau)+B\,\alpha_1(\tau)+C\,\alpha_3(\tau)+D[\alpha_4(\tau)+\tau\,\alpha_3(\tau)],$$
$$\sigma_{100}=A\gamma_1(\tau)+B\gamma_2(\tau)+C\gamma_4(\tau)+D[\gamma_3(\tau)+\tau\,\gamma_4(\tau)].$$

In order to satisfy the initial conditions we must have, at $\tau=0$,

$$\rho_{100}=1, \qquad \rho_{100}'=0, \qquad \sigma_{100}=0, \qquad \sigma_{100}'=0.$$

From these conditions we find that

$$\left.\begin{aligned} A\alpha_2(0)+D\alpha_4(0)&=1, & B\alpha_1'(0)+C\alpha_3'(0)&=0,\\ A\gamma_1'(0)+D[\gamma_3'(0)+\gamma_4(0)]&=0, & B\gamma_2(0)+C\gamma_4(0)&=0. \end{aligned}\right\} \tag{104}$$

The solution of these conditional equations is

$$\left.\begin{aligned} A=\frac{\gamma_3'(0)+\gamma_4(0)}{\Delta}=A_{100}^{(1)}, \qquad D=-\frac{\gamma_1'(0)}{\Delta}=A_{100}^{(2)}, \qquad B=C=0.\\ \Delta=\alpha_2(0)[\gamma_3'(0)+\gamma_4(0)]-\alpha_4(0)\,\gamma_1'(0). \end{aligned}\right\} \tag{105}$$

Hence the solution of equations (103) takes the form

$$\left.\begin{aligned} \rho_{100}&=A_{100}^{(1)}\,\alpha_2(\tau)+A_{100}^{(2)}[\alpha_4(\tau)+\tau\,\alpha_3(\tau)],\\ \sigma_{100}&=A_{100}^{(1)}\gamma_1(\tau)+A_{100}^{(2)}[\gamma_3(\tau)+\tau\gamma_4(\tau)]. \end{aligned}\right\} \tag{106}$$

Coefficients of δ. The terms of the first degree in δ must satisfy

$$\rho_{010}''+\theta_2\,\rho_{010}+\theta_3\,\sigma_{010}=0, \qquad \sigma_{010}''+\theta_4\,\sigma_{010}+\theta_3\,\rho_{010}=0. \tag{107}$$

These equations are the same as (103), and from the initial conditions we must have, at $\tau=0$,

$$\rho_{010}=0, \qquad \rho_{010}'=0, \qquad \sigma_{010}=0, \qquad \sigma_{010}'=1.$$

The solutions of equations (107) are therefore

$$\left.\begin{aligned} \rho_{010}&=A_{010}^{(1)}\,\alpha_2(\tau)+A_{010}^{(2)}\,\alpha_4[(\tau)+\tau\,\alpha_3(\tau)],\\ \sigma_{010}&=A_{010}^{(1)}\gamma_1(\tau)+A_{010}^{(2)}\gamma_3[(\tau)+\tau\,\gamma_4(\tau)], \end{aligned}\right\} \tag{108}$$

where

$$A_{010}^{(1)}=-\frac{\alpha_4(0)}{\Delta}, \qquad A_{010}^{(2)}=+\frac{\alpha_2(0)}{\Delta}.$$

Coefficients of ϵ. The differential equations for these terms are

$$\rho''_{001}+\theta_2\,\rho_{001}+\theta_3\,\sigma_{001}=\theta_{001}\,,\qquad \sigma''_{001}+\theta_4\,\sigma_{001}+\theta_3\,\rho_{001}=0. \tag{109}$$

The right member, θ_{001}, is a periodic function of τ with the period 2π. Furthermore, it involves only cosines of even multiples of τ. Consequently, by Theorem II of §71, the solution has the form

$$\left.\begin{aligned}\rho_{001}&=A\,\alpha_2(\tau)+B\,\alpha_1(\tau)+C\,\alpha_3(\tau)+D\,[\alpha_4(\tau)+\tau\,\alpha_3(\tau)]+\alpha_5(\tau)+a\,\tau\,\alpha_3(\tau),\\ \sigma_{001}&=A\gamma_1(\tau)+B\gamma_2(\tau)+C\gamma_4(\tau)+D\,[\gamma_3(\tau)+\tau\,\gamma_4(\tau)]+\gamma_5(\tau)+a\,\tau\,\gamma_4(\tau),\end{aligned}\right\} \tag{110}$$

where a is a constant depending on θ_{001}; $\alpha_5(\tau)$ is a cosine series involving only even multiples of τ; and $\gamma_5(\tau)$ involves only sines of odd multiples of τ.

From the initial conditions it follows that ρ_{001}, σ_{001}, ρ'_{001}, and σ'_{001} all vanish at $\tau=0$. On determining the constants of integration so as to satisfy these conditions, the solution is

$$\left.\begin{aligned}\rho_{001}&=A^{(1)}_{001}\,\alpha_2(\tau)+A^{(2)}_{001}\,[\alpha_4(\tau)+\tau\alpha_3(\tau)]+\alpha_6(\tau),\\ \sigma_{001}&=A^{(1)}_{001}\gamma_1(\tau)+A^{(2)}_{001}\,[\gamma_3(\tau)+\tau\gamma_4(\tau)]+\gamma_6(\tau),\end{aligned}\right\} \tag{111}$$

where

$$\begin{aligned}A^{(1)}_{001}&=\frac{1}{\Delta}\Big[\alpha_4(0)\,(\gamma'_5(0)+a\,\gamma_4(0))-\alpha_5(0)\,(\gamma'_3(0)+\gamma_4(0))\Big]\\ &=\frac{\alpha_4(0)\gamma'_6(0)-\alpha_6(0)(\gamma'_3(0)+\gamma_4(0))}{\Delta}\,,\\ A^{(2)}_{001}&=a+\frac{1}{\Delta}\Big[\alpha_5(0)\,\gamma'_1(0)-\alpha_2(0)\,(\gamma'_5(0)+a\,\gamma_4(0))\Big]\\ &=\frac{\alpha_6(0)\gamma'_1(0)-\alpha_2(0)\gamma'_6(0)}{\Delta}\,,\end{aligned}$$

$$\alpha_6(\tau)=\alpha_5(\tau)-a\alpha_4(\tau),\qquad \gamma_6(\tau)=\gamma_5(\tau)-a\gamma_3(\tau).$$

It will be seen at the end of §73 that the value of $\alpha_5(\tau)$ for $\tau=0$ plays an important rôle, and it is necessary for us to verify that it does not vanish. By hypothesis, $\alpha_5(\tau)$ is the periodic part of the solution for ρ_{001} in the differential equations (109). Let us put in these equations

$$\rho_{001}=\varphi(\tau)+a\tau\alpha_3(\tau),\qquad \sigma_{001}=\psi(\tau)+a\tau\gamma_4(\tau),$$

where φ and ψ are the periodic parts. We find

$$\begin{aligned}&\varphi''+\theta_2\,\varphi+\theta_3\,\psi=-2a\alpha'_3(\tau)+1+\Big[\Big(-3\theta_1^2+\tfrac{3}{4}\beta^2\Big)-\tfrac{3}{4}\beta^2\cos 2\tau\Big]\mu^2+\cdots,\\ &\psi''+\theta_4\,\psi+\theta_3\,\varphi=-2a\gamma'_4(\tau);\end{aligned}$$

or, using the explicit values of $\alpha'_3(\tau)$ and $\gamma'_4(\tau)$,

$$\begin{aligned}&\varphi''+\theta_2\,\varphi+\theta_3\,\psi=1+\Big[\Big(-3\theta_1^2+\tfrac{3}{4}\beta_1^2\Big)-\Big(2a_0+\tfrac{3}{4}\beta^2\Big)\cos 2\tau\Big]\mu^2+\cdots,\\ &\psi''+\theta_4\psi+\theta_3\,\varphi=-2a_0\beta\sin\tau\cdot\mu+\cdots.\end{aligned}$$

In these last equations we have put

$$a=a_0+a_2\mu^2+\cdots.$$

Let us put now

$$\varphi=\varphi_0+\varphi_2\mu^2+\cdots,\qquad \psi=\psi_1\mu+\psi_3\mu^3+\cdots,$$

and integrate as a power series in μ, having in mind that φ and ψ must be periodic. We find

$$\varphi_0''+\varphi_0=1,\qquad \varphi_0=1+c_0\cos\tau,$$
$$\psi_1''+\psi_1=(3+2a_0)\beta\sin\tau+\frac{3}{2}\beta c_0\sin 2\tau.$$

Since ψ_1 is periodic we must put $a_0=-3/2$, and then, after integrating, we have $\psi_1=c_1\sin\tau-1/2\,\beta c_0\sin 2\tau$. From the coefficient of μ^2 we obtain

$$\varphi_2''+\varphi_2=-\frac{1}{4}c_0\beta^2\cos\tau+\text{other terms}.$$

Since φ_2 is periodic we must take c_0 equal to zero. Therefore

$$\varphi=\alpha_5(\tau)=1+\text{power series in }\mu^2,\qquad a=-\frac{3}{2}+\text{power series in }\mu^2.$$

Consequently

$$\alpha_6(\tau)=\alpha_5(\tau)-a\,\alpha_4(\tau)=1+\text{power series in }\mu^2,\tag{112}$$

which does not vanish for $\tau=0$.

Coefficients of α^2. The terms of the second degree in α are defined by

$$\rho_{200}''+\theta_2\rho_{200}+\theta_3\sigma_{200}=R_{200},\qquad \sigma_{200}''+\theta_4\rho_{200}+\theta_3\rho_{200}=S_{200},\tag{113}$$

where the right members have the following expressions:

$$\begin{aligned}R_{200}=&+A_{100}^{(1)2}[\theta_{200}\alpha_2^2+\theta_{110}\alpha_2\gamma_1+\theta_{020}\gamma_1^2]\\&+A_{100}^{(1)}A_{100}^{(2)}[2\theta_{200}\alpha_2(\tau\alpha_3+\alpha_4)+\theta_{110}\{\alpha_2(\tau\gamma_4+\gamma_3)+\gamma_1(\tau\alpha_3+\alpha_4)\}+2\theta_{020}\gamma_1(\tau\gamma_4+\gamma_3)]\\&+A_{100}^{(2)2}[\theta_{200}(\tau\alpha_3+\alpha_4)^2+\theta_{110}(\tau\alpha_3+\alpha_4)(\tau\gamma_4+\gamma_3)+\theta_{020}(\tau\gamma_4+\gamma_3)^2],\end{aligned}$$

$$\begin{aligned}S_{200}=&+A_{100}^{(1)2}[\bar\theta_{200}\alpha_2^2+\bar\theta_{110}\alpha_2\gamma_1+\bar\theta_{020}\gamma_1^2]\\&+A_{100}^{(1)}A_{100}^{(2)}[2\bar\theta_{200}\alpha_2(\tau\alpha_3+\alpha_4)+\bar\theta_{110}\{\alpha_2(\tau\gamma_4+\gamma_3)+\gamma_1(\tau\alpha_3+\alpha_4)\}+2\bar\theta_{020}\gamma_1(\tau\gamma_4+\gamma_3)]\\&+A_{100}^{(2)2}[\bar\theta_{200}(\tau\alpha_3+\alpha_4)^2+\bar\theta_{110}(\tau\alpha_3+\alpha_4)(\tau\gamma_4+\gamma_3)+\bar\theta_{020}(\tau\gamma_4+\gamma_3)^2].\end{aligned}$$

By the initial conditions ρ_{200}, σ_{200}, and their first derivatives vanish at $\tau=0$.

Since the equations (113) are linear, the solutions have the form

$$\left.\begin{aligned}\rho_{200}=&A\alpha_2(\tau)+B\alpha_1(\tau)+C\alpha_3(\tau)+D[\alpha_4(\tau)+\tau\alpha_3(\tau)]\\&+A_{100}^{(1)2}\varphi_1(\tau)+A_{100}^{(1)}A_{100}^{(2)}\varphi_2(\tau)+A_{100}^{(2)2}\varphi_3(\tau),\\\sigma_{200}=&A\gamma_1(\tau)+B\gamma_2(\tau)+C\gamma_4(\tau)+D[\gamma_3(\tau)+\tau\gamma_4(\tau)]\\&+A_{100}^{(1)2}\psi_1(\tau)+A_{100}^{(1)}A_{100}^{(2)}\psi_2(\tau)+A_{100}^{(2)2}\psi_3(\tau).\end{aligned}\right\}\tag{114}$$

Upon imposing the initial conditions, we find

$$B=C=0,$$

$$A=\frac{\psi_1'(0)\alpha_4(0)-\varphi_1(0)[\gamma_4(0)+\gamma_3'(0)]}{\Delta}A_{100}^{(1)2}+\frac{\psi_2'(0)\alpha_4(0)-\varphi_2(0)[\gamma_4(0)+\gamma_3'(0)]}{\Delta}A_{100}^{(1)}A_{100}^{(2)}$$
$$+\frac{\psi_3'(0)\alpha_4(0)-\varphi_3(0)[\gamma_4(0)+\gamma_3'(0)]}{\Delta}A_{100}^{(2)2},$$

$$D=\frac{\varphi_1(0)\gamma_1'(0)-\psi_1'(0)\alpha_2(0)}{\Delta}A_{100}^{(1)2}+\frac{\varphi_2(0)\gamma_1'(0)-\psi_2'(0)\alpha_2(0)}{\Delta}A_{100}^{(1)}A_{100}^{(2)}$$
$$+\frac{\varphi_3(0)\gamma_1'(0)-\psi_3'(0)\alpha_2(0)}{\Delta}A_{100}^{(2)2},$$

where

$$\Delta=\alpha_2(0)[\gamma_4(0)+\gamma_3'(0)]-\gamma_1'(0)\alpha_4(0).$$

On substituting these values in (141), we have for the solutions

$$\left.\begin{aligned}\rho_{200}&=A_{100}^{(1)2}x_1(\tau)+A_{100}^{(1)}A_{100}^{(2)}x_2(\tau)+A_{100}^{(2)2}x_3(\tau),\\ \sigma_{200}&=A_{100}^{(1)2}y_1(\tau)+A_{100}^{(1)}A_{100}^{(2)}y_2(\tau)+A_{100}^{(2)2}y_3(\tau),\end{aligned}\right\}\qquad(115)$$

where

$$x_1(\tau)=\varphi_1(\tau)+\frac{\psi_1'(0)\alpha_4(0)-\varphi_1(0)[\gamma_4(0)+\gamma_3'(0)]}{\Delta}\alpha_2(\tau)$$
$$+\frac{\varphi_1(0)\gamma_1'(0)-\psi_1'(0)\alpha_2(0)}{\Delta}[\alpha_4(\tau)+\tau\alpha_3(\tau)],$$

$$y_1(\tau)=\psi_1(\tau)+\frac{\psi_1'(0)\alpha_4(0)-\varphi_1(0)[\gamma_4(0)+\gamma_3'(0)]}{\Delta}\gamma_1(\tau)$$
$$+\frac{\varphi_1(0)\gamma_1'(0)-\psi_1'(0)\alpha_2(0)}{\Delta}[\gamma_3(\tau)+\tau\gamma_4(\tau)];$$

and similar expressions for x_2, y_2, x_3, and y_3, the values of which we shall find we do not need. The properties of x_1 and y_1 are known with the exception of φ_1 and ψ_1, which we will now investigate. The functions φ_1 and ψ_1 are those portions of the solution of the differential equations which depend upon the coefficients of $A_{100}^{(1)2}$. These coefficients are homogeneous of the second degree in $\alpha_2(\tau)$ and $\gamma_1(\tau)$.

In R_{200} and S_{200} the expressions θ_{200}, $\overline{\theta}_{110}$, and θ_{020} contain only cosines of even multiples of τ; $\overline{\theta}_{200}$, θ_{110}, and $\overline{\theta}_{020}$ contain only sines of odd multiples of τ; $\alpha_2(\tau)$ has the form $\alpha_2=\sum_{n=0}^{\infty}a_n\cos[(2n+1)\pm\lambda]\tau$; $\gamma_1(\tau)$ has the form $\gamma_1=\sum_{n=0}^{\infty}b_n\sin[2n\pm\lambda]\tau$. Consequently, so far as the coefficient of $A_{100}^{(1)2}$ is concerned, R_{200} and S_{200} have the form

$$R_{200}=\sum_{n=0}^{\infty}a_n^{(1)}\cos 2n\tau+\sum_{n=0}^{\infty}a_n^{(2)}\cos[2n\pm 2\lambda]\tau,$$

$$S_{200}=\sum_{n=0}^{\infty}b_n^{(1)}\sin(2n+1)\tau+\sum_{n=0}^{\infty}b_n^{(2)}\sin[(2n+1)\pm 2\lambda]\tau.$$

By §30 terms involving multiples of $\lambda\tau$ give rise only to periodic terms in the solution. By §31 those parts of R_{200} and S_{200} which are independent of λ give rise to terms in the solution which have the form

$$\rho=p_1(\tau)+c_1'\tau\alpha_3(\tau), \qquad \sigma=p_2(\tau)+c_1'\tau\gamma_4(\tau),$$

where $p_1(\tau)$ and $p_2(\tau)$ are periodic with the period 2π. Consequently the functions $x_1(\tau)$ and $y_1(\tau)$ have the form

$$x_1(\tau)=P_1(\tau)+c_1\tau\alpha_3(\tau), \qquad y_1(\tau)=P_2(\tau)+c_1\tau\gamma_4(\tau), \tag{116}$$

where $P_1(\tau)$ and $P_2(\tau)$ are periodic with the period $2\kappa\pi$.

Coefficients of $\alpha\delta$. The differential equations for these terms are

$$\rho_{110}''+\theta_2\,\rho_{110}+\theta_3\,\sigma_{110}=R_{110}, \qquad \sigma_{110}''+\theta_4\,\sigma_{110}+\theta_3\,\rho_{110}=S_{110}, \tag{117}$$

where

$$\begin{aligned} R_{110}=\,&2A_{100}^{(1)}A_{010}^{(1)}[\theta_{200}\,\alpha_2^2+\theta_{110}\,\alpha_2\,\gamma_1+\theta_{020}\,\gamma_1^2]\\ &+[A_{100}^{(1)}A_{010}^{(2)}+A_{010}^{(1)}A_{100}^{(2)}][2\theta_{200}\,\alpha_2(\tau\alpha_3+\alpha_4)+\theta_{110}\{\gamma_1(\tau\alpha_3+\alpha_4)+\alpha_2(\tau\gamma_4+\gamma_3)\}\\ &\qquad\qquad+2\theta_{020}\,\gamma_1(\tau\gamma_4+\gamma_3)]\\ &+2A_{100}^{(2)}A_{010}^{(2)}[\theta_{200}(\tau\,\alpha_3+\alpha_4)^2+\theta_{110}(\tau\alpha_3+\alpha_4)(\tau\gamma_4+\gamma_3)+\theta_{020}(\tau\gamma_4+\gamma_3)^2]; \end{aligned}$$

and S_{110} is obtained from R_{110} by replacing θ_{ijk} by $\overline{\theta}_{ijk}$.

The functions R_{110} and S_{110} differ from R_{200} and S_{200} only in the constants A_{ijk}. The initial conditions impose the same conditional equations. Consequently the solutions differ only in the constants A_{ijk}, so that we can express them at once without computation in the form

$$\left.\begin{aligned} \rho_{110}&=2A_{100}^{(1)}A_{010}^{(1)}x_1(\tau)+[A_{100}^{(1)}A_{010}^{(2)}+A_{100}^{(2)}A_{010}^{(1)}]x_2(\tau)+2\,A_{100}^{(2)}A_{010}^{(2)}x_3(\tau),\\ \sigma_{110}&=2A_{100}^{(1)}A_{100}^{(1)}y_1(\tau)+[A_{100}^{(1)}A_{010}^{(2)}+A_{100}^{(2)}A_{010}^{(1)}]y_2(\tau)+2\,A_{100}^{(2)}A_{010}^{(2)}y_3(\tau), \end{aligned}\right\} \tag{118}$$

where the $x_i(\tau)$ and $y_i(\tau)$ are the same functions of τ as in (115).

Coefficients of δ^2. By symmetry with the coefficient of α^2, it is seen that

$$\left.\begin{aligned} \rho_{020}&=A_{010}^{(1)2}x_1(\tau)+A_{010}^{(1)}A_{010}^{(2)}x_2(\tau)+A_{010}^{(2)2}x_3(\tau),\\ \sigma_{020}&=A_{010}^{(1)2}y_1(\tau)+A_{010}^{(1)}A_{010}^{(2)}y_2(\tau)+A_{010}^{(2)2}y_3(\tau). \end{aligned}\right\} \tag{119}$$

Coefficients of ϵ^2. Since the coefficients of the first powers of α and δ are homogeneous of the first degree in the A's, the coefficients of α^2, $\alpha\delta$, and δ^2 are homogeneous of the second degree in the A's. The coefficients of the first power of ϵ are not homogeneous in the A's; hence the coefficients of the second power are not homogeneous. But if the functions α_6 and γ_6 were zero the coefficient of the first power of ϵ would be homogeneous, and therefore the second also. By symmetry, therefore, we can at once write down the terms involving the A's to the second degree. To these must be added terms in the first degree in the A's, and one term independent of the A's.

The differential equations for these terms are

$$\rho''_{002}+\theta_2\rho_{002}+\theta_3\sigma_{002}=R_{002}\,,\qquad\qquad \sigma''_{002}+\theta_4\sigma_{002}+\theta_3\rho_{002}=S_{002}\,,\qquad (120)$$

where

$$R_{002}=\theta_{200}\rho^2_{001}+\theta_{110}\rho_{001}\sigma_{001}+\theta_{020}\sigma^2_{001}+\theta_{101}\rho_{001}\,,$$
$$S_{002}=\overline{\theta}_{200}\rho^2_{001}+\overline{\theta}_{110}\rho_{001}\sigma_{001}+\overline{\theta}_{020}\sigma^2_{001}\,.$$

The terms involved in R_{002} are shown in the following table, where the coefficients of the constants given in the first line are the products of the functions in their respective columns and the functions of the same line in the last column. Thus, one of the coefficients of $A^{(1)2}_{001}$ is $\alpha_2\gamma_1\theta_{110}$, and this coefficient comes from the expansion of $\rho_{001}\sigma_{001}$.

Origin of term	$A^{(1)2}_{001}$	$A^{(1)}_{001}\ A^{(2)}_{001}$	$A^{(2)2}_{001}$	$A^{(1)}_{001}$	$A^{(2)}_{001}$	1	Multiplied by
ρ^2_{001}	α_2^2	$2\alpha_2(\tau\alpha_3+\alpha_4)$	$(\tau\alpha_3+\alpha_4)^2$	$2\alpha_2\alpha_6$	$2\alpha_6(\tau\alpha_3+\alpha_4)$	α_6^2	θ_{200}
$\rho_{001}\sigma_{001}$	$\alpha_2\gamma_1$	$\gamma_1(\tau\alpha_3+\alpha_4)+\alpha_2(\tau\gamma_4+\gamma_3)$	$(\tau\alpha_3+\alpha_4)(\tau\gamma_4+\gamma_3)$	$\gamma_1\alpha_6+\alpha_2\gamma_6$	$\alpha_6(\tau\gamma_4+\gamma_3)+\gamma_6(\tau\alpha_3+\alpha_4)$	$\alpha_6\gamma_6$	θ_{110}
σ^2_{001}	γ_1^2	$2\gamma_1(\tau\gamma_4+\gamma_3)$	$(\tau\gamma_4+\gamma_3)^2$	$2\gamma_1\gamma_6$	$2(\tau\gamma_4+\gamma_3)$	γ_6^2	θ_{020}
ρ_{001}				α_2	$(\tau\alpha_3+\alpha_4)$	α_6	θ_{101}

For the S_{002} it is necessary in the above table only to change the θ_{ijk} into $\overline{\theta}_{ijk}$ in the last column.

The solutions of equations (120) can be expressed in the form

$$\left.\begin{aligned}\rho_{002}&=A^{(1)2}_{001}x_1(\tau)+A^{(1)}_{001}A^{(2)}_{001}x_2(\tau)+A^{(2)2}_{001}x_3(\tau)+A^{(1)}_{001}x_4(\tau)+A^{(2)}_{001}x_5(\tau)+x_6(\tau),\\ \sigma_{002}&=A^{(1)2}_{001}y_1(\tau)+A^{(1)}_{001}A^{(2)}_{001}y_2(\tau)+A^{(2)2}_{001}y_3(\tau)+A^{(1)}_{001}y_4(\tau)+A^{(2)}_{001}y_5(\tau)+y_6(\tau),\end{aligned}\right\}\quad (121)$$

where x_1, x_2, x_3, y_1, y_2, and y_3 are the same functions as in (115).

The coefficients of $A^{(1)}_{001}$ in the differential equations (120) are homogeneous of the first degree in α_2 and γ_1, every term of which involves the first multiple of $\lambda\tau$. Hence the solutions for these terms, by Theorem IV, §71, involve non-periodic terms, and we can write

$$x_4(\tau)=P_3(\tau)+c_2\tau\alpha_1(\tau)+c_3\tau\alpha_3(\tau),\qquad y_4(\tau)=P_4(\tau)+c_2\tau\gamma_2(\tau)+c_3\tau\gamma_4(\tau),\quad (122)$$

where P_3 and P_4 are periodic with the period $2\kappa\pi$.

It is seen from the table that $x_6(\tau)$ and $y_6(\tau)$ do not involve the λ. They have, therefore, the form

$$x_6(\tau)=P_5(\tau)+c_4\tau\alpha_3(\tau),\qquad\qquad y_6(\tau)=P_6(\tau)+c_4\tau\gamma_4(\tau).\qquad (123)$$

It will be verified at the bottom of page 141 that we do not need to know the character of $x_5(\tau)$ and $y_5(\tau)$.

Coefficients of $\alpha\epsilon$. These terms satisfy the differential equations

$$\rho''_{101}+\theta_2\,\rho_{101}+\theta_3\,\sigma_{101}=R_{101}\,,\qquad \sigma''_{101}+\theta_4\,\sigma_{101}+\theta_3\,\rho_{101}=S_{101}\,, \tag{124}$$

where

$$R_{101}=\theta_{200}[2\rho_{100}\,\rho_{001}]+\theta_{110}[\rho_{100}\,\sigma_{001}+\rho_{001}\,\sigma_{100}]+\theta_{020}[2\sigma_{100}\,\sigma_{001}]+\theta_{101}\,\rho_{100}\,,$$
$$S_{101}=\overline{\theta}_{200}[2\rho_{100}\,\rho_{001}]+\overline{\theta}_{110}[\rho_{100}\,\sigma_{001}+\rho_{001}\,\sigma_{100}]+\overline{\theta}_{020}[2\sigma_{100}\,\sigma_{001}].$$

The following table for R_{101}, constructed like that on page 138, shows the character of the terms entering into these expressions:

Origin of term	$2A^{(1)}_{100}A^{(1)}_{001}$	$A^{(1)}_{100}A^{(2)}_{001}+A^{(2)}_{100}A^{(1)}_{001}$	$2A^{(2)}_{100}A^{(2)}_{001}$	$A^{(1)}_{100}$	$A^{(2)}_{100}$	Multiplied by
$\rho_{101}\,\rho_{010}$	α_2^2	$2\alpha_3(\tau\alpha_2+\alpha_4)$	$(\tau\alpha_2+\alpha_4)^2$	$2\alpha_2\,\alpha_6$	$2\alpha_6(\tau\alpha_2+\alpha_4)$	θ_{200}
$\rho_{100}\,\sigma_{001}+\rho_{001}\,\rho_{100}$	$\alpha_2\,\gamma_1$	$\gamma_1(\tau\alpha_3+\alpha_4)+\alpha_2(\tau\gamma_4+\gamma_3)$	$(\tau\alpha_2+\alpha_4)(\tau\gamma_4+\gamma_3)$	$\alpha_6\gamma_1+\alpha_2\,\gamma_6$	$\alpha_6(\tau\gamma_4+\gamma_2)+\gamma_6(\tau\alpha_2+\alpha_4)$	θ_{110}
$\sigma_{100}\,\sigma_{001}$	γ_1^2	$2\gamma_1(\tau\gamma_4+\gamma_3)$	$(\tau\gamma_4+\gamma_2)^2$	$2\gamma_1\,\gamma_6$	$2\gamma_6(\tau\gamma_4+\gamma_2)$	θ_{020}
ρ_{100}				α_2	$(\tau\alpha_2+\alpha_4)$	θ_{101}

In order to obtain S_{101} it is necessary only to change the θ_{ijk} into $\overline{\theta}_{ijk}$ in the last column of the table.

This table shows that R_{101} and S_{101} differ from R_{002} and S_{002} only in the constants A_{ijk}. Since the initial conditions impose the same conditional equations as for the coefficient of ϵ^2, the solution has the form

$$\left.\begin{aligned}\rho_{101}&=2A^{(1)}_{100}A^{(1)}_{001}\,x_1(\tau)+[A^{(1)}_{100}A^{(2)}_{001}+A^{(2)}_{100}A^{(1)}_{001}]\,x_2(\tau)\\&\qquad+2A^{(2)}_{100}A^{(2)}_{001}\,x_3(\tau)+A^{(1)}_{100}\,x_4(\tau)+A^{(2)}_{100}\,x_5(\tau),\\\sigma_{101}&=2A^{(1)}_{100}A^{(1)}_{001}\,y_1(\tau)+[A^{(1)}_{100}A^{(2)}_{001}+A^{(2)}_{100}A^{(1)}_{001}]\,y_2(\tau)\\&\qquad+2A^{(2)}_{100}A^{(2)}_{001}\,y_3(\tau)+A^{(1)}_{100}\,y_4(\tau)+A^{(2)}_{100}\,y_5(\tau),\end{aligned}\right\} \tag{125}$$

where the $x_i(\tau)$ and $y_i(\tau)$ are the same functions as in (121).

Coefficients of $\delta\epsilon$. These coefficients can be obtained by symmetry from the coefficient of $\alpha\epsilon$, and are

$$\left.\begin{aligned}\rho_{011}&=2A^{(1)}_{010}A^{(1)}_{001}\,x_1(\tau)+[A^{(1)}_{010}A^{(2)}_{001}+A^{(2)}_{010}A^{(1)}_{001}]\,x_2(\tau)\\&\qquad+2A^{(2)}_{010}A^{(2)}_{001}\,x_3(\tau)+A^{(1)}_{010}\,x_4(\tau)+A^{(2)}_{010}\,x_5(\tau),\\\sigma_{011}&=2A^{(1)}_{010}A^{(1)}_{001}\,y_1(\tau)+[A^{(1)}_{010}A^{(2)}_{001}+A^{(2)}_{010}A^{(1)}_{001}]\,y_2(\tau)\\&\qquad+2A^{(2)}_{010}A^{(2)}_{001}\,y_3(\tau)+A^{(1)}_{010}\,y_4(\tau)+A^{(2)}_{010}\,y_5(\tau).\end{aligned}\right\} \tag{126}$$

This concludes the computation of all terms up to the second order inclusive in α, δ, and ϵ. It is not necessary to carry the computation further.

73. Existence of Periodic Orbits having the Period $2\kappa\pi$.—We have chosen the initial conditions so that at $\tau=0$ the particle is crossing the ρ-axis orthogonally. It is obvious geometrically that if at any future time it again crosses the ρ-axis perpendicularly, the orbit will be a closed one and the motion in it will be periodic. The conditions that the particle shall cross the ρ-axis perpendicularly at $\tau=T$ are that at this epoch $\rho'=\sigma=0$.

The equations of variation have the period $2\kappa\pi$. Therefore we shall choose $T=\kappa\pi$. Since ρ is an even series in τ, and σ is an odd series, all the purely periodic terms in ρ' and σ are sines, and consequently vanish at $\tau=\kappa\pi$. The terms which do not vanish must carry τ as a factor. The conditions for periodicity give us the two equations

$$\left.\begin{aligned}
\rho'(\kappa\pi)=0=a_{100}\,\alpha+a_{010}\,\delta+a_{001}\,\epsilon+a_{200}\,\alpha^2+a_{110}\,\alpha\,\delta+a_{020}\,\delta^2+a_{011}\,\delta\,\epsilon& \\
+a_{101}\,\alpha\epsilon+a_{002}\,\epsilon^2+\cdots,& \\
\sigma(\kappa\pi)=0=b_{100}\,\alpha+b_{010}\,\delta+b_{001}\,\epsilon+b_{200}\,\alpha^2+b_{110}\,\alpha\,\delta+b_{020}\,\delta^2+b_{011}\,\delta\,\epsilon& \\
+b_{101}\,\alpha\epsilon+b_{002}\,\epsilon^2+\cdots,&
\end{aligned}\right\} \quad (127)$$

where a_{ijk} and b_{ijk} are the coefficients which were computed in §72. Their explicit values are as follows:

$$\left.\begin{aligned}
&a_{100}=A^{(2)}_{100}\,u, \qquad a_{010}=A^{(2)}_{010}\,u, \qquad a_{001}=A^{(2)}_{001}\,u,\\
&b_{100}=A^{(2)}_{100}\,v, \qquad b_{010}=A^{(2)}_{010}\,v, \qquad b_{001}=A^{(2)}_{001}\,v,\\
&a_{200}=A^{(1)2}_{100}\,\overline{x}_1+A^{(1)}_{100}\,A^{(2)}_{100}\,\overline{x}_2+A^{(2)2}_{100}\,\overline{x}_3\,,\\
&b_{200}=A^{(1)2}_{100}\,\overline{y}_1+A^{(1)}_{100}\,A^{(2)}_{100}\,\overline{y}_2+A^{(2)2}_{100}\,\overline{y}_3\,,\\
&a_{110}=2A^{(1)}_{100}\,A^{(1)}_{010}\,\overline{x}_1+[A^{(1)}_{100}\,A^{(2)}_{010}+A^{(2)}_{100}\,A^{(1)}_{010}]\,\overline{x}_2+2A^{(2)}_{100}\,A^{(2)}_{010}\,\overline{x}_3\,,\\
&b_{110}=2A^{(1)}_{100}\,A^{(1)}_{010}\,\overline{y}_1+[A^{(1)}_{100}\,A^{(2)}_{010}+A^{(2)}_{100}\,A^{(1)}_{010}]\,\overline{y}_2+2A^{(2)}_{100}\,A^{(2)}_{010}\,\overline{y}_3\,,\\
&a_{020}=A^{(1)2}_{010}\,\overline{x}_1+A^{(1)}_{010}\,A^{(2)}_{010}\,\overline{x}_2+A^{(2)2}_{010}\,\overline{x}_3\,,\\
&b_{020}=A^{(1)2}_{010}\,\overline{y}_1+A^{(1)}_{010}\,A^{(2)}_{010}\,\overline{y}_2+A^{(2)2}_{010}\,\overline{y}_3\,,\\
&a_{101}=2A^{(1)}_{100}A^{(1)}_{001}\overline{x}_1+[A^{(1)}_{100}A^{(2)}_{001}+A^{(2)}_{100}A^{(1)}_{001}]\overline{x}_2+2A^{(2)}_{100}A^{(2)}_{001}\overline{x}_3+A^{(1)}_{100}\overline{x}_4+A^{(2)}_{100}\overline{x}_5\,,\\
&b_{101}=2A^{(1)}_{100}A^{(1)}_{001}\overline{y}_1+[A^{(1)}_{100}A^{(2)}_{001}+A^{(2)}_{100}A^{(1)}_{001}]\overline{y}_2+2A^{(2)}_{100}A^{(2)}_{001}\overline{y}_3+A^{(1)}_{100}\overline{y}_4+A^{(2)}_{100}\overline{y}_5\,,\\
&a_{011}=2A^{(1)}_{010}A^{(1)}_{001}\overline{x}_1+[A^{(1)}_{010}A^{(2)}_{001}+A^{(2)}_{010}A^{(1)}_{001}]\overline{x}_2+2A^{(2)}_{010}A^{(2)}_{001}\overline{x}_3+A^{(1)}_{010}\overline{x}_4+A^{(2)}_{010}\overline{x}_5\,,\\
&b_{011}=2A^{(1)}_{010}A^{(1)}_{001}\overline{y}_1+[A^{(1)}_{010}A^{(2)}_{001}+A^{(2)}_{010}A^{(1)}_{001}]\overline{y}_2+2A^{(2)}_{010}A^{(2)}_{001}\overline{y}_3+A^{(1)}_{010}\overline{y}_4+A^{(2)}_{010}\overline{y}_5\,,\\
&a_{002}=A^{(1)2}_{001}\,\overline{x}_1+A^{(1)}_{001}\,A^{(2)}_{001}\,\overline{x}_2+A^{(2)2}_{001}\,\overline{x}_3+A^{(1)}_{001}\overline{x}_4+A^{(2)}_{001}\overline{x}_5+\overline{x}_6\,,\\
&b_{002}=A^{(1)2}_{001}\,\overline{y}_1+A^{(1)}_{001}\,A^{(2)}_{001}\,\overline{y}_2+A^{(2)2}_{001}\,\overline{y}_3+A^{(1)}_{001}\overline{y}_4+A^{(2)}_{001}\overline{y}_5+\overline{y}_6\,,
\end{aligned}\right\} \quad (128)$$

where

$$u=\kappa\pi\frac{d\alpha_3}{d\tau}, \qquad v=\kappa\pi\gamma_4\,, \qquad \overline{x}_i=\frac{dx_i}{d\tau}, \qquad \overline{y}_i=y_i \qquad \text{at } \tau=\kappa\pi. \qquad (129)$$

Let us solve the first equation of (127) for ϵ as a power series in α and δ. We obtain

$$\epsilon=\epsilon_{10}\,\alpha+\epsilon_{01}\,\delta+\epsilon_{20}\,\alpha^2+\epsilon_{11}\,\alpha\,\delta+\epsilon_{02}\,\delta^2+\cdots, \tag{130}$$

where the coefficients ϵ_{ij} have the values

$$\left.\begin{aligned}
&\epsilon_{10}=-\frac{a_{100}}{a_{001}}, \qquad \epsilon_{20}=-\frac{a_{200}\,a_{001}^2-a_{100}\,a_{101}\,a_{001}+a_{002}\,a_{100}^2}{a_{001}^3},\\
&\epsilon_{02}=-\frac{a_{020}\,a_{001}^2-a_{011}\,a_{010}\,a_{001}+a_{002}\,a_{010}^2}{a_{001}^3}, \qquad \epsilon_{01}=-\frac{a_{010}}{a_{001}},\\
&\epsilon_{11}=-\frac{a_{110}\,a_{001}^2-a_{011}\,a_{100}\,a_{001}-a_{101}\,a_{010}\,a_{001}+2\,a_{100}\,a_{002}\,a_{010}}{a_{001}^3}.
\end{aligned}\right\} \tag{131}$$

Solving the second equation of (127) for ϵ in terms of α and δ, we obtain

$$\epsilon=\overline{\epsilon}_{10}\,\alpha+\overline{\epsilon}_{01}\,\delta+\overline{\epsilon}_{20}\,\alpha^2+\overline{\epsilon}_{11}\,\alpha\,\delta+\overline{\epsilon}_{02}\,\delta_2+\cdots, \tag{132}$$

where the $\overline{\epsilon}_{ij}$ have the same expressions in the b_{ijk} as the ϵ_{ij} have in the a_{ijk}.

Upon subtracting (130) from (132), we have

$$0=[\overline{\epsilon}_{10}-\epsilon_{10}]\,\alpha+[\overline{\epsilon}_{01}-\epsilon_{01}]\,\delta+[\overline{\epsilon}_{20}-\epsilon_{20}]\,\alpha^2+[\overline{\epsilon}_{11}-\epsilon_{11}]\,\alpha\,\delta+[\overline{\epsilon}_{02}-\epsilon_{02}]\,\delta^2+\cdots \tag{133}$$

We must examine the coefficients of this series. The first two are

$$\overline{\epsilon}_{10}-\epsilon_{10}=\frac{a_{100}}{a_{001}}-\frac{b_{100}}{b_{001}}=\frac{A^{(2)}_{100}\,u}{A^{(2)}_{001}\,u}-\frac{A^{(2)}_{100}\,v}{A^{(2)}_{001}\,v}\equiv 0, \qquad \overline{\epsilon}_{01}-\epsilon_{01}=\frac{a_{010}}{a_{001}}-\frac{b_{010}}{b_{001}}=\frac{A^{(2)}_{010}\,u}{A^{(2)}_{001}\,u}-\frac{A^{(2)}_{010}\,v}{A^{(2)}_{001}\,v}\equiv 0. \tag{134}$$

Both of the linear terms therefore vanish.

The computation of the second degree terms is somewhat more complicated. It will simplify matters somewhat if we observe that the $\overline{\epsilon}_{jk}$ are the same expressions in v and $\overline{y}_i$ as the ϵ_{jk} are in u and $\overline{x}_i$. It will therefore be sufficient to compute one and derive the other from it. On substituting in the expression for ϵ_{20} in (131) the values of the a_{ijk} from (128), we get

$$\left.\begin{aligned}
\frac{a_{200}\,a_{001}^2}{a_{001}^3}&=\frac{1}{A^{(2)3}_{001}}\left[A^{(1)2}_{100}\,A^{(2)2}_{001}\frac{\overline{x}_1}{u}+A^{(1)}_{100}\,A^{(2)}_{100}\,A^{(2)2}_{001}\frac{\overline{x}_2}{u}+A^{(2)2}_{100}\,A^{(2)2}_{001}\frac{\overline{x}_3}{u}\right],\\
-\frac{a_{100}\,a_{101}\,a_{001}}{a_{001}^3}&=\frac{1}{A^{(2)3}_{001}}\Big[-2A^{(1)}_{100}\,A^{(2)}_{100}\,A^{(1)}_{001}\,A^{(2)}_{001}\frac{\overline{x}_1}{u}+\Big(-A^{(2)2}_{100}\,A^{(1)}_{001}A^{(2)}_{001}\\
&\qquad -A^{(1)}_{100}\,A^{(2)}_{100}\,A^{(2)2}_{001}\Big)\frac{\overline{x}_2}{u}-2A^{(2)2}_{100}\,A^{(2)2}_{001}\frac{\overline{x}_3}{u}\\
&\qquad -A^{(1)}_{100}\,A^{(2)}_{100}\,A^{(2)}_{001}\frac{\overline{x}_4}{u}-A^{(2)2}_{100}\,A^{(2)}_{001}\frac{\overline{x}_5}{u}\Big],\\
\frac{a_{002}\,a_{100}^2}{a_{001}^3}&=\frac{1}{A^{(2)3}_{001}}\Big[A^{(2)2}_{100}\,A^{(1)2}_{001}\frac{\overline{x}_1}{u}+A^{(2)2}_{100}\,A^{(1)}_{001}\,A^{(2)}_{001}\frac{\overline{x}_2}{u}+A^{(2)2}_{100}\,A^{(2)2}_{001}\frac{\overline{x}_3}{u}\\
&\qquad +A^{(2)2}_{100}\,A^{(1)}_{001}\frac{\overline{x}_4}{u}+A^{(2)2}_{100}\,A^{(2)}_{001}\frac{\overline{x}_5}{u}+A^{(2)2}_{100}\frac{\overline{x}_6}{u}\Big].
\end{aligned}\right\} \tag{135}$$

On forming the sum of these three expressions, there results

$$-\epsilon_{20}=\frac{1}{A^{(2)3}_{001}}\left\{\left[A^{(1)}_{100}\,A^{(2)}_{001}-A^{(2)}_{100}\,A^{(1)}_{001}\right]^2\frac{\overline{x}_1}{u}-\left[A^{(1)}_{100}\,A^{(2)}_{001}-A^{(2)}_{100}\,A^{(1)}_{001}\right]A^{(2)}_{100}\frac{\overline{x}_4}{u}+A^{(2)2}_{100}\frac{\overline{x}_6}{u}\right\},$$

the coefficients of $\overline{x}_2/u$, $\overline{x}_3/u$, and $\overline{x}_5/u$ being identically zero.

On changing the $\bar{x}_i$ into $\bar{y}_i$ and u into v, we get $-\bar{\epsilon}_{20}$. Hence

$$\left[\bar{\epsilon}_{20}-\epsilon_{20}\right]=\frac{1}{A_{001}^{(2)3}}\Big\{\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]^2\left[\frac{\bar{x}_1}{u}-\frac{\bar{y}_1}{v}\right]$$
$$-A_{100}^{(2)}\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right]+A_{100}^{(2)2}\left[\frac{\bar{x}_6}{u}-\frac{\bar{y}_6}{v}\right]\Big\}.$$

But $\left[\frac{\bar{x}_1}{u}-\frac{\bar{y}_1}{v}\right]$ and $\left[\frac{\bar{x}_6}{u}-\frac{\bar{y}_6}{v}\right]$ vanish since

$$\bar{x}_1=c_1 u, \qquad \bar{y}_1=c_1 v, \qquad \bar{x}_6=c_4 u, \qquad \bar{y}_6=c_4 v,$$

as is readily seen from (116) and (123). On referring to (122), it is also seen that $\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right]$ does not vanish, but is equal to $c_2\left[\frac{1}{u}\frac{d\alpha_1}{d\tau}-\frac{\gamma_2}{v}\right]_{\tau=\kappa\pi}$. Hence

$$\left[\bar{\epsilon}_{20}-\epsilon_{20}\right]=-\frac{A_{100}^{(2)}\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right]. \tag{136}$$

Without repeating the details of the computation, we find similarly

$$\left.\begin{aligned}\left[\bar{\epsilon}_{11}-\epsilon_{11}\right]&=-\frac{A_{010}^{(2)}\left[A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right]+A_{100}^{(2)}\left[A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}\right]}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right],\\ \left[\bar{\epsilon}_{02}-\epsilon_{02}\right]&=-\frac{A_{010}^{(2)}\left[A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}\right]}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right].\end{aligned}\right\} \tag{137}$$

On substituting in (133) the values obtained for the coefficients, we find that the second degree terms in α and δ are factorable, giving

$$\left.\begin{aligned}0=\frac{1}{A_{001}^{(2)3}}\left[\frac{\bar{x}_4}{u}-\frac{\bar{y}_4}{v}\right]\left[A_{100}^{(2)}\alpha+A_{010}^{(2)}\delta+\cdots\right]\Big[&\left(A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}\right)\alpha\\ &+\left(A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}\right)\delta+\cdots\Big].\end{aligned}\right\} \tag{138}$$

There are therefore two solutions for δ as power series in α.

On substituting the two solutions of (138) for δ in (130), we find the two corresponding values of ϵ. We thus obtain the two solutions

$$\left.\begin{aligned}\delta&=-\frac{A_{100}^{(1)}A_{001}^{(2)}-A_{100}^{(2)}A_{001}^{(1)}}{A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}}\alpha+\cdots\\ &=\frac{\gamma_5'(0)-a\gamma_3'(0)}{\alpha_5(0)-a\alpha_4(0)}\alpha+\cdots=\frac{\gamma_6'(0)}{\alpha_6(0)}\alpha+\cdots,\\ \epsilon&=-\frac{A_{100}^{(1)}A_{010}^{(2)}-A_{100}^{(2)}A_{010}^{(1)}}{A_{010}^{(1)}A_{001}^{(2)}-A_{010}^{(2)}A_{001}^{(1)}}\alpha+\cdots\\ &=\frac{1}{\alpha_5(0)-a\alpha_4(0)}\alpha+\cdots=\frac{1}{\alpha_6(0)}\alpha+\cdots;\end{aligned}\right\} \tag{139}$$

$$\left.\begin{aligned}\delta&=-\frac{A_{100}^{(2)}}{A_{010}^{(2)}}\alpha+\cdots &&=\frac{\gamma_1'(0)}{\alpha_2(0)}\alpha+\cdots,\\ \epsilon&=-\frac{A_{100}^{(2)}}{A_{001}^{(2)}}\alpha-\frac{A_{010}^{(2)}}{A_{001}^{(2)}}\delta+\cdots &&=\quad 0\cdot\alpha+\cdots,\end{aligned}\right\} \tag{140}$$

where α_6 and γ_6 are the quantities defined in (111), and $\gamma_6'(0)$ is the value of $d\gamma_6/d\tau$ for $\tau=0$. It was shown in (112) that $\alpha_6(0)$ is distinct from zero, and in (79) that $\alpha_2(0)$ is equal to unity. Thus one solution for ϵ begins with the first power of α, while the other certainly does not begin before the second, but in both solutions δ begins with the first power of α.

74. Construction of the Solutions with the Period $2\kappa\pi$.—We have proved the existence of series for ρ, σ, and ϵ proceeding in powers of the initial value of ρ, which we will now denote by e*. The series for ρ and σ are periodic in τ with the period $2\kappa\pi$, and since this condition holds for all values of e sufficiently small, each coefficient separately is periodic. The series for ρ is even in τ, and the series for σ is odd in τ. These series have the form

$$\left.\begin{aligned}\rho&=\rho_1\,e+\rho_2\,e^2+\rho_3\,e^3+\cdots,\\ \sigma&=\sigma_1\,e+\sigma_2\,e^2+\sigma_3\,e^3+\cdots,\\ \epsilon&=\epsilon_1\,e+\epsilon_2\,e^2+\epsilon_3\,e^3+\cdots.\end{aligned}\right\}\tag{141}$$

We shall substitute these series in equations (102) and integrate the coefficients of the powers of e in order, and determine the constants in such a way that ρ and σ shall be periodic and shall satisfy the initial conditions

$$\rho(0)=e,\qquad \sigma(0)=0,\qquad \rho'(0)=0,\qquad \sigma'(0)=\nu,$$

where ν is a constant which will be determined in the process.

On substituting the series (141) in the differential equations (102), we find for the coefficients of the first power of e

$$\rho_1''+\theta_2\,\rho_1+\theta_3\,\sigma_1=\theta_{001}\,\epsilon_1\,,\qquad \sigma_1''+\theta_4\,\sigma_1+\theta_3\,\rho_1=0.\tag{142}$$

By the condition of orthogonality ρ must be even in τ and σ odd in τ, and the solution complying with these conditions is

$$\left.\begin{aligned}\rho_1&=A^{(1)}\,\alpha_2(\tau)+D^{(1)}[\tau\alpha_3(\tau)+\alpha_4(\tau)]+\epsilon_1[a\tau\,\alpha_3(\tau)+\alpha_5(\tau)],\\ \sigma_1&=A^{(1)}\gamma_1(\tau)+D^{(1)}[\tau\gamma_4(\tau)+\gamma_3(\tau)]+\epsilon_1[a\tau\gamma_4(\tau)+\gamma_5(\tau)],\end{aligned}\right\}\tag{143}$$

where $\alpha_5(\tau)$ contains only cosines of even multiples of τ, and $\gamma_5(\tau)$ contains only sines of odd multiples of τ. In order that this solution shall be periodic it is necessary and sufficient that

$$D^{(1)}=-a\epsilon_1\,.$$

Upon substituting this value of $D^{(1)}$, the solution (143) becomes

$$\left.\begin{aligned}\rho_1&=A^{(1)}\,\alpha_2[(\tau)+\epsilon_1\alpha_5(\tau)-a\,\alpha_4(\tau)]=A^{(1)}\,\alpha_2(\tau)+\epsilon_1\,\alpha_6(\tau),\\ \sigma_1&=A^{(1)}\,\gamma_1[(\tau)+\epsilon_1\gamma_5(\tau)=a\,\gamma_3(\tau)]=A^{(1)}\,\gamma_1(\tau)+\epsilon_1\,\gamma_6(\tau).\end{aligned}\right\}\tag{144}$$

It remains to impose the initial condition that $\rho_1=1$ at $\tau=0$, which gives

$$1=A^{(1)}\,\bar{\alpha}_2+\epsilon_1\,\alpha_6\,,\tag{145}$$

where $\bar{\alpha}_2$ and $\bar{\alpha}_6$ denote the values of α_2 and α_6 for $\tau=0$.

*The reason for changing from a to e is that this parameter corresponds to the eccentricity in the two-body problem.

Coefficients of e^2. The coefficients of e^2 satisfy the equations

$$\left.\begin{aligned} \rho_2''+\theta_2\,\rho_2+\theta_3\,\sigma_2 &= \theta_{001}\,\epsilon_2+\theta_{101}\,\epsilon_1\,\rho_1+\theta_{200}\,\rho_1^2+\theta_{110}\,\rho_1\,\sigma_1+\theta_{020}\,\sigma_1^2=R_2\,, \\ \sigma_2''+\theta_4\,\sigma_2+\theta_3\,\rho_2 &= \qquad\qquad\qquad \bar\theta_{200}\,\rho_1^2+\bar\theta_{110}\,\rho_1\,\sigma_1+\bar\theta_{020}\,\sigma_1^2=S_2\,. \end{aligned}\right\} \tag{146}$$

Every term of R_2 and S_2 contains either $A^{(1)}$, ϵ_1, or ϵ_2 as a factor. Arranged in this manner, we have

$$\begin{aligned} R_2 = &+A^{(1)2}\,[\theta_{200}\,\alpha_2^2+\theta_{110}\,\alpha_2\,\gamma_1+\theta_{020}\,\gamma_1^2] \\ &+A^{(1)}\epsilon_1\,[2\theta_{200}\,\alpha_2\,\alpha_6+\theta_{110}(\gamma_1\,\alpha_6+\alpha_2\,\gamma_6)+2\theta_{020}\,\gamma_1\,\gamma_6+\theta_{101}\,\alpha_2] \\ &+\quad \epsilon_1^2\,[\theta_{200}\,\alpha_6^2+\theta_{110}\,\alpha_6\,\gamma_6+\theta_{020}\,\gamma_6^2 \qquad\qquad +\theta_{101}\,\alpha_6] \\ &+\quad \epsilon_2\,[\theta_{001}], \end{aligned}$$

$$\begin{aligned} S_2 = &+A^{(1)2}\,[\bar\theta_{200}\,\alpha_2^2+\bar\theta_{110}\,\alpha_2\,\gamma_1+\bar\theta_{020}\,\gamma_1^2] \\ &+A^{(1)}\epsilon_1\,[2\bar\theta_{200}\,\alpha_2\,\alpha_6+\bar\theta_{110}(\gamma_1\,\alpha_6+\alpha_2\,\gamma_6)+2\bar\theta_{020}\,\gamma_1\,\gamma_6] \\ &+\quad \epsilon_1^2\,[\bar\theta_{200}\,\alpha_6^2+\bar\theta_{110}\,\alpha_6\,\gamma_6+\bar\theta_{020}\,\gamma_6^2]. \end{aligned}$$

In order to understand the character of the solution of equations (146), we must examine the character of R_2 and S_2. The coefficient of $A^{(1)2}$ in both R_2 and S_2 is homogeneous of the second degree in α_2 and γ_1. Its expansion therefore involves terms carrying $2\lambda\tau$ and terms independent of $\lambda\tau$. By §30, the solution for the terms in $2\lambda\tau$ is periodic. The terms independent of $\lambda\tau$ are cosines of even multiples of τ in R_2, and sines of odd multiples of τ in S_2. These terms have the same character as those in the coefficients of ϵ_1^2 and ϵ_2, and will be considered under the discussion of those terms.

The coefficients of $A^{(1)}\epsilon_1$ in both R_2 and S_2 are homogeneous of the first degree in α_2 and γ_1, all terms of which carry the first multiple of $\lambda\tau$. By Theorem IV, §71, the expression for ρ_2 will carry the term $\tau\alpha_1(\tau)$, and for σ_2, the term $\tau\gamma_2(\tau)$. Non-periodic terms of this character do not arise elsewhere in the solution. Hence, in order to avoid them, we must take either $A^{(1)}=0$ or $\epsilon_1=0$. If we choose $A^{(1)}=0$, then, by (145), ϵ_1 is determined and has the value $\epsilon_1=1/\bar\alpha_6$, thus agreeing with the first solution (139) of the existence proof. But if we choose $\epsilon_1=0$, so that by (145) $A^{(1)}=1/\bar\alpha_2$, we are in agreement with the second solution (140) of the existence proof. We will commence by developing the first solution, in which

$$A^{(1)}=0, \qquad \epsilon_1=\frac{1}{\bar\alpha_6}.$$

FIRST SOLUTION.

Since $A^{(1)}=0$, all terms in R_2 and S_2 which carry $\lambda\tau$, or any multiple of $\lambda\tau$, vanish. There remain

$$\left.\begin{aligned} R_2 &= \epsilon_1^2[\theta_{200}\,\alpha_6^2+\theta_{110}\,\alpha_6\,\gamma_6+\theta_{020}\,\gamma_6^2+\theta_{101}\,\alpha_6]+\epsilon_2\,\theta_{101}\,, \\ S_2 &= \epsilon_1^2[\bar\theta_{200}\,\alpha_6^2+\bar\theta_{110}\,\alpha_6\,\gamma_6+\bar\theta_{020}\,\gamma_6^2]. \end{aligned}\right\} \tag{147}$$

We have also

$$\rho_1 = \frac{\alpha_6(\tau)}{\overline{\alpha}_6}, \qquad \sigma_1 = \frac{\gamma_6(\tau)}{\overline{\alpha}_6}, \qquad \epsilon_1 = \frac{1}{\overline{\alpha}_6}. \tag{148}$$

It follows from (147) that R_2 contains only cosines of even multiples of τ, and S_2 contains only sines of odd multiples of τ. Since ρ_2 is an even function of τ and σ_2 is an odd function of τ, the solution is

$$\left.\begin{aligned}\rho_2 &= A^{(2)}\,\alpha_2(\tau) + D^{(2)}[\tau\,\alpha_3(\tau) + \alpha_4(\tau)] + [\eta_2(\tau) + a_2\tau\,\alpha_3(\tau)] + \epsilon_2[\alpha_5(\tau) + a\tau\,\alpha_3(\tau)],\\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + D^{(2)}[\tau\gamma_4(\tau) + \gamma_3(\tau)] + [\zeta_2(\tau) + a_2\tau\gamma_4(\tau)] + \epsilon_2[\gamma_5(\tau) + a\tau\gamma_4(\tau)].\end{aligned}\right\}\tag{149}$$

In this solution the terms are grouped according to their origin. The first two terms are the complementary function. The third arises from the terms carrying ϵ_1^2 as a factor. The fourth arises from the terms having ϵ_2 as a factor, a_2 is a constant depending upon the coefficients of ϵ_1^2 in the differential equations, and $\alpha_5(\tau)$ and $\gamma_5(\tau)$ are the same functions as in the coefficient of the first power of e, $\eta_2(\tau)$ and $\zeta_2(\tau)$ are periodic functions of τ with the period 2π, and so constituted that $\eta_2(\tau)$ contains only cosines of even multiples of τ, and $\zeta_2(\tau)$ contains only sines of odd multiples of τ.

In order that ρ_2 and σ_2 shall be periodic we must have

$$D^{(2)} = -a_2 - a\epsilon_2,$$

which makes

$$\left.\begin{aligned}\rho_2 &= A^{(2)}\,\alpha_2(\tau) + \epsilon_2\,\alpha_6(\tau) + \eta_2(\tau) - a_2\,\alpha_4(\tau),\\ \sigma_2 &= A^{(2)}\,\gamma_1(\tau) + \epsilon_2\,\gamma_6(\tau) + \zeta_2(\tau) - a_2\,\gamma_3(\tau).\end{aligned}\right\}\tag{150}$$

In order that we may satisfy the initial conditions, we must have $\rho_2 = 0$ at $\tau = 0$, which determines ϵ_2 by the equation

$$\epsilon_2 = \frac{a_2\,\overline{\alpha}_4 - \overline{\eta}_2 - A^{(2)}\,\overline{\alpha}_2}{\alpha_6}.$$

It is obvious that A_2, which so far is arbitrary, must be zero, for in the coefficient of e^3 it will give rise to terms involving the first multiple of $\lambda\tau$. All such terms will carry $A^{(2)}$ as a factor; hence to avoid non-periodic terms of this character, we choose $A^{(2)} = 0$. Anticipating this step, we have

$$\left.\begin{aligned}\rho_2 &= \frac{a_2\,\overline{\alpha}_4 - \overline{\eta}_2}{\overline{\alpha}_6}\,\alpha_6(\tau) + \eta_2(\tau) - a_2\,\alpha_4(\tau),\\ \sigma_2 &= \frac{a_2\,\overline{\alpha}_4 - \overline{\eta}_2}{\overline{\alpha}_6}\,\gamma_6(\tau) + \zeta_2(\tau) - a_2\,\gamma_3(\tau),\end{aligned}\right\}\tag{151}$$

so that ρ_2 contains only cosines of even multiples of τ, and σ_2 contains only sines of odd multiples of τ.

It only remains to show that this process of integration can be carried on indefinitely. On assuming that up to and including ρ_{i-1} and σ_{i-1} every ρ_j and σ_j is periodic with the period 2π, and that the ρ_j contain only cosines of even multiples of τ and the σ_j only sines of odd multiples of τ, except that

$\rho_{\iota-1}$ contains the term $A^{(\iota-1)}\alpha_2(\tau)$ and $\sigma_{\iota-1}$ contains the term $A^{(\iota-1)}\gamma_1(\tau)$, it will be shown that the same conditions will obtain for the next succeeding step. For ρ_ι and σ_ι we have, from the differential equations (102),

$$\left.\begin{aligned}\rho_\iota''+\theta_2\rho_\iota+\theta_3\sigma_\iota&=\theta_{001}\,\epsilon_\iota+A^{(\iota-1)}[2\theta_{200}\,\rho_1\alpha_2+\theta_{110}(\rho_1\gamma_1+\sigma_1\alpha_2)\\&\qquad+2\theta_{020}\,\sigma_1\gamma_1+\theta_{101}\,\epsilon_1\alpha_2]+\Phi_\iota\,,\\\sigma_\iota''+\theta_4\sigma_\iota+\theta_3\rho_\iota&=\qquad A^{(\iota-1)}[2\overline{\theta}_{200}\,\rho_1\alpha_2+\overline{\theta}_{110}(\rho_\iota\gamma_1+\sigma_1\alpha_2)+2\overline{\theta}_{020}\,\sigma_1\gamma_1]+\Psi_\iota.\end{aligned}\right\}\quad(152)$$

From the properties of the differential equations it is readily seen that Φ_ι contains only known terms all of which are cosines of even multiples of τ, and that Ψ_ι contains only known terms all of which are sines of odd multiples of τ. The coefficients of $A^{(\iota-1)}$ are homogeneous of the first degree in α_2 and γ_1, and consequently each term involves a first multiple of $\lambda\tau$. They give rise to non-periodic terms of the form $\tau\alpha_1(\tau)$ and $\tau\gamma_2(\tau)$ in the solution. They carry $A^{(\iota-1)}$ as a factor, and since terms of this type arise nowhere else, we can make them disappear only by putting $A^{(\iota-1)}=0$. The solution for (152) then has the form

$$\left.\begin{aligned}\rho_\iota&=A^{(\iota)}\alpha_2(\tau)+D^{(\iota)}[\tau\,\alpha_3(\tau)+\alpha_4(\tau)]+[\eta_\iota(\tau)+a_\iota\tau\,\alpha_3(\tau)]+\epsilon_\iota[\alpha_5(\tau)+a\tau\,\alpha_3(\tau)],\\\sigma_\iota&=A^{(\iota)}\gamma_1(\tau)+D^{(\iota)}[\tau\gamma_4(\tau)+\gamma_3(\tau)]+[\zeta_\iota(\tau)+a_\iota\tau\gamma_4(\tau)]+\epsilon_\iota[\gamma_5(\tau)+a\tau\gamma_4(\tau)],\end{aligned}\right\}\quad(153)$$

where $\eta_\iota(\tau)$ and $\zeta_\iota(\tau)$ are periodic with the period 2π, and where by Theorem II, §71, $\eta_\iota(\tau)$ contains only cosines of even multiples of τ, and $\zeta_\iota(\tau)$ contains only sines of odd multiples of τ.

In order that ρ_ι and σ_ι shall be periodic it is necessary and sufficient that

$$D^{(\iota)}=-a_\iota-a\epsilon_\iota\,,$$

which makes

$$\left.\begin{aligned}\rho_\iota&=A^{(\iota)}\,\alpha_2(\tau)+\eta_\iota(\tau)-a_\iota\alpha_4(\tau)+\epsilon_\iota\alpha_6(\tau),\\\sigma_\iota&=A^{(\iota)}\,\gamma_1(\tau)+\xi_\iota(\tau)-a_\iota\gamma_3(\tau)+\epsilon_\iota\gamma_6(\tau).\end{aligned}\right\}\quad(154)$$

From the initial conditions we must have $\rho_\iota=0$ at $\tau=0$, which determines ϵ_ι by the equation

$$\epsilon_\iota=\frac{a_\iota\,\overline{\alpha}_4-\overline{\eta}_\iota-A^{(\iota)}\overline{\alpha}_2}{\overline{\alpha}_6}\,.$$

Thus the constants are uniquely determined. The ρ_ι and σ_ι have the properties assumed for those having smaller subscripts, and the process of integration can be continued indefinitely. Every $A^{(\iota)}$ is zero. Since no terms involving the $\lambda\tau$ enter, the solution has the period 2π, and the orbits represented belong to the class of generating orbits with which we started. In other words, we set out with a generating orbit for which the initial distance was, let us say, r_0, and we have found another generating orbit for which the initial distance is r_0+e (e arbitrary). There is nothing surprising in this, for r_0 is a function of an arbitrary constant β.

Let us suppose we had started with a definite value of β, for example β_0, which gives a definite generating orbit with a definite initial distance r_0. Let us seek now the generating orbit for which the initial distance is r_0+e. If e is sufficiently small we can evidently give an increment ϵ to β_0, which will increase r_0 by the amount e. We have

$$r_0=f(\beta_0), \qquad r_0+e=f(\beta_0+\epsilon).$$

Expanding the right member of the second equation in powers of ϵ, we have

$$e=\frac{\partial f}{\partial \beta_0}\epsilon+\frac{1}{2}\frac{\partial^2 f}{\partial \beta_0^2}\epsilon^2+\cdots,$$

which gives, by inversion, a series of the form

$$\epsilon=c_1 e+c_2 e^2+\cdots.$$

Then, by substituting $\beta=\beta_0+c_1 e+c_2 e^2+\cdots$ in the generating orbit and arranging the solutions as a power series in e, we obtain the orbit in which the initial distance is r_0+e. As these are the same conditions that were imposed when we sought new orbits through the equations of variation, it was to have been expected that one of the class of generating orbits would satisfy them.

SECOND SOLUTION.

We return now to equation (146), and continue with the second solution, in which $\epsilon_1=0$ and $A^{(1)}=1/\bar{a}_2$. From (82) it is seen that $\bar{a}_2=a_2(0)=1$, and therefore $A^{(1)}=1$. Hence in the second solution

$$\rho_1=a_2(\tau), \qquad \sigma_1=\gamma_1(\tau). \tag{155}$$

On using these values of $A^{(1)}$ and ϵ_1, R_2 and S_2 of (146) become

$$\left.\begin{aligned} R_2&=[\theta_{200}\, a_2^2+\theta_{110}\, a_2\gamma_1+\theta_{020}\,\gamma_1^2]+\epsilon_2\theta_{001},\\ S_2&=[\bar{\theta}_{200}\, a_2^2+\bar{\theta}_{110}\, a_2\gamma_1+\bar{\theta}_{020}\,\gamma_1^2]. \end{aligned}\right\} \tag{156}$$

All of the terms in these expressions except $\epsilon_2\theta_{001}$ are of the second degree in a_2 and γ_1. Therefore they involve only terms carrying $2\lambda\tau$ and terms independent of $\lambda\tau$, and θ_{001} is independent of $\lambda\tau$. In the solutions the terms depending upon $2\lambda\tau$ are periodic. As for the terms independent of λ, R_2 contains only cosines of even multiples of τ, and S_2 contains only sines of odd multiples of τ. These terms give rise to non-periodic terms in the solution, which has the form

$$\left.\begin{aligned} \rho_2=A^{(2)}a_2(\tau)&+D^{(2)}[\tau a_3(\tau)+a_4(\tau)]+\varphi_2(\lambda,\tau)\\ &+[\eta_2(\tau)+a_2\tau a_3(\tau)]+\epsilon_2[a_5(\tau)+a\tau a_3(\tau)],\\ \sigma_2=A^{(2)}\gamma_1(\tau)&+D^{(2)}[\tau\gamma_4(\tau)+\gamma_3(\tau)]+\psi_2(\lambda,\tau)\\ &+[\zeta_2(\tau)+a_2\tau\gamma_4(\tau)]+\epsilon_2[\gamma_5(\tau)+a\tau\gamma_4(\tau)], \end{aligned}\right\} \tag{157}$$

where $\varphi_2(\lambda,\tau)$ and $\psi_2(\lambda,\tau)$ are the periodic terms involving λ; η_2 and ζ_2 are the periodic terms with the period 2π; a_2 is the constant belonging to the non-periodic part; and the coefficients of ϵ_2 are the solutions depending

on the coefficient of ϵ_2 in the differential equations. In order that this solution shall be periodic, it is necessary that

$$D^{(2)} = -a_2 - a\epsilon_2 ,$$

which reduces ρ_2 and σ_2 to

$$\left.\begin{aligned}\rho_2 &= A^{(2)}\alpha_2(\tau) + \varphi_2(\lambda, \tau) + \eta_2(\tau) - a_2\alpha_4(\tau) + \epsilon_2\alpha_6(\tau),\\ \sigma_2 &= A^{(2)}\gamma_1(\tau) + \psi_2(\lambda, \tau) + \zeta_2(\tau) - a_2\gamma_3(\tau) + \epsilon_2\gamma_6(\tau).\end{aligned}\right\} \qquad (158)$$

To satisfy the initial conditions we must have $\rho_2(0) = 0$. Hence, since $\alpha_2(0) = 1$, $A^{(2)}$ is defined by

$$A^{(2)} = -\varphi_2(0) - \eta_2(0) + a_2\alpha_4(0) - \epsilon_2\alpha_6(0),$$

where ϵ_2 is a constant which is determined by the periodicity condition on the coefficient of ϵ^3.

Coefficients of e^3. The coefficients of the third degree terms are defined by

$$\rho_3'' + \theta_2\rho_3 + \theta_3\sigma_3 = R_3, \qquad \sigma_3'' + \theta_4\sigma_3 + \theta_3\,\rho_3 = S_3, \qquad (159)$$

where

$$\begin{aligned}R_3 = \theta_{001}\,\epsilon_3 + \theta_{101}\,\epsilon_2\rho_1 &+ 2\theta_{200}\,\rho_1\rho_2 + \theta_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\theta_{020}\,\sigma_1\sigma_2 + \theta_{300}\,\rho_1^3\\ &+ \theta_{210}\,\rho_1^2\sigma_1 + \theta_{120}\,\sigma_1^2\rho_1 + \theta_{030}\,\sigma_1^3,\\ S_3 = \qquad\qquad &+ 2\overline{\theta}_{200}\,\rho_1\rho_2 + \overline{\theta}_{110}[\sigma_2\rho_1 + \sigma_1\rho_2] + 2\overline{\theta}_{020}\,\sigma_1\sigma_2 + \overline{\theta}_{300}\,\rho_1^3\\ &+ \overline{\theta}_{210}\,\rho_1^2\sigma_1 + \overline{\theta}_{120}\,\sigma_1^2\rho_1 + \overline{\theta}_{030}\,\sigma_1^3.\end{aligned}$$

In classifying the terms which belong to the expansion of R_3 and S_3, we bear in mind that

1. The θ_{ijk} in R_3 involve only cosines of even multiples of τ, except those which are coefficients of odd powers of σ (*i. e.*, where j is odd), and these involve only sines of odd multiples. The opposite is the case in the $\overline{\theta}_{ijk}$ of S_3. If j is even, the $\overline{\theta}_{ijk}$ involve only sines of odd multiples of τ. If j is odd, the $\overline{\theta}_{ijk}$ involve only cosines of even multiples.
2. The terms independent of λ involve only cosines of even multiples of τ in the expressions for ρ_1 and ρ_2, and only sines of odd multiples of τ in the expressions for σ_1 and σ_2.

It is seen, then, that in those terms of R_3 which are independent of λ only cosines of even multiples of τ enter; and in those terms of S_3 which are independent of λ only sines of odd multiples enter. In the process of integration, therefore, two types of non-periodic terms arise. First, those coming from the terms which involve the first multiple of $\lambda\tau$, and secondly, those coming from the terms which are independent of λ. It is important, therefore, to separate the various terms into three classes, (*a*) terms independent of λ, (*b*) terms involving first multiple of $\lambda\tau$ only, (*c*) terms involving multiples of $\lambda\tau$ higher than the first.

We rewrite, then, the differential equations (159) in the form

$$\left.\begin{aligned} \rho_3''+\theta_2\,\rho_3+\theta_3\,\sigma_3&=\theta_{001}\,\epsilon_3+\epsilon_2 f_1(\lambda,\,\tau)+f_2(\lambda,\,\tau)+f_3(\tau)+f_4(\kappa\lambda,\,\tau),\\ \sigma_3''+\theta_4\,\sigma_3+\theta_3\,\rho_3&=\qquad\qquad \epsilon_2\,g_1(\lambda,\,\tau)+g_2(\lambda,\,\tau)+g_3(\tau)+g_4(\kappa\lambda,\,\tau),\end{aligned}\right\}\qquad(160)$$

where

$$\begin{aligned} f_1(\lambda,\,\tau)&=\theta_{101}\,a_2+2\theta_{200}\,a_6a_2+\theta_{110}[a_2\gamma_6+\gamma_1a_6]+2\theta_{020}\gamma_1\gamma_6\,,\\ g_1(\lambda,\,\tau)&=\qquad\quad 2\overline{\theta}_{200}a_6a_2+\overline{\theta}_{110}[a_2\gamma_6+\gamma_1a_6]+2\overline{\theta}_{020}\gamma_1\gamma_6\,.\end{aligned}$$

The f_1 and g_1 terms are homogeneous of the first degree in a_2 and γ_1, and consequently involve only terms which carry the first multiple of $\lambda\tau$; they are considered separately from other terms of the same character, because they carry the undetermined constant ϵ_2 as a factor. The solution for these terms has the form

$$\rho=F_1(\lambda,\,\tau)+b_1\tau a_1(\tau),\qquad\qquad \sigma=G_1(\lambda,\,\tau)+b_1\tau\gamma_2(\tau),$$

where F_1 and G_1 are periodic and involve only terms carrying the first multiple of $\lambda\tau$; b_1 is a constant depending upon f_1 and g_1, and is distinct from zero.

The $f_2(\lambda,\,\tau)$ and $g_2(\lambda,\,\tau)$ have the same properties as f_1 and g_1. They are considered separately, since they do not carry ϵ_2 in their coefficients. Their solutions may be written

$$\rho=F_2(\lambda,\,\tau)+b_2\tau a_1(\tau),\qquad\qquad \sigma=G_2(\lambda,\,\tau)+b_2\tau\gamma_2(\tau),$$

where F_2 and G_2 are periodic.

The $f_3(\tau)$ and $g_3(\tau)$ are independent of λ, and f_3 carries only cosines of even multiples of τ, while g_3 carries only sines of odd multiples of τ. The solution for these terms has the form

$$\rho=F_3(\tau)+b_3\tau a_3(\tau),\qquad\qquad \sigma=G_3(\tau)+b_3\tau\gamma_4(\tau),$$

where F_3 and G_3 are periodic.

The $f_4(\kappa\lambda,\,\tau)$ and $g_4(\kappa\lambda,\,\tau)$ involve only terms which carry multiples of $\lambda\tau$ higher than the first. The solution for these terms is periodic and may be written

$$\rho=F_4(\kappa\lambda,\,\tau),\qquad\qquad \sigma=G_4(\kappa\lambda,\,\tau).$$

The complete solution is therefore

$$\begin{aligned} \rho_3=A^{(3)}a_2(\tau)+D^{(3)}[\tau a_3(\tau)+a_4(\tau)]+\epsilon_3[a_5(\tau)+a\tau a_3(\tau)]+\epsilon_2[F_1(\lambda,\,\tau)+b_1\tau a_1(\tau)]\\ +[F_2(\lambda,\,\tau)+b_2\tau a_1(\tau)]+[F_3(\tau)+b_3\tau a_3(\tau)]+F_4(\kappa\lambda,\,\tau),\end{aligned}$$

$$\begin{aligned} \sigma_3=A^{(3)}\gamma_1(\tau)+D^{(3)}[\tau\gamma_4(\tau)+\gamma_3(\tau)]+\epsilon_3[\gamma_6(\tau)+a\tau\gamma_4(\tau)]+\epsilon_2[G_1(\lambda,\,\tau)+b_1\tau\lambda_4(\tau)]\\ +[G_2(\lambda,\,\tau)+b_2\gamma_2(\tau)]+[G_3(\tau)+b_3\tau\gamma_4(\tau)]+G_4(\kappa\lambda,\,\tau).\end{aligned}$$

All the functions $a_i(\tau)$, $\gamma_i(\tau)$, $F_i(\tau)$, and $G_i(\tau)$ are periodic. In order that ρ_3 and σ_3 shall be periodic, it is necessary and sufficient that the coefficient of $\tau a_3(\tau)$ and $\tau\gamma_4(\tau)$, and the coefficient of $\tau a_1(\tau)$ and $\tau\gamma_2(\tau)$ be zero; whence

$$D^{(3)}=-b_3-a\epsilon_3\,,\qquad\qquad \epsilon_2=-\frac{b_2}{b_1}.$$

Consequently the value of ϵ_2 is determined. In order to satisfy the initial conditions, we must have $\rho_3 = 0$ at $\tau = 0$, which determines $A^{(3)}$ by the equation

$$A^{(3)} = b_3 \, a_4(0) - \epsilon_3 \, a_6(0) + \frac{b_2}{b_1} F_1(0) - F_2(0) - F_3(0) - F_4(0) \, .$$

Thus all the constants are determined except ϵ_3, and the solution is

$$\rho_3 = A^{(3)} \, a_2(\tau) - b_3 a_4(\tau) + \epsilon_3 \, a_6(\tau) - \frac{b_2}{b_1} F_1(\lambda, \tau) + F_2(\lambda, \tau) + F_3(\tau) + F_4(\kappa\lambda, \tau),$$

$$\sigma_3 = A^{(3)} \gamma_1(\tau) - b_3 \gamma_3(\tau) + \epsilon_3 \gamma_6(\tau) - \frac{b_2}{b_1} G_1(\lambda, \tau) + G_2(\lambda, \tau) + G_3(\tau) + G_4(\kappa\lambda, \tau).$$

The constant ϵ_3 will be determined in satisfying the periodicity condition for the coefficients of e^4. It is obvious that this process of integration can be continued indefinitely. The ρ_3 and σ_3 have the same properties that had been found for ρ_1 and ρ_2. It is evident from the properties of the differential equations that these properties persist for ρ_4 and σ_4, and so on indefinitely. The coefficient for ϵ_{i-1}, in so far as it carries the first multiple of $\lambda\tau$, is always the same as for ϵ_2. Therefore the arbitrary constant ϵ_{i-1} can always be determined so as to avoid non-periodic terms of the type $\tau a_1(\tau)$ and $\tau\gamma_2(\tau)$. The constant $D^{(i)}$ of integration can always be determined so as to destroy non-periodic terms of the type $\tau a_3(\tau)$ and $\tau\gamma_4(\tau)$. The constant $A^{(i)}$ can always be determined so as to satisfy the initial conditions. The analysis of the types of terms entering is the same as for the subscript 3.

We have, therefore, a periodic solution with the period $2\kappa\pi$ which does not belong to the class of generating orbits from which we set out, for the particle makes many revolutions before its orbit re-enters.

After substituting the value of r in the equation

$$\frac{dv}{d\tau} = \frac{c}{r^2}$$

and integrating, the solution contains five arbitrary (except for the restriction that λ shall be rational) constants corresponding to the mean distance, the eccentricity, the inclination, the longitude of the node, and the epoch. One more, a constant corresponding to the longitude of the perihelion, is necessary for a general solution of the differential equations. The periodic orbits developed here are special in that they are all symmetrical with respect to the equatorial plane of the oblate spheroid.

CHAPTER V.

OSCILLATING SATELLITES ABOUT THE STRAIGHT-LINE EQUILIBRIUM POINTS.

FIRST METHOD.*

75. Statement of Problem.—Lagrange has shown† that if any two finite spherical bodies revolve about their common center of mass in circles, then there are three points in the line of these masses such that, if infinitesimal bodies be placed at them and projected so as to be instantaneously fixed relatively to the revolving system, they will always remain fixed relatively to the revolving system. There are also collinear solutions in which only the ratios of the mutual distances of the three masses remain constant, but in this chapter we shall consider only the case in which the distances themselves are constant. In Chapter VII the more general case will be treated. The three positions which the infinitesimal body may occupy are separated by the finite bodies; *i. e.*, starting from minus infinity, the order is an equilibrium point, a finite body, an equilibrium point, the second finite body, and the third equilibrium point. It is not necessary that one of the three masses shall be infinitesimal, but we shall limit ourselves at present to this case. In Chapter VIII it will be shown that the problem can be generalized to n masses. There are also solutions in which the bodies lie at the vertices of an equilateral triangle, and the oscillations about these points will be treated by Dr. Buck in Chapter IX.

If the sun, earth, and moon were so placed as to satisfy the conditions for a straight-line solution, and if the earth were between the sun and moon, then, as Laplace first pointed out, the moon would always be full, and either the sun or the moon would always be above the horizon of every observer. But these conditions would not be preserved unless the moon were in a position of stable equilibrium. If the position were one of complete stability and the moon were slightly disturbed from it, then it would perpetually oscillate about the point of equilibrium; if the position were one of complete instability, a slight disturbance of the moon would cause it to depart widely from the point of equilibrium. In the intermediate case of incomplete stability and also incomplete instability, the moon would either oscillate about the point of equilibrium, at least for some time, or it would speedily

*Read before the American Mathematical Society, June 28, 1900; abstract in *Bulletin of the American Mathematical Society*, vol. VII (1900), p. 12. The second method is given in Chap. VI.

†Lagrange's *Collected Works*, vol. VI, pp. 229–324; Tisserand's *Mécanique Céleste*, vol. I, Chap. 8; Moulton's *Introduction to Celestial Mechanics*, Chap. 7.

depart from it, according to the character of the disturbance. If it is given such an initial displacement that it revolves in the vicinity of the point of equilibrium in an orbit closed relatively to the moving system, it is called an *oscillating satellite*; for, as seen from the earth, it oscillates in the neighborhood of the equilibrium point in an apparently closed orbit. We shall consider here the motion of infinitesimal satellites oscillating in the vicinity of each of the three collinear points of equilibrium.

The literature of oscillating satellites is quite extensive, but in most of the papers the differential equations have been limited to their linear terms. In the discussion of the stability of a solution, it may be justifiable to neglect all except the linear terms when the differential equations are infinite power series; but with these restrictions, which are inadmissible in a treatment aiming at rigor, it is not possible to determine whether or not periodic solutions exist. Poincaré made a few remarks* upon this subject, relating his methods to the equations of Hill, which lack the parallactic terms. Burrau discovered several orbits in a special case from successive trial computations by mechanical quadratures.† Perchot and Mascart treated the special case in which the finite masses are equal.‡ Sir George Darwin found examples of these orbits about two of the points of equilibrium in his celebrated memoir on Periodic Orbits.|| His methods, like those of Burrau, were purely numerical. Under the assumption that the orbits exist, Plummer gave a convenient literal development of expressions for the coördinates.§ His method is simple, but apparently it is not easily extensible to most of the more complicated cases. All of the writers mentioned have treated the problem only in the plane of motion of the finite masses. It would be practically impossible to discover three-dimensional orbits by numerical processes, but there would be no difficulty in applying Plummer's method to infinitesimal satellites oscillating in three dimensions when the finite bodies describe circular orbits.

76. The Differential Equations of Motion.—Let us take the origin at the center of gravity of the system and refer the motion of the infinitesimal body to a set of axes, ξ, η, ζ. We will choose the ξ and η-axes in the plane of motion of the finite bodies, and suppose that they rotate in the direction of motion of the system, with the same angular velocity. The initial position of the axes will be determined so that the finite bodies continually lie on the ξ-axis. The distance between the finite bodies will be taken as the unit of length, the sum of the masses as the unit of mass, and the unit of time will be chosen so that the Gaussian constant is unity. Let the masses

Les Méthodes Nouvelles de la Mécanique Céleste, vol. I (1892), p. 159.

†*Astronomische Nachrichten*, Nos. 3230, 3251 (1894).

‡*Bulletin Astronomique*, vol. XII (1895), p. 329. Apparently their work is vitiated by an error in establishing the existence of the solutions, and their construction fails where they stopped.

||*Acta Mathematica*, vol. XXI (1897), p. 99.

§*Monthly Notices, Royal Astronomical Society*, vol. LXIII (1903), p. 436, and vol. LXIV (1903), p. 98.

of the finite bodies be $1-\mu$ and μ. The units have been chosen so that the angular velocity of revolution is unity. The differential equations of motion of the infinitesimal body are then*

$$
\left.\begin{aligned}
&\frac{d^2\xi}{dt^2}-2\frac{d\eta}{dt}=\frac{\partial U}{\partial\xi}, \qquad \frac{d^2\eta}{dt^2}+2\frac{d\xi}{dt}=\frac{\partial U}{\partial\eta}, \qquad \frac{d^2\zeta}{dt^2}=\frac{\partial U}{\partial\zeta},\\
&2U=\xi^2+\eta^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}=(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\mu\left(r_2^2+\frac{2}{r_2}\right)-\zeta^2-\mu(1-\mu),\\
&r_1=\sqrt{(\xi-\xi_1)^2+\eta^2+\zeta^2}, \quad r_2=\sqrt{(\xi-\xi_2)^2+\eta^2+\zeta^2}, \quad \xi_1=-\mu, \quad \xi_2=1-\mu.
\end{aligned}\right\} \tag{1}
$$

Necessary and sufficient conditions for a solution in which the infinitesimal body is at rest relatively to the finite masses are

$$
\frac{\partial U}{\partial\xi}=\frac{\partial U}{\partial\eta}=\frac{\partial U}{\partial\zeta}=0. \tag{2}
$$

The second and third of these equations are satisfied by $\eta=\zeta=0$, whatever ξ may be. The first equation has three solutions:† (*a*) one between $+\infty$ and the finite mass μ, (*b*) one between μ and $1-\mu$, and (*c*) one between $1-\mu$ and $-\infty$.

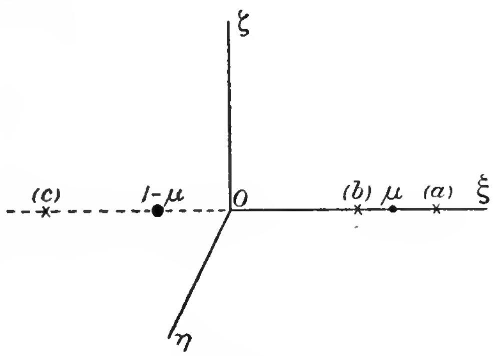

FIG. 2.

These three solutions are the real positive roots of the quintic equations

$$
\left.\begin{aligned}
&(a) \quad r_2^5+(3-\mu)r_2^4+(3-2\mu)r_2^3-\mu r_2^2-2\mu r_2-\mu=0,\\
&(b) \quad r_2^5-(3-\mu)r_2^4+(3-2\mu)r_2^3-\mu r_2^2+2\mu r_2-\mu=0,\\
&(c) \quad \rho^5-(7+\mu)\rho^4+(19+6\mu)\rho^3-(24+13\mu)\rho^2+(12+14\mu)\rho-7\mu=0,
\end{aligned}\right\} \tag{3}
$$

where $\rho=2-r_2$, and where r_2 is the distance from μ to the equilibrium point. The real positive solutions of (3) are respectively

$$
\left.\begin{aligned}
&(a) \quad r_2^{(0)}=\left(\frac{\mu}{3}\right)^{\frac{1}{3}}+\frac{1}{3}\left(\frac{\mu}{3}\right)^{\frac{2}{3}}-\frac{1}{9}\left(\frac{\mu}{3}\right)^{\frac{3}{3}}\cdots,\\
&(b) \quad r_2^{(0)}=\left(\frac{\mu}{3}\right)^{\frac{1}{3}}-\frac{1}{3}\left(\frac{\mu}{3}\right)^{\frac{2}{3}}-\frac{1}{9}\left(\frac{\mu}{3}\right)^{\frac{3}{3}}\cdots,\\
&(c) \quad r_2^{(0)}=2-\frac{7}{12}\mu-\frac{23\times 7^2}{12^4}\mu^2\cdots.
\end{aligned}\right\} \tag{4}
$$

*Moulton's *Introduction to Celestial Mechanics*, p. 185.

†See *Introduction to Celestial Mechanics*, Art. 121, and especially Charlier's *Die Mechanik des Himmels*, vol. II, pp. 102–111, for a detailed discussion.

Suppose the coördinates of (a), (b), or (c) are $\xi=\xi_0$, $\eta=0$, $\zeta=0$, the value of ξ_0 depending upon which point is in question. It will not be necessary to distinguish among them except in numerical computation. Now give the infinitesimal body a small displacement from one of these points, and a small velocity with respect to the finite masses such that

$$\left.\begin{aligned} &\xi=\xi_0+x', && \eta=0+y', && \zeta=0+z', \\ &\frac{d\xi}{dt}=0+\frac{dx'}{dt}, && \frac{d\eta}{dt}=0+\frac{dy'}{dt}, && \frac{d\zeta}{dt}=0+\frac{dz'}{dt}. \end{aligned}\right\} \quad (5)$$

The differential equations (1) are transformed by these relations into

$$\left.\begin{aligned} &\frac{d^2x'}{dt^2}-2\frac{dy'}{dt}=\frac{\partial U}{\partial x'}=+P_1(x', y'^2, z'^2), \\ &\frac{d^2y'}{dt^2}+2\frac{dx'}{dt}=\frac{\partial U}{\partial y'}=y'P_2(x', y'^2, z'^2), \\ &\frac{d^2z'}{dt^2}=+Z=\frac{\partial U}{\partial z'}=z'P_3(x', y'^2, z'^2); \\ &U=\tfrac{1}{2}(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\tfrac{1}{2}\mu\left(r_2^2+\frac{2}{r_2}\right)-\tfrac{1}{2}z'^2-\tfrac{1}{2}\mu(1-\mu), \\ &r_1=\sqrt{(\xi_0+x'+\mu)^2+y'^2+z'^2}, \qquad r_2=\sqrt{(\xi_0-1+x'+\mu)^2+y'^2+z'^2}, \end{aligned}\right\} \quad (6)$$

where P_1, P_2, and P_3 are power series in x', y'^2, and z'^2.

77. Regions of Convergence of the Series P_1, P_2, P_3.—It follows from the form of U in equations (6) that P_1, P_2, and P_3 converge for the common region of convergence of the expansions of $1/r_1$ and $1/r_2$. We are considering only real values of x', y', and z', and consequently the conditions for the convergence of the expansions of $1/r_1$ and $1/r_2$ as power series in x', y', and z' are respectively

$$\left.\begin{aligned} &-1<\frac{2x'}{\xi_0+\mu}+\frac{x'^2+y'^2+z'^2}{(\xi_0+\mu)^2}<+1, \\ &-1<\frac{2x'}{\xi_0-1+\mu}+\frac{x'^2+y'^2+z'^2}{(\xi_0-1+\mu)^2}<+1. \end{aligned}\right\} \quad (7)$$

The surfaces which bound the regions of convergence of the expansions are obtained by replacing these inequalities by equalities. For the convergence of the expansion of $1/r_1$, the equations of the bounding surfaces are

$$\left.\begin{aligned} &x'^2+y'^2+z'^2+2(\xi_0+\mu)x'+(\xi_0+\mu)^2=0, \\ &x'^2+y'^2+z'^2+2(\xi_0+\mu)x'-(\xi_0+\mu)^2=0. \end{aligned}\right\} \quad (8)$$

The first is the equation of the point occupied by the finite body $1-\mu$. The second is the equation of a sphere whose center is at $1-\mu$ and whose radius is $\sqrt{2(\xi_0+\mu)^2}$. The convergence of the expansion of $1/r_1$ holds for the space between the point and this sphere.

The equations of corresponding surfaces for the expansion of $1/r_2$ are

$$\left.\begin{aligned} x'^2+y'^2+z'^2+2(\xi_0-1+\mu)\,x'+(\xi_0-1+\mu)^2=0,\\ x'^2+y'^2+z'^2+2(\xi_0-1+\mu)\,x'-(\xi_0-1+\mu)^2=0. \end{aligned}\right\} \qquad (9)$$

These are respectively the equations of the point occupied by the mass μ and of a sphere whose center is at μ and whose radius is $\sqrt{2(\xi_0-1+\mu)^2}$. The convergence of the expansion of $1/r_2$ holds for all points within this sphere except the center.

The distances from $1-\mu$ and μ to the point (a) are respectively $\xi_0+\mu$ and $\xi_0-1+\mu$. The radii of the spheres which have been defined in (8) and (9) are $\sqrt{2}$ times these distances. Since $\sqrt{2}\,(\xi_0+\mu)-1>\sqrt{2}\,(\xi_0-1+\mu)$, the sphere around $1-\mu$ as a center is entirely outside of the one around μ as a center. Consequently, the series P_1, P_2, and P_3 converge in the case of the transformation to the point (a) for all points within the sphere whose center is at μ and whose radius is $\sqrt{2}\,(\xi_0+\mu)$, except the point μ itself.

The distances from $1-\mu$ and μ to the point (b) are $\xi_0+\mu$ and $\sqrt{(\xi_0-1+\mu)^2}$. The radii of the two spheres are $\sqrt{2}$ times these distances, and hence they both include the point (b) in their interiors. In this case the two spheres intersect unless μ is small, when one will be entirely within the other.

The distances from $1-\mu$ and μ to the point (c) are $-\xi_0-\mu$ and $1-\xi_0-\mu$. Since $\sqrt{2}\,(1-\xi_0-\mu)-1>\sqrt{2}\,(-\xi_0-\mu)$, the sphere around μ as a center includes in its interior the one around $1-\mu$ as a center. The latter includes (c) in its interior, and everywhere within it, except at $1-\mu$, the series P_1, P_2, and P_3 converge.

78. Introduction of the Parameters ϵ and δ.—Let us now make the transformations

$$x'=x\epsilon', \qquad y'=y\epsilon', \qquad z'=z\epsilon' \qquad (\epsilon'\neq 0), \qquad t-t_0=(1+\delta)\,\tau, \qquad (10)$$

where ϵ' and δ are constant, but at present undetermined, parameters. Then equations (6) become

$$\left.\begin{aligned} \frac{d^2x}{d\tau^2}-2(1+\delta)\frac{dy}{d\tau} &= (1+\delta)^2P_1(x,\,y^2,\,z^2)\\ &= (1+\delta)^2\,[X_1+X_2\,\epsilon'+\cdots+X_n\,(\epsilon')^{n-1}+\cdots],\\ \frac{d^2y}{d\tau^2}+2(1+\delta)\frac{dx}{d\tau} &= (1+\delta)^2\,yP_2(x,\,y^2,\,z^2)\\ &= (1+\delta)^2\,[Y_1+Y_2\,\epsilon'+\cdots+Y_n\,(\epsilon')^{n-1}+\cdots],\\ \frac{d^2z}{d\tau^2} = (1+\delta)^2Z &= (1+\delta)^2zP_3(x,\,y^2,\,z^2)\\ &= (1+\delta)^2\,[Z_1+Z_2\,\epsilon'+\cdots+Z_n\,(\epsilon')^{n-1}+\cdots], \end{aligned}\right\} \qquad (11)$$

where X_n, Y_n, Z_n are homogeneous functions of x, y, and z of degree n. These differential equations are valid for all values of x, y, z, and ϵ' satisfying the conditions for convergence which have been developed.

We shall now generalize the parameter ϵ' (see §13) by replacing it everywhere by ϵ, where ϵ may have the value zero or any value in its neighborhood. When $\epsilon \neq \epsilon'$, the differential equations belong to a purely mathematical problem; but when $\epsilon = \epsilon'$ they belong to the physical problem. Since the value of ϵ' has not been specified except that it is distinct from zero, the generalization may appear trivial, but the same method can be used where the parameter corresponding to ϵ' does not have this arbitrary character, and where the device is of the highest importance. We have therefore to consider the differential equations

$$\left.\begin{aligned}
\frac{d^2x}{d\tau^2} - 2(1+\delta)\frac{dy}{d\tau} &= (1+\delta)^2[X_1+X_2\epsilon+ \cdots +X_n\epsilon^{n-1}+ \cdots], \\
\frac{d^2y}{d\tau^2} + 2(1+\delta)\frac{dx}{d\tau} &= (1+\delta)^2[Y_1+Y_2\epsilon+ \cdots +Y_n\epsilon^{n-1}+ \cdots], \\
\frac{d^2z}{d\tau^2} = (1+\delta)^2 Z &= (1+\delta)^2[Z_1+Z_2\epsilon+ \cdots +Z_n\epsilon^{n-1}+ \cdots].
\end{aligned}\right\} \quad (12)$$

79. Jacobi's Integral.—Equations (1) admit the integral

$$\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 = 2U - C,$$

where C is the constant of integration. This integral was first given by Jacobi in *Comptes Rendus de l'Académie des Sciences de Paris*, vol. III, p. 59. For equations (12) there is the corresponding integral

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 = 2(1+\delta)^2U - C, \quad (13)$$

where U is now a power series in ϵ.

80. The Symmetry Theorem.—Let us consider the solution of equations (12) and suppose that at $\tau = 0$ we have

$$x = c_1, \qquad \frac{dy}{d\tau} = c_2, \qquad \frac{dz}{d\tau} = c_3, \qquad y = z = \frac{dx}{d\tau} = 0;$$

that is, that the infinitesimal body crosses the x-axis perpendicularly at $\tau = 0$. The solution will have the form

$$\begin{aligned}
x &= f_1(\tau), & y &= f_2(\tau), & z &= f_3(\tau), \\
\frac{dx}{d\tau} &= f_1'(\tau), & \frac{dy}{d\tau} &= f_2'(\tau), & \frac{dz}{d\tau} &= f_3'(\tau).
\end{aligned}$$

Now transform equations (12) by the substitution

$$\begin{aligned}
x &= +x', & y &= -y', & z &= -z', & \tau &= -\tau', \\
\frac{dx}{d\tau} &= -\frac{dx'}{d\tau'}, & \frac{dy}{d\tau} &= +\frac{dy'}{d\tau'}, & \frac{dz}{d\tau} &= +\frac{dz'}{d\tau'}.
\end{aligned}$$

The equations in the new variables are precisely the same as in the old; consequently, if the values of the dependent variables at $\tau'=0$ are

$$x'=c_1, \qquad \frac{dy'}{d\tau'}=c_2, \qquad \frac{dz'}{d\tau'}=c_3, \qquad y'=z'=\frac{dx'}{d\tau'}=0,$$

the solution is

$$x'=f_1(\tau'), \qquad y'=f_2(\tau'), \qquad z'=f_3(\tau'),$$

$$\frac{dx'}{d\tau'}=f_1'(\tau), \qquad \frac{dy'}{d\tau'}=f_2'(z'), \qquad \frac{dz'}{d\tau'}=f_3'(\tau').$$

Now it follows from the relations between the two sets of variables that

$$\begin{aligned} f_1(\tau)&=+f_1(\tau')=+f_1(-\tau), & f_1'(\tau)&=-f_1'(\tau')=-f_1'(-\tau),\\ f_2(\tau)&=-f_2(\tau')=-f_2(-\tau), & f_2'(\tau)&=+f_2'(\tau')=+f_2'(-\tau),\\ f_3(\tau)&=-f_3(\tau')=-f_3(-\tau), & f_3'(\tau)&=+f_3'(\tau')=+f_3'(-\tau).\end{aligned}$$

Therefore, if the infinitesimal body is projected perpendicularly from the x-axis, then x, $dy/d\tau$, and $dz/d\tau$ are even functions of τ, and $dx/d\tau$, y, and z are odd functions of τ; that is, the orbit is geometrically symmetrical with respect to the x-axis, and it is symmetrical in τ with respect to the time of crossing.

81. Outline of Steps for Proving the Existence of Periodic Solutions of Equations (12).—In (12) we put $\delta=\epsilon=0$ and find the general solutions of the resulting equations. For special values of the constants of integration there are periodic solutions. Then we change the initial values of the dependent variables by small amounts and take $\delta\neq 0$, $\epsilon\neq 0$. The equations are integrated as power series in δ and ϵ and in the increments to the initial values of the dependent variables. By §11, these parameters can be taken so small in numerical value that the solutions will converge for all τ in any preassigned range, and in particular for the periods of the periodic solutions obtained when $\epsilon=0$.

After having formed the solutions as power series in the parameters, the conditions are imposed that the solutions shall be periodic with the same period in τ(not in t) as the generating solutions have for $\epsilon=0$. These conditions are that the orbit shall re-enter at the end of the period; or, in the case of the symmetrical orbits, that they shall cross the x-axis perpendicularly at the half period. These periodicity conditions are relations imposed upon the initial values of the dependent variables and upon δ. It is shown that these conditions can be satisfied by expressing δ and the initial values of the dependent variables as power series in ϵ, and these series converge for the modulus of ϵ sufficiently small.

82. General Solutions of Equations (12) for $\delta=\epsilon=0$.—On referring to equations (1) and the succeeding transformations, we find that equations (12), for $\delta=\epsilon=0$, become explicitly*

$$\left.\begin{aligned} \frac{d^2x}{d\tau^2}-2\frac{dy}{d\tau} &= X_1=(1+2A)x, \\ \frac{d^2y}{d\tau^2}+2\frac{dx}{d\tau} &= Y_1= (1-A)\,y, \\ \frac{d^2z}{d\tau^2} &= Z_1=-A\,z, \end{aligned}\right\} \tag{14}$$

where

$$A=\frac{1-\mu}{r_1^{(0)3}}+\frac{\mu}{r_2^{(0)3}}=\frac{1-\mu}{[(\xi_0+\mu)^2]^{\frac{3}{2}}}+\frac{\mu}{[(\xi_0-1+\mu)^2]^{\frac{3}{2}}}. \tag{15}$$

The third equation of (14) is independent of the first two, and its general solution is

$$z=c_1\cos\sqrt{A}\,\tau+c_2\sin\sqrt{A}\,\tau, \tag{16}$$

where c_1 and c_2 are the constants of integration.

The first two equations of (14) are linear and homogeneous, and they have constant coefficients. To find their solution, let

$$x=Ke^{\lambda\tau}, \qquad y=Le^{\lambda\tau}, \tag{17}$$

where K and L are constants. The conditions that these expressions shall identically satisfy (14) are

$$[\lambda^2-(1+2A)]\,K-2\lambda L=0, \qquad 2\lambda K+[\lambda^2-(1-A)]\,L=0. \tag{18}$$

In order that these equations may have a solution for K and L other than the trivial one $K=L=0$, we must impose the condition

$$\Delta\equiv\begin{vmatrix} \lambda^2-(1+2A), & -2\lambda \\ +2\lambda, & \lambda^2-(1-A) \end{vmatrix}=\lambda^4+(2-A)\lambda^2+(1-A)(1+2A)=0. \tag{19}$$

We shall now discuss the roots of this biquadratic equation in λ. Its discriminant is

$$D=(2-A)^2-4(1-A)(1+2A)=(9A-8)A. \tag{20}$$

We shall show that $1-A$ is negative for each of the points (a), (b), and (c) for all values of μ†, and therefore that D is positive. From (15) we have

$$1-A=1-\frac{1-\mu}{r_1^{(0)3}}-\frac{\mu}{r_2^{(0)3}}=(1-\mu)\left[1-\frac{1}{r_1^{(0)3}}\right]+\mu\left[1-\frac{1}{r_2^{(0)3}}\right]. \tag{21}$$

At the points (a), (b), and (c) we have

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x}+\frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x}=0. \tag{22}$$

*See also *Introduction to Celestial Mechanics*, and Charlier's *Mechanik des Himmels*, vol. II, pp. 117–137.

†First proved for all μ by H. C. Plummer, *Monthly Notices of Royal Astronomical Society*, vol. LXII (1901).

It is found from the definition of U in (1) that

$$\frac{\partial U}{\partial r_1}=(1-\mu)\left(r_1-\frac{1}{r_1^2}\right),\qquad \frac{\partial U}{\partial r_2}=\mu\left(r_2-\frac{1}{r_2^2}\right). \tag{23}$$

For the point (a) the relation between $r_1^{(0)}$ and $r_2^{(0)}$ is $r_1^{(0)}=1+r_2^{(0)}$, and therefore

$$\frac{\partial r_1^{(0)}}{\partial x}=\frac{\partial r_2^{(0)}}{\partial x}.$$

Hence, equations (22) and (23) give for this point

$$\frac{\partial U}{\partial r_1^{(0)}}=-\frac{\partial U}{\partial r_2^{(0)}},\qquad (1-\mu)\left[r_1^{(0)}-\frac{1}{r_1^{(0)2}}\right]=-\mu\left[r_2^{(0)}-\frac{1}{r_2^{(0)2}}\right].$$

Therefore, since the first two factors are positive while the third is negative, (21) becomes, for the equilibrium point (a),

$$1-A=(1-\mu)\left(1-\frac{1}{r_1^{(0)3}}\right)\left(1-\frac{r_1^{(0)}}{r_2^{(0)}}\right)<0. \tag{24}$$

Similarly, since only the second factor in the expression for $1-A$ is negative, for the point (b) we find

$$\left.\begin{aligned}&r_1^{(0)}=1-r_2^{(0)},\qquad \frac{\partial r_1^{(0)}}{\partial x}=-\frac{\partial r_2^{(0)}}{\partial x},\qquad \frac{\partial U}{\partial r_1^{(0)}}=\frac{\partial U}{\partial r_2^{(0)}},\\ &(1-\mu)\left(r_1^{(0)}-\frac{1}{r_1^{(0)2}}\right)=\mu\left(r_2^{(0)}-\frac{1}{r_2^{(0)2}}\right),\quad 1-A=(1-\mu)\left(1-\frac{1}{r_1^{(0)3}}\right)\left(1+\frac{r_1^{(0)}}{r_2^{(0)}}\right)<0.\end{aligned}\right\} \tag{25}$$

For the point (c) we have the corresponding equations

$$\left.\begin{aligned}&r_1^{(0)}=-1+r_2^{(0)},\qquad \frac{\partial r_1^{(0)}}{\partial x}=\frac{\partial r_2^{(0)}}{\partial x},\qquad \frac{\partial U}{\partial r_1^{(0)}}=-\frac{\partial U}{\partial r_2^{(0)}},\\ &(1-\mu)\left(r_1^{(0)}-\frac{1}{r_1^{(0)2}}\right)=-\mu\left(r_2^{(0)}-\frac{1}{r_2^{(0)2}}\right),\quad 1-A=\mu\left(1-\frac{1}{r_2^{(0)3}}\right)\left(1-\frac{r_2^{(0)}}{r_1^{(0)}}\right).\end{aligned}\right\} \tag{26}$$

Then $1-A$ is negative because the third factor alone is negative.

Since $1-A$ is negative in every case for $0 \gtreqless \mu \leqq 0.5$, it follows that two of the roots of (19) are real and equal numerically but opposite in sign, and that the other two are conjugate pure imaginaries. Let the real roots be $\pm\rho$ and the imaginary $\pm\sigma\sqrt{-1}$. For each of these roots there is a particular solution (17), and the general solution is

$$\left.\begin{aligned}x&=K_1\,e^{\sigma\sqrt{-1}\tau}+K_2\,e^{-\sigma\sqrt{-1}\tau}+K_3\,e^{\rho\tau}+K_4\,e^{-\rho\tau},\\ y&=L_1\,e^{\sigma\sqrt{-1}\tau}+L_2\,e^{-\sigma\sqrt{-1}\tau}+L_3\,e^{\rho\tau}+L_4\,e^{-\rho\tau},\end{aligned}\right\} \tag{27}$$

where, from (18),

$$\left.\begin{aligned}L_1&=\sqrt{-1}\,\frac{[\sigma^2+1+2A]}{2\sigma}K_1=\sqrt{-1}\,n\,K_1=\frac{-K_1}{K_2}L_2,\\ L_3&=\quad+\frac{[\rho^2-1-2A]}{2\rho}K_3=\quad+m\,K_3=\frac{-K_3}{K_4}L_4.\end{aligned}\right\} \tag{28}$$

The constants m and n are defined by these equations.

83. Periodic Solutions when $\delta=\epsilon=0$.—The general solution of equations (14) is contained in equations (16) and (27). One periodic solution is

$$x=y=0, \qquad z=c_1\cos\sqrt{A}\tau+c_2\sin\sqrt{A}\tau,$$

the period of this solution being $2\pi/\sqrt{A}$. The constants c_1 and c_2, and the t_0 on which τ depends, are not independent. We shall suppose t_0 is taken so that $c_1=0$, $c_2=c/\sqrt{A}$. Then one of the periodic generating solutions which we have to consider is

$$x=y=0, \qquad z=\frac{c}{\sqrt{A}}\sin\sqrt{A}\,\tau. \tag{29}$$

These will be called *Orbits of Class A*.

Another periodic solution of the differential equations (14) is

$$x=K_1\,e^{\sigma\sqrt{-1}\tau}+K_2\,e^{-\sigma\sqrt{-1}\tau}, \qquad y=n\sqrt{-1}\,(K_1\,e^{\sigma\sqrt{-1}\tau}-K_2\,e^{-\sigma\sqrt{-1}\tau}), \qquad z=0.$$

If the initial conditions are real, as we suppose, then K_1 and K_2 are conjugate complex quantities. We shall suppose t_0 is chosen so that the imaginary part of K_1 and K_2 is zero. Let $a/2$ represent their real part. Then we have, as the second periodic generating solution,

$$x=a\cos\sigma\tau, \qquad y=-na\sin\sigma\tau, \qquad z=0, \tag{30}$$

the period being $2\pi/\sigma$. These will be called *Orbits of Class B*.

A third periodic solution will exist if σ and $\sqrt{A}$ are commensurable. We shall first prove the possibility of their being commensurable. The condition for commensurability is $\sigma/\sqrt{A}=p/q$, where p and q are positive integers. Since $\sigma\sqrt{-1}$ satisfies (19), we have from this relation

$$\frac{p^4}{q^4}A^2-(2-A)A\frac{p^2}{q^2}+(1-A)(1+2A)=0.$$

The solution of this quadratic equation for A is found to be

$$A=\frac{2\frac{p^2}{q^2}-1\pm\sqrt{9-8\frac{p^2}{q^2}}}{2\left(\frac{p^2}{q^2}+2\right)\left(\frac{p^2}{q^2}-1\right)}. \tag{31}$$

In order to establish the possibility of the commensurability of σ and $\sqrt{A}$, it is sufficient to show that p and q can be assigned such positive integral values that the A defined in (31) shall have a value equal to that obtained from (15) for some μ between 0 and 0.5.

It is to be observed first that the solutions of (3) are continuous functions of μ, and consequently A, as defined by (15), is a continuous function of μ. Therefore, if there are positive integral values of p and q such that the A defined by (31) lies in the range of values of A as defined by (15), there are infinitely many values of μ for which σ and $\sqrt{A}$ are commensurable.

On taking the upper sign in (31), we find that in order that A may be real and positive we must have $1<p^2/q^2<9/8$. For values of p and q satisfying these inequalities, A lies between $+\infty$ and $8/5$. On taking the lower sign in (31), we must have $0<p^2/q^2<9/8$ in order that A may be real and positive. The values of A for these limits are unity and $8/5$. Equation (31) takes the indeterminate form $0\div 0$ for $p=q$, but it is easily found that the corresponding value of A is unity. For $0<p^2/q^2<1$, the value of A, defined by (31) with the lower sign, is less than unity. On taking both the upper and lower signs, it follows that A takes values in every finite interval from 1 to ∞ as p^2/q^2 goes over all rational fractions from 0 to $9/8$.

It was proved in the preceding article that $1-A<0$ for each one of the three solution points (a), (b), and (c). Therefore for these points and $0 \overline{\gtrless} \mu<0.5$, we have $A>1$. Consequently, there are infinitely many values of μ between 0 and 0.5, such that σ and $\sqrt{A}$ are commensurable. When the commensurability relation is satisfied we have the periodic solution

$$
\begin{aligned}
x &= K_1 e^{\sigma\sqrt{-1}\,\tau}+K_2 e^{-\sigma\sqrt{-1}\,\tau},\\
y &= n\sqrt{-1}\,(K_1 e^{\sigma\sqrt{-1}\,\tau}-K_2 e^{-\sigma\sqrt{-1}\,\tau}),\\
z &= c_1\cos\sqrt{A}\,\tau+c_2\sin\sqrt{A}\,\tau.
\end{aligned}
$$

We can choose t_0 so that $c_1=0$, and let a represent twice the real part of K_1 and K_2, and b twice the imaginary part of $-K_1$ and K_2. Then this solution becomes

$$
x=a\cos\sigma\tau+b\sin\sigma\tau,\quad y=na\sin\sigma\tau+nb\cos\sigma\tau,\quad z=\frac{c}{\sqrt{A}}\sin\sqrt{A}\,\tau. \tag{32}
$$

The period of this solution is $P=2\pi p/\sigma=2\pi q/\sqrt{A}$. In this period x and y make p complete oscillations, and z makes q complete oscillations. These will be called *Orbits of Class C.*

84. Normal Form for the Differential Equations.—We are about to prove that when $\epsilon\neq 0$ the initial values of x, y, z, and their derivatives can be so determined, depending on ϵ, that periodic solutions having the periods of (29) and (30) exist for all values of ϵ sufficiently small, and reduce to these solutions for $\epsilon=0$. In this discussion it is convenient to have the differential equations in a normal form, and it is necessary to compute the first terms of the solutions as power series in δ, ϵ, and the increments to the initial values of the dependent variables.

The linear terms of (12) are found by (14) to be

$$
\begin{aligned}
&\frac{d^2x}{d\tau^2}-2(1+\delta)\frac{dy}{d\tau}-(1+\delta)^2(1+2A)x=0,\\
&\frac{d^2y}{d\tau^2}+2(1+\delta)\frac{dx}{d\tau}-(1+\delta)^2(1-A)y=0,\\
&\frac{d^2z}{d\tau^2}+(1+\delta)^2Az=0.
\end{aligned}
$$

The general solution of the first two of these equations is

$$x=K_1\,e^{\sigma\sqrt{-1}(1+\delta)\tau}+K_2\,e^{-\sigma\sqrt{-1}(1+\delta)\tau}+K_3\,e^{\rho(1+\delta)\tau}+K_4\,e^{-\rho(1+\delta)\tau},$$
$$y=n\sqrt{-1}\,[K_1\,e^{\sigma\sqrt{-1}(1+\delta)\tau}-K_2\,e^{-\sigma\sqrt{-1}(1+\delta)\tau}]+m[K_3\,e^{\rho(1+\delta)\tau}-K_4\,e^{-\rho(1+\delta)\tau}].$$

Therefore we see that the transformation

$$\left.\begin{aligned}
x &=(u_1+u_2)+(u_3+u_4),\\
\frac{dx}{d\tau} &=\sigma(1+\delta)\sqrt{-1}\,(u_1-u_2)+\rho(1+\delta)(u_3-u_4),\\
y &=n\sqrt{-1}\,(u_1-u_2)+m(u_3-u_4),\\
\frac{dy}{d\tau} &=-n\sigma(1+\delta)(u_1+u_2)+m\rho(1+\delta)(u_3+u_4),
\end{aligned}\right\}\tag{33}$$

changes (12) into

$$\left.\begin{aligned}
&\frac{du_1}{d\tau}-\sigma(1+\delta)iu_1=\frac{+m(1+\delta)[\ ]\,\epsilon}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{(1+\delta)\{\ \}\,\epsilon}{2(m\rho+n\sigma)}\qquad(i=\sqrt{-1}),\\
&\frac{du_2}{d\tau}+\sigma(1+\delta)iu_2=\frac{-m(1+\delta)[\ ]\,\epsilon}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{(1+\delta)\{\ \}\,\epsilon}{2(m\rho+n\sigma)},\\
&\frac{du_3}{d\tau}-\rho(1+\delta)\,u_3=\frac{-n(1+\delta)[\ ]\,\epsilon}{2(m\sigma-n\rho)}+\frac{(1+\delta)\{\ \}\,\epsilon}{2(m\rho+n\sigma)},\\
&\frac{du_4}{d\tau}+\rho(1+\delta)\,u_4=\frac{+n(1+\delta)[\ ]\,\epsilon}{2(m\sigma-n\rho)}+\frac{(1+\delta)\{\ \}\,\epsilon}{2(m\rho+n\sigma)},\\
&\frac{d^2z}{d\tau^2}+(1+\delta)^2Az=(1+\delta)^2 3Bxz\epsilon+\frac{3}{2}(1+\delta)^2C[-4x^2z+y^2z+z^3]\epsilon^2+\cdots,
\end{aligned}\right\}\tag{34}$$

where

$$\left.\begin{aligned}
&[\ ]=\frac{3}{2}B[-2x^2+y^2+z^2]+2C[2x^3-3xy^2-3xz^2]\,\epsilon+\cdots,\\
&\{\ \}=3B\{xy\}+\frac{3}{2}C\{-4x^2y+y^3+yz^2\}\,\epsilon+\cdots,\\
&B=\begin{smallmatrix}+\\ \pm\end{smallmatrix}\frac{1-\mu}{r_1^{(0)4}}\begin{smallmatrix}+\\ \pm\\ -\end{smallmatrix}\frac{\mu}{r_2^{(0)4}},\qquad C=\frac{1-\mu}{r_1^{(0)5}}+\frac{\mu}{r_2^{(0)5}}.
\end{aligned}\right\}\tag{35}$$

In B the upper, middle, or lower signs are to be used according as solutions in the vicinity of the point (a), (b), or (c) are being treated. This transformation is always valid, since we find from (28) that

$$2(m\sigma-n\rho)=-(1+2A)\,\frac{(\sigma^2+\rho^2)}{\sigma\rho}\neq 0,\qquad 2(m\rho+n\sigma)=(\sigma^2+\rho^2)\neq 0.\tag{36}$$

It is not advantageous to transform the z-equation, nor [ ] and { }.

It follows from (1) that whenever the infinitesimal body is displaced from the xy-plane, it is always subject to a component of acceleration *toward* this plane. Therefore it can not revolve in a closed orbit entirely on one side of the xy-plane. Hence we may determine t_0 so that $z=0$ when $\tau=0$. That is, without loss of generality we can take the initial value of z as zero.

Let the initial conditions be

$$u_i = a_i + \alpha_i \qquad (i=1, \ldots, 4), \qquad z=0, \qquad \frac{dz}{d\tau} = c+\gamma,$$

where the a_i and c can be given such values that we shall have, for $\epsilon = 0$, either (29), (30), or (32). The α_i and γ are to be determined in terms of ϵ so that the solutions shall remain periodic for $\epsilon \neq 0$.

Instead of integrating (34) directly in powers of all the parameters $\alpha_1, \ldots, \alpha_4, \gamma, \delta$, and ϵ, we can more conveniently integrate them as power series in ϵ; the parameter δ can be introduced in connection with τ, since it always occurs in the combination $(1+\delta)\tau$; and the parameters $\alpha_1, \ldots, \alpha_4$, and γ can be introduced when the constants of integration are determined.

The terms which are independent of ϵ are defined by the equations

$$\left.\begin{aligned} &\frac{du_1^{(0)}}{d\tau} - \sigma(1+\delta)\sqrt{-1}\,u_1^{(0)} = 0, \qquad && \frac{du_3^{(0)}}{d\tau} - \rho(1+\delta)u_3^{(0)} = 0, \\ &\frac{du_2^{(0)}}{d\tau} + \sigma(1+\delta)\sqrt{-1}\,u_2^{(0)} = 0, && \frac{du_4^{(0)}}{d\tau} + \rho(1+\delta)u_4^{(0)} = 0, \\ &\qquad\qquad \frac{d^2 z_0}{d\tau^2} + A(1+\delta)^2 z_0 = 0. \end{aligned}\right\} \qquad (37)$$

The solutions of these equations which satisfy the initial conditions are

$$\left.\begin{aligned} &u_1^{(0)} = (a_1+\alpha_1)\, e^{+\sigma(1+\delta)\sqrt{-1}\tau}, \qquad && u_3^{(0)} = (a_3+\alpha_3)\, e^{+\rho(1+\delta)\tau}, \\ &u_2^{(0)} = (a_2+\alpha_2)\, e^{-\sigma(1+\delta)\sqrt{-1}\tau}, && u_4^{(0)} = (a_4+\alpha_4)\, e^{-\rho(1+\delta)\tau}, \\ &\qquad z_0 = \frac{(c+\gamma)}{\sqrt{A}(1+\delta)} \sin\sqrt{A}(1+\delta)\tau. \end{aligned}\right\} \qquad (38)$$

These expressions can at once be expanded as power series in δ. The coefficients of higher powers of ϵ can be found by the usual process, but we shall not need them in proving the existence of periodic solutions.

85. Existence of Periodic Orbits of Class A.—For $\epsilon = 0$ the coördinates in these orbits are given in (29). Therefore, since the determinant of the transformation (33), viz., $\Delta = 4(m\rho + n\sigma)(m\sigma - n\rho)\sqrt{-1}$, is distinct from zero, it follows that in this case $a_1 = a_2 = a_3 = a_4 = 0$. The general solutions of (34) are of the form

$$\left.\begin{aligned} u_i &= P_i(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \tau) \qquad (i=1, \ldots, 4), \\ z &= P_5(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \tau), \\ z' &= P_6(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \tau), \end{aligned}\right\} \qquad (39)$$

where the P_i are power series in $\alpha_1, \ldots, \alpha_4, \gamma, \delta$, and ϵ, and where z' denotes the derivative of z with respect to τ.

Since equations (34) do not involve τ explicitly, sufficient conditions that the solutions (39) shall be periodic in τ with the period $2\pi/\sqrt{A}$ are

$$P_i\left(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \frac{2\pi}{\sqrt{A}}\right) - P_i(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; 0) = 0. \qquad (40)$$

These conditions are not all necessary, for it can be shown that the last one is a consequence of the first five. If we make the transformation

$$u_i = \alpha_i + v_i, \qquad z = \frac{(c+\gamma)}{\sqrt{A}} \sin\sqrt{A}\,\tau + \zeta, \qquad z' = (c+\gamma)\cos\sqrt{A}\,\tau + \zeta',$$

the integral (13) may be written

$$F\left(\alpha_i + v_i, \frac{c+\gamma}{\sqrt{A}} \sin\sqrt{A}\,\tau + \zeta, (c+\gamma)\cos\sqrt{A}\,\tau + \zeta', \delta, \epsilon\right) - F(\alpha_i, 0, c+\gamma, \delta, \epsilon) = 0. \quad (41)$$

This equation is satisfied at $\tau = 2\pi/\sqrt{A}$ by $v_i = \zeta = \zeta' = 0$, and we find from the explicit form of F in (13) that for these values

$$\frac{\partial F}{\partial \zeta'} = 2(c+\gamma) \neq 0.$$

It follows that (41) can be solved for ζ' as a power series in α_i, γ, δ, ϵ, $v_1, \ldots, v_4$, ζ, which vanishes with $v_i = \zeta = 0$. That is, if $u_1, \ldots, u_4$, and z retake their initial values at $\tau = 2\pi/\sqrt{A}$, z' also retakes its initial value. Hence we can suppress the last equation of (40) and consider the solution of the first five equations.

It follows from (38) that the explicit forms of the first terms of (40) are

$$\left.\begin{aligned}
&0 = \alpha_1\left[e^{+\sigma(1+\delta)\sqrt{-1}\frac{2\pi}{\sqrt{A}}} - 1\right] + \epsilon Q_1, \qquad && 0 = \alpha_3\left[e^{+\rho(1+\delta)\frac{2\pi}{\sqrt{A}}} - 1\right] + \epsilon Q_3,\\
&0 = \alpha_2\left[e^{-\sigma(1+\delta)\sqrt{-1}\frac{2\pi}{\sqrt{A}}} - 1\right] + \epsilon Q_2, \qquad && 0 = \alpha_4\left[e^{-\rho(1+\delta)\frac{2\pi}{\sqrt{A}}} - 1\right] + \epsilon Q_4,\\
&\qquad\qquad 0 = \frac{c+\gamma}{\sqrt{A}(1+\delta)} \sin 2\pi(1+\delta) + \epsilon Q_5, &&
\end{aligned}\right\} \quad (42)$$

where the Q_i are power series in the α_i, γ, δ, and ϵ. The coefficients of α_3 and α_4 are always distinct from zero, and the parts of the coefficients of α_1 and α_2 which are independent of δ vanish only if $\sigma/\sqrt{A}$ is an integer. We shall suppose at present that this ratio is not an integer, and that it is incommensurable. Part of the discussion becomes quite different when it is commensurable, and this case will be taken up when we discuss, in §96, the question of the existence of orbits of Class C.

It follows from (34) that since [ ] and { } involve terms in z^2 alone, and since z_0 does not vanish identically for $\alpha_i = \gamma = \delta = 0$, the Q_i carry terms in ϵ alone. The determinant of the linear terms in $\alpha_1, \ldots, \alpha_4$ of the first four equations of (42) is the product of the coefficients of $\alpha_1, \ldots, \alpha_4$, and is distinct from zero. Therefore these equations can be solved for $\alpha_1, \ldots, \alpha_4$ in the form

$$\alpha_i = \epsilon\, R_i(\gamma, \delta, \epsilon), \quad (43)$$

where the R_i are power series in γ, δ, and ϵ. When these results are substituted in the last equation of (42), we have

$$\frac{c+\gamma}{\sqrt{A}(1+\delta)} \sin 2\pi(1+\delta) = \epsilon\, P(\gamma, \delta, \epsilon). \quad (44)$$

The solution of this equation gives us the periodic orbits in question. We have the two arbitrary parameters γ and δ, and we shall show first that we can not give δ an arbitrary value and solve the equation for γ as a power series in ϵ, vanishing with ϵ.

Suppose that δ is neither zero nor an integer. Then equation (44) is not satisfied by $\gamma=\epsilon=0$, and the solution can not be made. Now suppose δ is zero or an integer. Then the left member of (44) vanishes, and the equation is divisible by ϵ. It is, in fact, divisible by ϵ^2. It is seen from (34) that x and y do not enter in the last equation except in terms involving ϵ as a factor, and since the a_i defined in (43) enter only through x and y, the part of ϵP coming from the first four equations is divisible by ϵ^2. It is seen also that the part of the right member of the last equation of (34) which is independent of x and y is multiplied by ϵ^2. Therefore the right member of (44) is divisible by ϵ^2. We shall now prove that after ϵ^2 has been divided out there is left a term which is independent of γ and ϵ, and which is distinct from zero.

Terms in ϵ^2 in the right member of (44) are introduced both through the a_i defined in (43), and directly in the integration of the last equation of (34). The terms obtained in the former way involve B as a factor and depend upon σ and ρ; the terms entering in the latter way carry C as a factor. Hence, if the coefficient of ϵ^2 is to be identically zero, the parts involving B and C as factors separately must be zero. We shall verify that the part involving C as a factor is distinct from zero.

The coefficient of ϵ^2 in (44), so far as it is independent of B and γ, is defined by the equation

$$\left.\begin{aligned}\frac{d^2z_2}{d\tau^2}+(1+\delta)^2Az_2=\frac{3}{2}Cz_0^3=\frac{9Cc^3}{8A^{\frac{3}{2}}(1+\delta)^3}&\sin\sqrt{A}(1+\delta)\tau\\ -\frac{3Cc^3}{8A^{\frac{3}{2}}(1+\delta)^3}&\sin 3\sqrt{A}(1+\delta)\tau.\end{aligned}\right\}\tag{45}$$

The solution of this equation satisfying the conditions $z_2=0$, $z_2'=0$, at $\tau=0$, is

$$\left.\begin{aligned}z_2=\frac{27Cc^3}{64A^{\frac{5}{2}}(1+\delta)^5}\sin\sqrt{A}(1+\delta)\tau-\frac{9Cc^3}{16A^2(1+\delta)^4}\tau\cos\sqrt{A}(1+\delta)\tau\\ +\frac{3Cc^3}{64A^{\frac{5}{2}}(1+\delta)^5}\sin 3\sqrt{A}(1+\delta)\tau.\end{aligned}\right\}\tag{46}$$

Consequently the last equation of (42) becomes

$$z\left(\frac{2\pi}{\sqrt{A}}\right)-z(0)=\left[-\frac{9\pi Cc^3}{8A^2(1+\delta)^4}+\cdot\cdot\cdot\right]\epsilon^2=0. \tag{47}$$

Hence, after division by ϵ^2, there is a term independent of both γ and ϵ, and the solution for γ as a power series in ϵ, vanishing with ϵ, does not exist. That is, periodic solutions of the type in question do not exist.

Now let us give γ an arbitrary value and attempt to solve equations (42) for $\alpha_1, \ldots, \alpha_4$, and δ as power series in ϵ, vanishing with ϵ. Since c is arbitrary, we may put γ equal to zero without loss of generality. Or, more conveniently for the construction of the periodic solution, we may give γ such a value that $z'=c/\sqrt{A}$ for $\tau=0$, whatever ϵ may be. But for simplicity in writing we shall suppose that γ is included in c. The first four equations can be solved for $\alpha_1, \ldots, \alpha_4$ in terms of δ and ϵ, and the results substituted in the last. The result differs from that above only in the terms multiplied by ϵ, and, as before, we find that the lowest term in ϵ alone has ϵ^2 as a factor. There is a linear term in δ alone whose coefficient is $2\pi c/\sqrt{A}$. Therefore, after $\alpha_1, \ldots, \alpha_4$ have been eliminated by means of the first four equations, the last equation can be solved for δ as power series in ϵ, *the term of lowest degree being* ϵ^2. When this result is substituted in the solutions of the first four equations, we have $\alpha_1, \ldots, \alpha_4$ expressed as power series in ϵ, vanishing with ϵ. That is, when $\gamma=0$ the solutions of (42) have the form

$$\delta=\epsilon^2 p(\epsilon), \qquad \alpha_i=\epsilon p_i(\epsilon) \qquad (i=1, \ldots, 4), \quad (48)$$

where p and the p_i are power series in ϵ. When these results are substituted in (39), we have

$$z=z_0+\epsilon^2 q(\epsilon;\,\tau), \qquad u_i=\epsilon q_i(\epsilon;\,\tau) \qquad (i=1, \ldots, 4), \quad (49)$$

where q and the q_i are power series in ϵ and are periodic in τ with the period $2\pi/\sqrt{A}$. These series converge for $|\epsilon|$ sufficiently small. The circle of convergence is determined by the singularities which are present in the differential equations (34), which are introduced in forming (39), and which are introduced in the solution of (42). Since the right members of (49) converge and are periodic for all $|\epsilon|$ sufficiently small, *the coefficient of each power of ϵ separately is periodic.*

86. Some Properties of Solutions of Class A.—It will now be shown that the orbits under consideration are re-entrant after one revolution, and that they cross the x-axis perpendicularly.

Let us find the orbits whose periods are $2\nu\pi/\sqrt{A}$, ν being an integer. We form equations analogous to (42) simply by replacing 2π by $2\nu\pi$. The determinant of the linear terms in $\alpha_1, \ldots, \alpha_4$ is distinct from zero unless $\nu\sigma/\sqrt{A}$ is an integer. We exclude this case here and treat it when we consider orbits of Class C. Therefore the first four equations can be solved for $\alpha_1, \ldots, \alpha_4$ as power series in γ, δ, and ϵ. On substituting the results in the last equation, we find, as before, that the solution can not be made for γ, taking δ arbitrary, but that the solution for δ as a power series in ϵ is unique. That is, the solution for $\alpha_1, \ldots, \alpha_4$, and δ as power series in ϵ, vanishing with ϵ, is unique. Hence for a given value of ϵ there is a single orbit of Class A having the period $2\nu\pi/\sqrt{A}$. We have shown also

that for a given value of ϵ there is *one* orbit of Class A having the period $2\pi/\sqrt{A}$. Since an orbit of period $2\pi/\sqrt{A}$ has also the period $2\nu\pi/\sqrt{A}$, the latter are included in the former. It follows, therefore, from the uniqueness of both orbits for a given ϵ, and from the fact that, for $\epsilon=0$, they re-enter after the period $2\pi/\sqrt{A}$, that all orbits of this class re-enter after a single revolution.

Let us now suppose that, at $\tau=0$, we have $dx/d\tau=y=z=0$; that is, that the orbit crosses the x-axis perpendicularly at $\tau=0$. It follows from equation (33) that

$$u_1(0)-u_2(0)=0, \qquad u_3(0)-u_4(0)=0; \tag{50}$$

whence

$$\alpha_1=\alpha_2\,, \qquad\qquad \alpha_3=\alpha_4\,.$$

The equations corresponding to (39) will now be power series in α_1, α_3, γ, δ, and ϵ. We may suppose $\gamma=0$ on the start, for orbits that may be found in this way will be included in those found from more general initial conditions, under which it was always permissible to put γ equal to zero. The orbits obtained with these initial conditions will be symmetrical with respect to the x-axis. Therefore necessary and sufficient conditions for periodicity are that the infinitesimal body shall cross the x-axis perpendicularly at the half period. These conditions are that at $\tau=\pi/\sqrt{A}$

$$\frac{dx}{d\tau}=y=z=0.$$

It follows from (33) that these conditions imply that, at $\tau=\pi/\sqrt{A}$,

$$u_1-u_2=0, \qquad u_3-u_4=0, \qquad z=0. \tag{51}$$

These conditions give us, in place of (42), the equations

$$\left.\begin{aligned} 0&=\alpha_1\left[e^{\sigma(1+\delta)\sqrt{-1}\frac{\pi}{\sqrt{A}}}-e^{-\sigma(1+\delta)\sqrt{-1}\frac{\pi}{\sqrt{A}}}\right]+\epsilon Q_1'\,,\\ 0&=\alpha_3\left[e^{\rho(1+\delta)\frac{\pi}{\sqrt{A}}}-e^{-\rho(1+\delta)\frac{\pi}{\sqrt{A}}}\right]+\epsilon Q_3'\,,\\ 0&=\frac{c}{\sqrt{A}\,(1+\delta)}\sin 2\pi(1+\delta)+\epsilon Q_5'\,,\end{aligned}\right\} \tag{52}$$

where Q_1', Q_3', and Q_5' are power series in α_1, α_3, δ, and ϵ. It is easy to see that these equations are solvable uniquely for α_1, α_3, and δ as power series in ϵ, vanishing with ϵ. Therefore, for a given value of ϵ there is one, and but one, of these symmetrical periodic orbits of this class. Since for a given value of ϵ there is but one periodic orbit for unrestricted initial conditions, it follows that *all orbits of Class A cross the x-axis perpendicularly at every half period.*

87. Direct Construction of the Solutions for Class A.—In the practical construction of the solutions for the orbits of Class A it is most convenient to use equations (12). The explicit values of the right members are

$$\left.\begin{aligned} &X_1=(1+2A)x, && X_2=\tfrac{3}{2}B[-2x^2+y^2+z^2], && X_3=2C[2x^3-3xy^2-3xz^2],\ldots,\\ &Y_1=(1-A)y, && Y_2=3Bxy, && Y_3=\tfrac{3}{2}C[-4x^2y+y^3+yz^2],\ldots,\\ &Z_1=-Az, && Z_2=3Bxz, && Z_3=\tfrac{3}{2}C[-4x^2z+y^2z+z^3],\ldots, \end{aligned}\right\} \quad (53)$$

the A, B, and C being constants which are defined in (15) and (35).

The x, y, z, and δ can be expanded uniquely as series of the form

$$x=\sum_{i=1}^{\infty}x_i(\tau)\epsilon^i,\quad y=\sum_{i=1}^{\infty}y_i(\tau)\epsilon^i,\quad z=\frac{c}{\sqrt{A}}\sin\sqrt{A}\,\tau+\sum_{i=1}^{\infty}z_i(\tau)\epsilon^i,\quad \delta=\sum_{i=2}^{\infty}\delta_i\epsilon^i,\quad (54)$$

where the $x_i(\tau)$, $y_i(\tau)$, and $z_i(\tau)$ are each periodic with the period $2\pi/\sqrt{A}$. On substituting these expressions in (12) and making use of (53), we obtain a series of sets of equations for the determination of the x_i, y_i, z_i, δ_i.

The coefficients of ϵ in (54) must satisfy the equations

$$\left.\begin{aligned} &\frac{d^2x_1}{d\tau^2}-2\frac{dy_1}{d\tau}-(1+2A)x_1=\tfrac{3}{2}B[-2x_0^2+y_0^2+z_0^2],\\ &\frac{d^2y_1}{d\tau^2}+2\frac{dx_1}{d\tau}-(1-A)y_1=3Bx_0y_0,\\ &\frac{d^2z_1}{d\tau^2}+Az_1=3Bx_0z_0. \end{aligned}\right\} \quad (55)$$

But $x_0=y_0=0$, $z_0=c/\sqrt{A}\sin\sqrt{A}\tau$. Therefore the solution of (55) which satisfies the conditions that x_1, y_1, and z_1 shall be periodic with the period $2\pi/\sqrt{A}$, and that $z=0$, $z'=c$, at $\tau=0$, whence $z_1(0)=z_1'(0)=0$, is

$$\left.\begin{aligned} &x_1=\frac{-3Bc^2}{4A(1+2A)}+\frac{3B(1+3A)c^2}{4A(1-7A+18A^2)}\cos 2\sqrt{A}\tau,\\ &y_1=\frac{-3Bc^2}{\sqrt{A}\,(1-7A+18A^2)}\sin 2\sqrt{A}\tau, \qquad z_1=0. \end{aligned}\right\} \quad (56)$$

Since in all cases $A>1$, the coefficients are always finite.

The coefficients of ϵ^2 in (54) must satisfy the differential equations

$$\left.\begin{aligned} &\frac{d^2x_2}{d\tau^2}-2\frac{dy_2}{d\tau}-(1+2A)x_2=0, \qquad \frac{d^2y_2}{d\tau^2}+2\frac{dx_2}{d\tau}-(1-A)y_2=0,\\ &\frac{d^2z_2}{d\tau^2}+Az_2=-2A\delta_2z_0+3Bx_1z_0+\tfrac{3}{2}Cz_0^3. \end{aligned}\right\} \quad (57)$$

Upon substituting the values of z_0 and x_1, the third equation becomes

$$\begin{aligned} \frac{d^2z_2}{d\tau^2}+Az_2=&\left[-2\sqrt{A}c\delta_2-\frac{27B^2(1-3A+14A^2)c^3}{8A^{\frac{3}{2}}(1+2A)(1-7A+18A^2)}+\frac{9Cc^3}{8A^{\frac{3}{2}}}\right]\sin\sqrt{A}\tau\\ &+\left[\frac{9B^2(1+3A)c^3}{8A^{\frac{3}{2}}(1-7A+18A^2)}-\frac{3Cc^3}{8A^{\frac{3}{2}}}\right]\sin 3\sqrt{A}\,\tau. \end{aligned}$$

The solution of this equation will not be periodic unless we impose the condition that the coefficient of $\sin\sqrt{A}\tau$ shall vanish. This condition is satisfied by $c=0$, but this leads to the trivial solution $x\equiv y\equiv z\equiv 0$. If we reject this solution, we may use the condition for the determination of δ_2. After it has been satisfied, the periodic solution of (57), having the period $2\pi/\sqrt{A}$ and fulfilling the conditions that $z_2=z_2'=0$ at $\tau=0$, is

$$\left.\begin{aligned} &x_2=y_2=0, \qquad \delta_2=\frac{-27B^2(1-3A+14A^2)c^2}{16A^2(1+2A)(1-7A+18A^2)}+\frac{9Cc^2}{16A^2}, \\ &z_2=\frac{3c^3}{64A^{\frac{5}{2}}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2}-C\right]\left[3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau\right]. \end{aligned}\right\} \quad (58)$$

In this manner the construction of the periodic solution can be continued as far as may be desired. We shall prove this statement by induction, and at the same time we shall derive certain general properties of the solution which are satisfied by the terms already computed.

Suppose $x_0, \ldots, x_{n-1};\ y_0, \ldots, y_{n-1};\ z_0, \ldots, z_{n-1};\ \delta_2, \ldots, \delta_{n-1}$ have been determined and that they have the following properties:

1. The x_{2j} and y_{2j} are identically zero, j an integer.
2. The z_{2j+1} are identically zero, j an integer.
3. The function x_{2j+1} is a sum of cosines of even multiples of $\sqrt{A}\tau$, and the highest multiple is $2j+2$.
4. The function y_{2j+1} is a sum of sines of even multiples of $\sqrt{A}\tau$, and the highest multiple is $2j+2$.
5. The function z_{2j} is a sum of sines of odd multiples of $\sqrt{A}\tau$, and the highest multiple is $2j+1$.
6. The δ_{2j+1} are zero.

It will now be shown that these properties hold for x_n, y_n, z_n, and δ_n.

The terms x_n, y_n, z_n, and δ_n satisfy the differential equations

$$\left.\begin{aligned} &\frac{d^2x_n}{d\tau^2}-2\frac{dy_n}{d\tau}-(1+2A)x_n=P_n(x_j, y_j, z_j, y_j', \delta_j), \\ &\frac{d^2y_n}{d\tau^2}+2\frac{dx_n}{d\tau}-(1-A)\,y_n=Q_n(x_j, y_j, z_j, x_j', \delta_j), \\ &\frac{d^2z_n}{d\tau^2}+Az_n=-\ 2A\,z_0\ \ \delta_n+R_n(x_j, y_j, z_j, \delta_j), \end{aligned}\right\} \quad (59)$$

where P_n, Q_n, and R_n are polynomials in $x_j, \ldots, \delta_j$ $(j=1, \ldots, n-1)$.

It is seen from (12) that P_n and Q_n involve y_j' and x_j' only in the products $y_j'\delta_{n-j}$ and $x_j'\delta_{n-j}$. If n is even, these terms are zero by properties 1 and 6, for then either j must be even or $n-j$ must be odd. But if n is odd, they are in general not zero.

We shall now prove that $P_n \equiv 0$ if n is even. The general term of P_n is

$$T_n = x_{\lambda_1}^{\mu_1} \cdots x_{\lambda_\kappa}^{\mu_\kappa} \cdot y_{\lambda_1'}^{\mu_1'} \cdots y_{\lambda_{\kappa'}'}^{\mu_{\kappa'}'} \cdot z_{\lambda_1''}^{\mu_1''} \cdots z_{\lambda_{\kappa''}''}^{\mu_{\kappa''}''} \cdot \delta_{p_1}^{q_1} \delta_{p_2}^{q_2}, \tag{60}$$

where $\lambda_1, \ldots, \lambda_\kappa, \ldots, \lambda_{\kappa''}''$; $\mu_1, \ldots, \mu_{\kappa''}''$; p_1, p_2, q_1, and q_2 are all integers. Since the δ_j enter only through $(1+\delta)^2$, the exponents q_1 and q_2 satisfy the relation

$$q_1 + q_2 \lesseqgtr 2. \tag{61}$$

The exponents and subscripts of (60) satisfy the following relations:

(a) $\mu_1' + \cdots + \mu_{\kappa'}'$ is an even integer because the right member of the first equation of (12) is a function of y^2.
(b) $\mu_1'' + \cdots + \mu_{\kappa''}''$ is an even integer for a similar reason.
(c) $\lambda_1, \ldots, \lambda_\kappa, \lambda_1', \ldots, \lambda_{\kappa'}'$ are odd integers by property 1.
(d) $\lambda_1'', \ldots, \lambda_{\kappa''}''$ are even integers by property 2.
(e) p_1 and p_2 are even integers by property 6.
(f) $\mu_1\lambda_1 + \cdots + \mu_\kappa\lambda_\kappa + \mu_1'\lambda_1' + \cdots + \mu_{\kappa'}'\lambda_{\kappa'}' + \mu_1''\lambda_1'' + \cdots + \mu_{\kappa''}''\lambda_{\kappa''}'' + p_1 q_1 + p_2 q_2 + \mu_1 + \cdots + \mu_\kappa + \mu_1' + \cdots + \mu_{\kappa'}' + \mu_1'' + \cdots + \mu_{\kappa''}'' - 1 = n$ because the term of degree $\mu_1 + \cdots + \mu_\kappa + \mu_1' + \cdots + \mu_{\kappa'}' + \mu_1'' + \cdots \mu_{\kappa''}''$ in x, y, and z has ϵ as a factor to a degree one less than this sum, the terms $x_{\lambda_1}^{\mu_1}, \ldots$ introduce ϵ to the degree $\mu_1\lambda_1, \ldots,$ and the sum of the exponents of ϵ must equal n.

There are two sub-cases, according as $\mu_1 + \cdots + \mu_\kappa$ is an even integer or an odd integer. When $\mu_1 + \cdots + \mu_\kappa$ is an even integer, the following statements are true:

(α) There is an even number of odd $\mu_1, \ldots, \mu_\kappa$.
(β) $\mu_1\lambda_1 + \cdots + \mu_\kappa\lambda_\kappa$ is an even integer by (c) and (α).
(γ) $\mu_1'\lambda_1' + \cdots + \mu_{\kappa'}'\lambda_{\kappa'}'$ is an even integer by (a) and (c).
(δ) $\mu_1''\lambda_1'' + \cdots + \mu_{\kappa''}''\lambda_{\kappa''}''$ is an even integer by (d).
(ϵ) $p_1 q_1 + p_2 q_2$ is an even integer by (e).

It follows from the assumption that $\mu_1 + \cdots + \mu_\kappa$ is even, and from (a), (b), (α), . . . , (ϵ) that the left member of (f) is odd. Therefore in this case T_n is identically zero if n is even, and in general is not identically zero if n is odd.

Suppose now that $\mu_1 + \cdots + \mu_\kappa$ is an odd integer. Then

(α') There is an odd number of odd $\mu_1, \ldots, \mu_\kappa$.
(β') $\mu_1\lambda_1 + \cdots + \mu_\kappa\lambda_\kappa$ is an odd integer by (c) and (α').

The properties (γ'), (δ'), and (ϵ') are the same as (γ), (δ), and (ϵ) respectively. Therefore the left member of (f) is again odd, and hence every T_n is identically zero if n is even, and in general not identically zero if n is odd. It follows that *P_n is identically zero if n is even, and in general is not zero if n is odd.*

The treatment of the general term of Q_n can be made in a similar way. The only differences are that in (a) and (γ) the sums are individually odd instead of even. But since (f) involves their sum, the result is that *Q_n is identically zero if n is even, and in general is not zero if n is odd.*

The general term in R_n has the form of (60), where the subscripts and exponents satisfy the relations:

(a″) $\mu' + \cdots + \mu'_{\kappa'}$ is an even integer because the right member of the third equation of (12) is a function of y^2.

(b″) $\mu''_1 + \cdots + \mu''_{\kappa''}$ is an odd integer because the right member of this equation involves only odd powers of z.

(c″), (d″), (e″), and (f″) are the same as (c), (d), (e), and (f) respectively.

Suppose $\mu_1 + \cdots + \mu_\kappa$ is an even integer. Then (α''), (β''), (γ''), (δ''), and (ϵ'') are the same as (α), (β), (γ), (δ), and (ϵ) respectively. Therefore in this case the left member of (f″) is an even integer. It is shown similarly that the same result is true when $\mu_1 + \cdots + \mu_\kappa$ is an odd integer. Therefore, R_n *is identically zero if n is odd, and in general is not identically zero if n is even.*

The discussion now naturally divides into two cases, viz., where n is even, and where n is odd. We shall treat them separately.

Case I. We shall prove that if n is even, R_n is a sum of sines of odd multiples of $\sqrt{A}\,\tau$, the highest multiple being $n+1$. Consider the general term (60). The x_{λ_i} are all cosines of even multiples of $\sqrt{A}\,\tau$; therefore the product $x_{\lambda_1}^{\mu_1} \cdots x_{\lambda_\kappa}^{\mu_\kappa}$ is a sum of cosines of even multiples. Because of property 3 and the properties of the products of cosines of multiples of an argument, it follows that the highest multiple which occurs is

$$\mu_1(\lambda_1+1) + \cdots + \mu_\kappa(\lambda_\kappa+1) = \mu_1\lambda_1 + \cdots + \mu_\kappa\lambda_\kappa + \mu_1 + \cdots + \mu_\kappa. \tag{62}$$

Similarly, from properties 4 and (a″), it follows that $y_{\lambda'_1}^{\mu'_1} \cdots y_{\lambda'_{\kappa'}}^{\mu'_{\kappa'}}$ is a sum of cosines of even multiples of $\sqrt{A}\tau$, the highest multiple being

$$\mu'_1(\lambda'_1+1) + \cdots + \mu'_{\kappa'}(\lambda'_{\kappa'}+1) = \mu'_1\lambda'_1 + \cdots + \mu'_{\kappa'}\lambda'_{\kappa'} + \mu'_1 + \cdots + \mu'_{\kappa'}. \tag{63}$$

From properties 5 and (b″), it follows that $z_{\lambda''_1}^{\mu''_1} \cdots z_{\lambda''_{\kappa''}}^{\mu''_{\kappa''}}$ is a sum of sines of odd multiples of $\sqrt{A}\,\tau$, the highest multiple being

$$\mu''_1(\lambda''_1+1) + \cdots + \mu''_{\kappa''}(\lambda''_{\kappa''}+1) = \mu''_1\lambda''_1 + \cdots + \mu''_{\kappa''}\lambda''_{\kappa''} + \mu''_1 + \cdots + \mu''_{\kappa''}. \tag{64}$$

On taking the product of these three sets of terms, we find that R_n is a sum of sines of odd multiples of $\sqrt{A}\,\tau$, the highest multiple being

$$\left.\begin{aligned} N = \mu_1\lambda_1 + \cdots + \mu_\kappa\lambda_\kappa + \mu'_1\lambda'_1 + \cdots + \mu'_{\kappa'}\lambda'_{\kappa'} + \mu''_1\lambda''_1 + \cdots + \mu''_{\kappa''}\lambda''_{\kappa''} \\ + \mu_1 + \cdots + \mu_\kappa + \mu'_1 + \cdots + \mu'_{\kappa'} + \mu''_1 + \cdots + \mu''_{\kappa''}. \end{aligned}\right\} \tag{65}$$

By (f″), which is of the same form as (f), we have

$$N = n+1-(p_1\,q_1 + p_2\,q_2).$$

For those terms in which $q_1 = q_2 = 0$, we have, as the largest value of N,

$$N = n+1. \tag{66}$$

Hence, for n even, equations (59) become

$$\left.\begin{aligned}
&\frac{d^2x_n}{d\tau^2}-2\frac{dy_n}{d\tau}-(1+2A)x_n=0,\\
&\frac{d^2y_n}{d\tau^2}+2\frac{dx_n}{d\tau}-(1-\ \ A)y_n=0,\\
&\frac{d^2z_n}{d\tau^2}+Az_n=(-2\sqrt{A}c\delta_n+C_1^{(n)})\sin\sqrt{A}\tau+C_3^{(n)}\sin 3\sqrt{A}\tau+\cdots\\
&\qquad\qquad +C_{2j+1}^{(n)}\sin(2j+1)\sqrt{A}\tau+\cdots+C_{n+1}^{(n)}\sin(n+1)\sqrt{A}\tau,
\end{aligned}\right\} \tag{67}$$

where the $C_{2j+1}^{(n)}$ are known constants which depend upon the coefficients of the terms with lower subscripts. The solution of these equations which satisfies the periodicity conditions and $z_n=z_n'=0$ at $\tau=0$, is

$$\left.\begin{aligned}
&x_n=y_n=0, \qquad\qquad \delta_n=\frac{C_1^{(n)}}{2\sqrt{A}c},\\
&z_n=\gamma_1^{(n)}\sin\sqrt{A}\tau+\gamma_3^{(n)}\sin 3\sqrt{A}\tau+\cdots+\gamma_{n+1}^{(n)}\sin(n+1)\sqrt{A}\tau,\\
&\gamma_{2j+1}^{(n)}=\frac{-C_{2j+1}^{(n)}}{[(2j+1)^2-1]A}=\frac{-C_{2j+1}^{(n)}}{4j(j+1)A} \qquad \left(j=1,\ldots,\frac{n}{2}\right),\\
&\gamma_1^{(n)}=\sum_{j=1}^{n/2}\frac{(2j+1)C_{2j+1}^{(n)}}{4j(j+1)\sqrt{A}}.
\end{aligned}\right\} \tag{68}$$

Case II. We shall prove that, if n is odd, P_n is a sum of cosines of even multiples of $\sqrt{A}\tau$, the highest multiple being $n+1$. From properties 1 and 3 it follows that $x_{\lambda_1}^{\mu_1}\cdots x_{\lambda_\kappa}^{\mu_\kappa}$ is a sum of cosines of even multiples of $\sqrt{A}\tau$, the highest being given by (62). From properties 1, 4, and (a) it follows that $y_{\lambda_1'}^{\mu_1'}\cdots y_{\lambda_{\kappa'}'}^{\mu_{\kappa'}'}$ is a sum of cosines of even multiples of $\sqrt{A}\tau$, the highest multiple being given by (63). From properties 2, 5, and (b) it follows that $z_{\lambda_1''}^{\mu_1''}\cdots z_{\lambda_{\kappa''}''}^{\mu_{\kappa''}''}$ is also a sum of cosines of even multiples of $\sqrt{A}\tau$, the highest multiple being given by (64). Therefore P_n is a sum of cosines of even multiples of $\sqrt{A}\tau$, and from (62), (63), (64), and (f) it follows that the highest multiple is $n+1$.

Similarly, it can be shown that Q_n is a sum of sines of even multiples of $\sqrt{A}\tau$, the highest multiple being $n+1$.

In this case $R_n=0$. Therefore, for n odd, equations (59) become

$$\left.\begin{aligned}
&\frac{d^2x_n}{d\tau^2}-2\frac{dy_n}{d\tau}-(1+2A)x_n=A_0^{(n)}+A_2^{(n)}\cos 2\sqrt{A}\tau+\cdots\\
&\qquad\qquad +A_{2j}^{(n)}\cos 2j\sqrt{A}\tau+\cdots+A_{n+1}^{(n)}\cos(n+1)\sqrt{A}\tau,\\
&\frac{d^2y_n}{d\tau^2}+2\frac{dx_n}{d\tau}-(1-A)y_n=B_2^{(n)}\sin 2\sqrt{A}\tau+\cdots\\
&\qquad\qquad +B_{2j}^{(n)}\sin 2j\sqrt{A}\tau+\cdots+B_{n+1}^{(n)}\sin(n+1)\sqrt{A}\tau,\\
&\frac{d^2z_n}{d\tau^2}+Az_n=-2\sqrt{A}c\delta_n\sin\sqrt{A}\tau.
\end{aligned}\right\} \tag{69}$$

The solution of these equations which satisfies the periodicity condition and the conditions $z_n = z_n' = 0$ at $\tau = 0$, is

$$
\left.
\begin{aligned}
x_n &= a_0^{(n)} + a_2^{(n)} \cos 2\sqrt{A}\tau + \cdots + a_{2j}^{(n)} \cos 2j\sqrt{A}\tau + \cdots \\
&\qquad\qquad + a_{n+1}^{(n)} \cos(n+1)\sqrt{A}\,\tau, \\
y_n &= \beta_2^{(n)} \sin 2\sqrt{A}\tau + \cdots + \beta_{2j}^{(n)} \sin 2j\sqrt{A}\tau + \cdots \\
&\qquad\qquad + \beta_{n+1}^{(n)} \sin(n+1)\sqrt{A}\,\tau, \\
z_n &= \delta_n = 0, \qquad a_0^{(n)} = -\frac{A_0^{(n)}}{1+2A}, \\
a_{2j}^{(n)} &= \frac{-[(4j^2-1)A+1]A_{2j}^{(n)} + 4j\sqrt{A}\,B_{2j}^{(n)}}{2(8j^4+2j^2-1)A^2-(8j^2-1)A+1} \qquad \left(j=1,\ \ldots,\ \frac{n+1}{2}\right), \\
\beta_{2j}^{(n)} &= \frac{+4j\sqrt{A}\,A_{2j}^{(n)} - [2(2j^2+1)A+1]\,B_{2j}^{(n)}}{2(8j^4+2j^2-1)A^2-(8j^2-1)A+1} \qquad \left(j=1,\ \ldots,\ \frac{n+1}{2}\right).
\end{aligned}
\right\} \quad (70)
$$

Since $A > 1$ these denominators can not vanish for any integral j.

It is obvious that in practice it is not necessary to refer to the differential equations at each step. The most convenient method to follow is to substitute as many terms of (54) in the right members of (12) as will be required in carrying the computation to the desired order in ϵ, and to arrange the results as power series in ϵ of the form

$$P_1\epsilon + P_2\epsilon^2 + \cdots + P_n\epsilon^n + \cdots,$$

and similar series for the other equations. From the P_n, Q_n, and R_n the $A_j^{(n)}$, $B_j^{(n)}$, and $C_j^{(n)}$ can be computed sequentially with respect to n without explicit reference to the left members of the differential equations. The coefficients of the solutions are given by (68) and (70). The whole process is unique and can be continued as far as may be desired.

88. Additional Properties of Orbits of Class A.—It will be observed that, so far as the computations have been carried, x_j, y_j, and z_j carry c^{j+1} as a factor and that δ_j carries c^j as a factor. We shall prove that this is a general property.

Suppose it is true for $j=0, \ldots, n-1$, and consider the question for $j=n$. The terms of order n are defined by equations (59). In P_n there are terms $y_j'\delta_{n-j}$. It follows from the assumed properties of the y_j and δ_j that this term carries c^{n+1} as a factor. Similarly the $x_j'\delta_{n-j}$ occurring in Q_n carry c^{n+1} as a factor. Now consider the general term (60). It follows from the assumed properties of x_j, y_j, z_j, and δ_j that this term carries c as a factor to the power

$$
\begin{aligned}
N = \mu_1(\lambda_1+1) + \cdots + \mu_\kappa(\lambda_\kappa+1) + \mu_1'(\lambda_1'+1) + \cdots + \mu_{\kappa'}'(\lambda_{\kappa'}'+1) \\
+ \mu_1''(\lambda_1''+1) + \cdots + \mu_{\kappa''}''(\lambda_{\kappa''}''+1) + p_1q_1 + p_2q_2.
\end{aligned}
$$

It follows from (f) that $N = n+1$, and therefore this property is general.

In order to obtain the coördinates in the physical problem, we must replace ϵ by ϵ' and multiply x, y, and z by ϵ' [equations (10)]. Then ϵ' and c occur in every term in x', y', and z' to the same degree, and are equivalent to a single parameter. That is, without loss of generality we may put c equal to unity, and the value of ϵ' will determine the dimensions of the orbit. Or, if we put ϵ' equal to unity, c will determine the dimensions of the orbit. It follows from these results that the coördinates and δ are expansible as power series in c, and the solutions could have been derived in this way without the introduction of ϵ and ϵ', but the discussion would have been less simple.

The explicit expressions for the periodic solution, so far as they have been worked out in (29), (56), and (58), are*

$$\left.\begin{aligned}
x' &= 0\,\epsilon' + \left[\frac{-3B}{4A(1+2A)} + \frac{3B(1+3A)}{4A(1-7A+18A^2)}\cos 2\sqrt{A}\,\tau\right]\epsilon'^2 + \cdots,\\
y' &= 0\,\epsilon' + \left[\frac{-3B}{\sqrt{A}(1-7A+18A^2)}\sin 2\sqrt{A}\,\tau\right]\epsilon'^2 + \cdots,\\
z' &= \left[\frac{1}{\sqrt{A}}\sin\sqrt{A}\,\tau\right]\epsilon' + 0\,\epsilon'^2\\
&\quad + \frac{3}{64A^{\frac{5}{2}}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2} - C\right]\left[3\sin\sqrt{A}\,\tau - \sin\sqrt{A}\,\tau\right]\epsilon'^3 + \cdots,\\
\delta &= 0\,\epsilon' - \frac{9}{16A^2}\left[\frac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)} - C\right]\epsilon'^2 + \cdots.
\end{aligned}\right\} \quad (71)$$

Since x' and y' are sums of cosines and sines respectively of even multiples of $\sqrt{A}\tau$, and since z' is a sum of sines of odd multiples of $\sqrt{A}\tau$, it follows that x' and y' are periodic with half the period of z'.

Since the relations

$$x'(\tau) = x'(-\tau), \qquad y'(\tau) = -y'(-\tau), \qquad z'(\tau) = -z'(-\tau)$$

are satisfied, the orbits are symmetrical with respect to the x-axis, as was shown in the existence proof.

Let T equal half the period. Then

$$x'(\tau) = x'(\mathrm{T}+\tau), \qquad y'(\tau) = y'(\mathrm{T}+\tau), \qquad z'(\tau) = -z'(\mathrm{T}+\tau).$$

Therefore the orbits are symmetrical with respect to the $x'y'$-plane. Similarly, since

$$x'(\tau) = x'(\mathrm{T}-\tau), \qquad y'(\tau) = -y'(\mathrm{T}-\tau), \qquad z'(\tau) = z'(\mathrm{T}-\tau),$$

the orbits are symmetrical with respect to the $x'z'$-plane.

It follows from the form of (71) that a change of the sign of ϵ' is equivalent to changing τ by a half period.

The period of the solutions in τ is $2\pi/\sqrt{A}$, but it follows from the last equation of (10) that in the time t the period is

$$P = \frac{2\pi(1+\delta)}{\sqrt{A}} = \frac{2\pi}{\sqrt{A}}\left\{1 - \frac{9}{16A^2}\left[\frac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)} - C\right]\epsilon'^2 \cdots\right\} \quad (72)$$

*The x', y', and z' are the actual coördinates and not their derivatives.

It is found from (71) that the equation of the projection of the orbit on the $x'y'$-plane is, up to terms of the fourth degree in ϵ',

$$\frac{16A^2}{(1+3A)^2}\left[x'+\frac{3B\epsilon'^2}{4A(1+2A)}\right]^2+Ay'^2=\frac{9B^2\epsilon'^4}{(1-7A+18A^2)^2}. \tag{73}$$

This is the equation of an ellipse whose center is at $x'=\dfrac{-3B\epsilon'^2}{4A(1+2A)}$, $y'=0$, and whose semi-axes are

$$\frac{3B(1+3A)\epsilon'^2}{4A(1-7A+18A^2)}, \qquad \frac{3B\epsilon'^2}{\sqrt{A}\,(1-7A+18A^2)}. \tag{74}$$

The equation of the projection of the orbits on the $y'z'$-plane is approximately

$$y'=\frac{-6Bz'}{(1-7A+17A^2)\epsilon'}\sqrt{1-\frac{Az'^2}{\epsilon'^2}}. \tag{75}$$

This is the equation of a figure-of-eight curve with its center at the origin, touching the y'-axis at no other point, and having two other intersections with the z'-axis.

The equation of the projection of the orbit on the $x'z'$-plane is approximately

$$x'=\frac{3B\epsilon'^2}{4A}\left[\frac{-1}{1+2A}+\frac{(1+3A)}{1-7A+18A^2}\left(1-\frac{2Az'^2}{\epsilon'^2}\right)\right]. \tag{76}$$

The orbit is a parabola whose axis is the x'-axis, and whose vertex is at

$$x'=\frac{9B(1-A)\epsilon'^2}{(1+2A)(1-7A+18A^2)}, \qquad y'=z'=0.$$

Only that part of the parabola for which $z'^2<\epsilon'^2/A$ belongs to the orbit, the infinite branches having been introduced in eliminating τ.

It is at the vertex of this parabola that the orbit has the double intersection with the x'-axis. In all cases $1-A<0$, and therefore $1-7A+18A^2>0$. It is seen from (35) that B is positive for the point (a). Hence these orbits intersect the x'-axis between (a) and the finite mass μ.

It follows from (35) and the second equation of (4) that B is negative for the point (b), at least if μ is small. Hence these orbits intersect the x'-axis between the point (b) and the finite mass μ. Similarly, those orbits near (c) intersect the x'-axis between (c) and the finite body $1-\mu$.

The vertices of these parabolas are the ends of the ellipses whose axes are given in (74). It is seen from the expressions for the coördinates of the centers of the ellipses and the ends of the parabolas, that the distance from the vertices of the parabolas to the centers of the ellipses is

$$\frac{+3B(1+3A)\epsilon'^2}{4A(1-7A+18A^2)}.$$

It follows from the signs of B that the orbits in all cases open out away from the points of equilibrium near which they lie.

From the properties which have been derived it is possible to infer the geometric character of these orbits. In a general way they have the shape of the handles of ice-tongs, one of the two handles being situated on one side of the $\xi\eta$-plane, and the other symmetrically on the other side of this plane. The place of the hinge is where they cross the ξ-axis. In the case of the points (*a*) and (*b*) they open toward the finite mass μ, and in the case of (*c*), toward the finite mass $1-\mu$.

In Fig. 3 an orbit of each class is shown. No attempt has been made to represent them to scale for any particular case, but the figures show their general positions and the directions of motion in them. The curves are drawn on elliptic cylinders to make the figures as clear as possible.

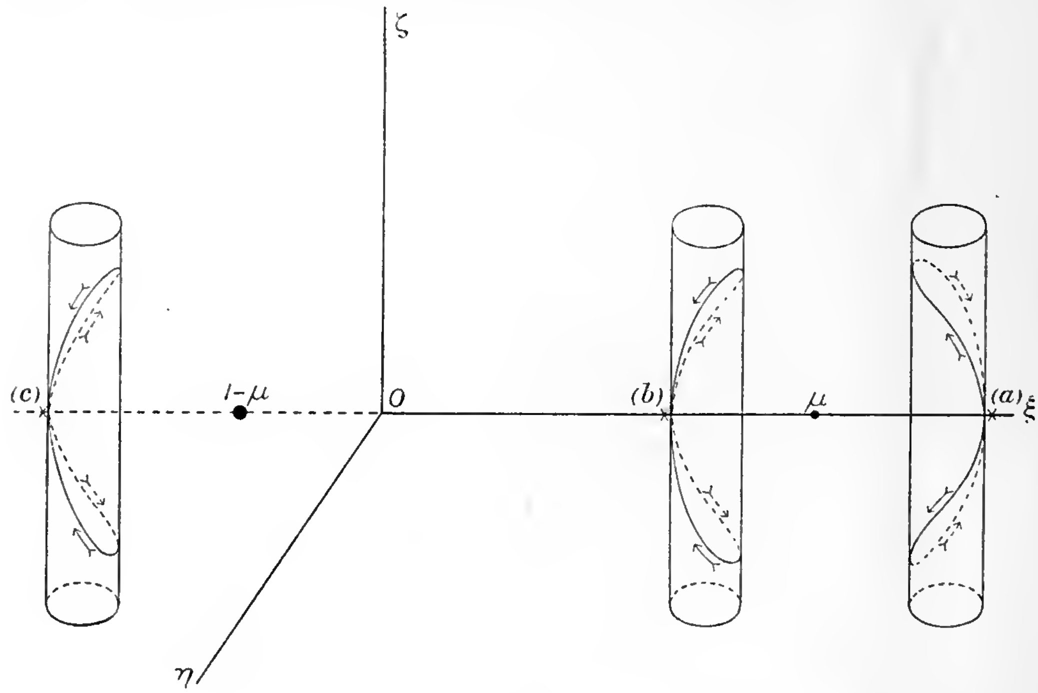

FIG. 3.

89. Application of Jacobi's Integral to the Orbits of Class A.—The original differential equations admit the integral (13), which holds for all orbits, and therefore in particular for the periodic orbits which we are discussing. It has already been seen that it plays an important rôle in the proof of the existence of the periodic solutions when we start from general initial conditions, and we shall now show that it is almost equally important in the construction of these solutions. We are illustrating, in a particularly simple problem, a new and valuable use to which integrals may be put.

The explicit form of the integral (13) is

$$\left.\begin{aligned} F \equiv \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 - (1+\delta)^2 \Big\{ x^2 + y^2 + A(2x^2 - y^2 - z^2) \\ + B(-2x^3 + 3y^2 + 3z^2)x\epsilon + \cdots \Big\} = \text{constant.} \end{aligned}\right\} \quad (77)$$

If we substitute the solutions (49), or rather the equivalent series for x, y, z, and their derivatives, and the series for δ in (77), and then arrange as a power series in ϵ, we have

$$\left.\begin{aligned} F=F_0\left(x_0,\ \ldots,\ \frac{dz_0}{d\tau}\right)+F_1\left(x_0,\ x_1,\ \ldots,\ \frac{dz_1}{d\tau}\right)\epsilon+\ \cdots \\ +F_n\left(x_0,\ \ldots,\ x_n,\ \ldots,\ \frac{dz_n}{d\tau},\ \delta_2,\ \ldots,\ \delta_n\right)\epsilon^n+\ \cdots\ =\text{constant.} \end{aligned}\right\} \quad (78)$$

Since this is an identity in ϵ, each coefficient is a constant. We shall now work out the form of the general term of (78). It is seen from (77) that the function F_n will contain terms of the types

$$\frac{dx_j}{d\tau}\frac{dx_{n-j}}{d\tau},\qquad \frac{dy_j}{d\tau}\frac{dy_{n-j}}{d\tau},\qquad \frac{dz_j}{d\tau}\frac{dz_{n-j}}{d\tau},\qquad \text{and (60).}$$

It follows from the properties of x, y, and z that the terms of the first three types are zero unless n is even.

Now consider the term of type (60). All the properties of its exponents and subscripts are the same as when it belonged to P_n except that in the relation (f) the -1 in the left member must be replaced by -2. Hence we see that this term is also identically zero unless n is even.

Since F_n involves only even powers of the y_j and z_j, it is a sum of cosines of even multiples of $\sqrt{A}\tau$. It follows from the relations (a), . . . , (f) (the last one modified as indicated) that the highest multiple is $n+2$. Hence we may write

$$\left.\begin{aligned} F_n=D_0'^{(n)}+D_2'^{(n)}\cos 2\sqrt{A}\tau+\ \cdots\ +D_{2j}'^{(n)}\cos 2j\sqrt{A}\tau \\ +\ \cdots\ +D_{n+2}'^{(n)}\cos(n+2)\sqrt{A}\tau=\text{constant.} \end{aligned}\right\} \quad (79)$$

Since F_n is identically a constant, we have

$$D_0'^{(n)}=\text{constant},\quad D_2'^{(n)}=\ \cdots\ =D_{n+2}'^{(n)}=0. \qquad (80)$$

The quantities $D_2'^{(n)}$, . . . , $D_{n+2}'^{(n)}$ depend upon $\alpha_j^{(1)}$, . . . , $\alpha_j^{(n)}$; $\beta_j^{(1)}$, . . . , $\beta_j^{(n)}$; $\gamma_j^{(1)}$, . . . , $\gamma_j^{(n)}$. Suppose all the α_j, β_j, γ_j up to $\alpha_j^{(n-2)}$, $\beta_j^{(n-2)}$, $\gamma_j^{(n-1)}$ have been computed and are known to be accurate. Equations (80) can then be used in two ways, as we shall show. First, they test the accuracy of the computation of the $\alpha_j^{(n-1)}$, $\beta_j^{(n-1)}$, $\gamma_j^{(n-1)}$, for these quantities must have such values that the equations shall be satisfied. And secondly, we can compute the $\alpha_j^{(n-1)}$, $\beta_j^{(n-1)}$ from equations of the type of (69), and then find the $\gamma_j^{(n)}$ and δ_n directly from (80) without referring to the differential equations of the type of (67).

The first use is obvious and we need to consider further only the second. We are working under the hypothesis that n is even. Therefore the $\alpha_j^{(n)}$ and the $\beta_j^{(n)}$ are identically zero. It is seen from (77) that the only terms which can introduce the $\gamma_j^{(n)}$ and δ_n are

$$2\frac{dz_0}{d\tau}\frac{dz_n}{d\tau}+2Az_0z_n+2A\delta_n z_0^2. \qquad (81)$$

From the form of z_n given in (68), we have

$$\left.\begin{aligned}
2\frac{dz_0}{d\tau}\frac{dz_n}{d\tau} &= c\sqrt{A}\Big\{\gamma_1^{(n)}+[\gamma_1^{(n)}+3\gamma_3^{(n)}]\cos 2\sqrt{A}\tau+\cdots \\
&\quad +[(2j-1)\gamma_{2j-1}^{(n)}+(2j+1)\gamma_{2j+1}^{(n)}]\cos 2j\sqrt{A}\tau+\cdots \\
&\quad +(n+1)\gamma_{n+1}^{(n)}\cos(n+2)\sqrt{A}\tau\Big\}, \\
2Az_0z_n &= c\sqrt{A}\Big\{\gamma_1^{(n)}+[-\gamma_1^{(n)}+\gamma_3^{(n)}]\cos 2\sqrt{A}\tau+\cdots \\
&\quad +[-\gamma_{2j-1}^{(n)}+\gamma_{2j+1}^{(n)}]\cos 2j\sqrt{A}\tau \\
&\quad -\gamma_{n+1}^{(n)}\cos(n+2)\sqrt{A}\tau\Big\}+\cdots, \\
2A\delta_n z_0^2 &= c^2\delta_n - c^2\delta_n\cos 2\sqrt{A}\tau.
\end{aligned}\right\} \quad (82)$$

Therefore equations (80) become

$$\left.\begin{aligned}
D_0^{\prime(n)} &= 2c\sqrt{A}\,\gamma_1^{(n)}+c^2\delta_n+D_0^{(n)} = \text{constant}, \\
D_2^{\prime(n)} &= 4c\sqrt{A}\,\gamma_3^{(n)}-c^2\delta_n+D_2^{(n)} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_{2j}^{\prime(n)} &= 2c\sqrt{A}(j-1)\gamma_{2j-1}^{(n)}+2c\sqrt{A}(j+1)\gamma_{2j+1}^{(n)}+D_{2j}^{(n)} = 0 \quad \left(j=2,\ldots,\frac{n}{2}\right), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_{n+2}^{\prime(n)} &= c\sqrt{A}\,n\,\gamma_{n+1}^{(n)}+D_{n+2}^{(n)} = 0,
\end{aligned}\right\} \quad (83)$$

where $D_0^{(n)}, \ldots, D_{n+2}^{(n)}$ are known constants depending upon $\alpha_j^{(1)}, \ldots, \alpha_j^{(n-1)}$; $\beta_j^{(1)}, \ldots, \beta_j^{(n-1)}$; $\gamma_j^{(0)}, \ldots, \gamma_j^{(n-2)}$.

The last $n/2$ equations, beginning with the last one, can be solved for $\gamma_{n+1}^{(n)}, \ldots, \gamma_3^{(n)}$ in order. Then the second equation gives δ_n uniquely. The results of these solutions are

$$\left.\begin{aligned}
\gamma_{n+1}^{(n)} &= -\frac{D_{n+2}^{(n)}}{c\sqrt{A}n}, \\
\gamma_{2j-1}^{(n)} &= -\frac{j+1}{j-1}\gamma_{2j+1}^{(n)}-\frac{D_{2j}^{(n)}}{2(j-1)c\sqrt{A}} \quad \left(j=\frac{n}{2},\ldots,3\right), \\
\gamma_3^{(n)} &= -3\gamma_5^{(n)}-\frac{D_4^{(n)}}{2c\sqrt{A}}, \qquad \delta_n = \frac{4c\sqrt{A}\,\gamma_3^{(n)}+D_2^{(n)}}{c^2}.
\end{aligned}\right\} \quad (84)$$

All the constants are uniquely determined except $\gamma_1^{(n)}$, which is defined by the condition that z_n' shall be zero at $\tau=0$. This condition gives

$$\gamma_1^{(n)} = -\sum_{j=1}^{n/2}(2j+1)\gamma_{2j+1}^{(n)}. \quad (85)$$

Thus we see that in orbits of Class A we can suppress the z-equation, if we wish, and compute the $\gamma_j^{(n)}$ from the integral; or, we may use the integral, step by step, as a check on the computations.

90. Numerical Examples of Orbits of Class A.—No periodic orbits of this class have so far been published. It is clear that it is practically impossible to discover them by numerical experiment. We shall suppose the ratio of the finite masses is ten to one, or $1-\mu=10/11$, $\mu=1/11$. Then, in the computation of the coefficients of the series for the solutions in the vicinity of the points (a), (b), and (c), the following results are found:

Coefficient.	Point (a).	Point (b).	Point (c).
$r_2^{(0)}$ [Equations (4)]	$+0.347$	$+0.282$	$+1.947$
$r_1^{(0)}$ [Equations (4)]	$+1.347$	$+0.718$	$+0.947$
A [Equation (15)]	$+2.548$	$+6.510$	$+1.082$
σ^2 [Equation (19)]	$+2.811$	$+6.820$	$+1.144$
ρ^2 [Equation (19)]	$+3.359$	$+11.330$	$+0.226$
n [Equation (28)]	$+2.657$	$+3.990$	$+2.014$
m [Equation (28)]	-0.747	-0.397	-3.091
B [Equation (35)]	$+6.548$	-10.961	-1.136
C [Equation (35)]	$+18.283$	$+55.740$	$+1.196$
$\dfrac{-3B}{4A(1+2A)}$ [(56)]	-0.316	$+0.090$	$+0.249$
$\dfrac{+3B(1+3A)}{4A(1-7A+18A^2)}$ [(56)]	$+0.151$	-0.036	-0.230
$\dfrac{-3B}{\sqrt{A}(1-7A+18A^2)}$ [(56)]	-0.112	$+0.018$	$+0.226$
$\dfrac{+3}{64A^{5/2}}\left[\dfrac{3B^2(1+3A)}{1-7A+18A^2}-C\right]$ (58)	-0.037	-0.020	-0.002
$\dfrac{-9}{16A^2}\left[\dfrac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)}-C\right]$	$+0.184$	$+0.467$	$+0.001$
P=period	$3.936(1+0.184\epsilon'^2+\cdots)$	$2.463(1+0.467\epsilon'^2+\cdots)$	$6.041(1+0.001\epsilon'^2+\cdots)$

From these results we find that the solutions in the neighborhood of the three points of equilibrium are

$$\left.\begin{array}{l}
(a)\begin{cases}
x'=[-0.316+0.151\cos 2\sqrt{A}\tau]\,\epsilon'^2+\cdots, \\
y'=[-0.112\sin 2\sqrt{A}\tau]\,\epsilon'^2+\cdots, \\
z'=[+0.626\sin\sqrt{A}\tau]\,\epsilon'+[-0.037\,(3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau)]\,\epsilon'^3+\cdots;
\end{cases}\\
(b)\begin{cases}
x'=[+0.090-0.036\cos 2\sqrt{A}\tau]\,\epsilon'^2+\cdots, \\
y'=[+0.018\sin 2\sqrt{A}\tau]\,\epsilon'^2+\cdots, \\
z'=[+0.392\sin\sqrt{A}\tau]\,\epsilon'+[-0.020\,(3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau)]\,\epsilon'^3+\cdots;
\end{cases}\\
(c)\begin{cases}
x'=[+0.249-0.230\cos 2\sqrt{A}\tau]\,\epsilon'^2+\cdots, \\
y'=[+0.226\sin 2\sqrt{A}\tau]\,\epsilon'^2+\cdots, \\
z'=[+0.961\sin\sqrt{A}\tau]\,\epsilon'+[-0.002\,(3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau)]\,\epsilon'^3+\cdots.
\end{cases}
\end{array}\right\}\quad(86)$$

If we regard the motion of the finite bodies as direct, and consider the projections of the motion of the infinitesimal body upon the $x'y'$-plane, we find that in all cases the motion in orbits of Class A is retrograde.

91. Construction of a Prescribed Orbit of Class A.—Suppose the masses of the finite bodies are given. Then a periodic orbit of Class A for the infinitesimal body may be defined (1) by the place it crosses the x'-axis, (2) by the y' or z'-component of velocity with which it crosses, (3) by the greatest value of the y' or z'-coördinate, (4) by the constant of the Jacobian integral, and (5) by the period. It is understood, of course, that these various quantities are arbitrary only within such limits that the series for the coördinates converge.

For $\epsilon'=0$ the Jacobian constant C and the period have definite values depending only upon μ. The increments to these values and all the other defining quantities enumerated above can be developed as power series in ϵ', vanishing with ϵ'. These series are odd or even, depending upon which other quantity is taken as defining the orbit. If s represents any of these quantities, we can write

$$s=s_1\,\epsilon'+s_2\,\epsilon'^2+s_3\,\epsilon'^3+\cdots,$$

where the coefficients $s_1, s_2, s_3, \ldots$ are constants which depend upon μ alone. If s is assigned numerically the inversion of this series gives ϵ'. This value of ϵ' substituted in (71) gives the desired orbit. Thus, the methods which have been developed not only prove the existence of the periodic orbits and give convenient processes for constructing them and testing the accuracy of the computations, but they furnish a ready means of finding any particular orbit that may be desired.

92. Existence of Orbits of Class B.—For $\epsilon=0$ the coördinates in these orbits are given by (30). Therefore $a_1=a_2\neq 0$, $a_3=a_4=c=0$ in (38). Sufficient conditions that the solutions (39) shall be periodic with the period $2\pi/\sigma$ are

$$\left.\begin{aligned} &u_i\left(\frac{2\pi}{\sigma}\right)-u_i(0)=0 &&(i=1,\ldots,4),\\ &z\left(\frac{2\pi}{\sigma}\right)-z(0)=0, &&z'\left(\frac{2\pi}{\sigma}\right)-z'(0)=0.\end{aligned}\right\}\quad(87)$$

If we let

$$\begin{aligned} &u_1=(a_1+\alpha_1)\,e^{+\sigma\sqrt{-1}\tau}+v_1, &&u_3=\alpha_3+v_3, &&z=0+\zeta,\\ &u_2=(a_1+\alpha_2)\,e^{-\sigma\sqrt{-1}\tau}+v_2, &&u_4=\alpha_4+v_4, &&z'=\gamma+\zeta',\end{aligned}$$

the integral (13) may be written

$$\left.\begin{aligned} &F\Big[(a_1+\alpha_1)e^{\rho\sqrt{-1}\tau}+v_1,\ (a_1+\alpha_2)\,e^{-\rho\sqrt{-1}\tau}+v_2,\ \alpha_3+v_3,\ \alpha_4+v_4,\ \zeta,\\ &\qquad\gamma+\zeta',\ \delta,\ \epsilon\Big]-F\Big[a_1+\alpha_1,\ a_1+\alpha_2,\ \alpha_3,\ \alpha_4,\ 0,\ \gamma,\ \delta,\ \epsilon\Big]=0.\end{aligned}\right\}\quad(88)$$

This equation is satisfied, at $\tau=2\pi/\sigma$, by $v_i=\zeta=\zeta'=0$; and we find from the explicit form of F and the transformation (33) that, for these values of the variables and $\alpha_1=\cdots=\alpha_4=\delta=\epsilon=0$,

$$\frac{\partial F}{\partial v_1}=4\,a_1[n^2\sigma^2-1-2A].$$

But from equations (19) and (28) we have

$$n=\frac{\sigma^2+1+2A}{2\sigma}=\frac{2\sigma}{\sigma^2+1-A}.$$

Therefore $n^2\sigma^2-1-2A=\sigma^2+n^2(A-1)$, which is always positive. Consequently, for $\tau=2\pi/\sigma$, (88) can be solved uniquely for v_1 in terms of α_i, γ, δ, ϵ, v_2, v_3, v_4, ζ, ζ', and this solution vanishes for $v_2=v_3=v_4=\zeta=\zeta'=0$. Hence, if we impose the condition that $v_2=v_3=v_4=\zeta=\zeta'=0$ at $\tau=2\pi/\sigma$, the equation $v_1=0$ at $\tau=2\pi/\sigma$ will be satisfied. Therefore the first equation is redundant, and it will be suppressed.

It will be shown that the orbits of Class B lie in the xy-plane. It follows from the form of the last equation of (34) and the initial values of z and z' that P_5 and P_6 contain γ as a factor. Therefore the last two equations of (88) contain γ as a factor. The explicit form of the next to the last one is

$$P_5\left(\frac{2\pi}{\sigma}\right)-P_5(0)=\frac{\gamma}{\sqrt{A}}\sin\frac{2\pi\sqrt{A}}{\sigma}+\gamma P(\alpha_j,\gamma,\delta,\epsilon)=0. \tag{89}$$

When $\sqrt{A}/\sigma$ is not an integer the only solution of this equation, vanishing with the parameters in terms of which the solution is made, is $\gamma=0$. The case where $\sqrt{A}/\sigma$ is commensurable will be considered in connection with the orbits of Class C. Therefore $z=0$, and the orbits are plane curves.

Necessary and sufficient conditions for the existence of the periodic solutions of Class B reduce to (87), where $i=2, 3, 4$. The explicit forms of these equations are

$$\left.\begin{aligned}
0&=(a_1+\alpha_2)\left[e^{-2\pi(1+\delta)\sqrt{-1}}-1\right]+\epsilon\,Q_2(\alpha_1,\ldots,\alpha_4,\delta,\epsilon),\\
0&=\quad\alpha_3\left[e^{+2\pi\frac{\rho}{\sigma}(1+\delta)}-1\right]+\epsilon\,Q_3(\alpha_1,\ldots,\alpha_4,\delta,\epsilon),\\
0&=\quad\alpha_4\left[e^{-2\pi\frac{\rho}{\sigma}(1+\delta)}-1\right]+\epsilon\,Q_4(\alpha_1,\ldots,\alpha_4,\delta,\epsilon).
\end{aligned}\right\} \tag{90}$$

We have three equations to satisfy and five arbitrary parameters, besides ϵ, at our disposal. The parameter α_1 enters only in the combination $a_1+\alpha_1$, and since a_1 is as yet subject only to the condition that it shall not vanish, we may let it absorb α_1. We may determine t_0, which enters in the

definition of τ, so that at $\tau=0$ we shall have $x'=0$, a condition which is fulfilled in all closed orbits in which the coördinates have continuous derivatives. By (33) this condition becomes

$$-\sigma\sqrt{-1}\ \alpha_2+\rho(\alpha_3-\alpha_4)=0,$$

which we may regard as eliminating α_2.

We now consider the solution of (90) for α_3, α_4, and δ as power series in ϵ, vanishing with ϵ. The determinant of the linear terms in α_3, α_4, and δ is

$$2\pi a_1\sqrt{-1}\left[e^{2\pi\frac{\rho}{\sigma}}-1\right]\left[e^{-2\pi\frac{\rho}{\sigma}}-1\right], \tag{91}$$

which is not zero. Therefore equations (90) have a unique solution for α_3, α_4, and δ as power series in ϵ, vanishing with ϵ. When these results are substituted in the first four equations of (39), the latter become power series in ϵ which are periodic in τ with the period $2\pi/\sigma$.

It will now be shown that all orbits of this class are symmetrical with respect to the x-axis. We choose t_0 so that we have $y=x'=0$ at $\tau=0$. Therefore it follows from equations (33) and from the initial values of the u_i that $\alpha_2=0$, $\alpha_3=\alpha_4$. Necessary and sufficient conditions that these symmetrical solutions shall be periodic are

$$\frac{dx}{d\tau}=y=0, \text{ at } \tau=\frac{\pi}{\sigma}\cdot$$

It follows from (33) that these equations are equivalent to $u_1=u_2$, $u_3=u_4$ at $\tau=\pi/\sigma$. The explicit expressions for the latter become

$$\left.\begin{aligned}0&=a_1\left[e^{\pi(1+\delta)\sqrt{-1}}-e^{-\pi(1+\delta)\sqrt{-1}}\right]+\epsilon\, Q_1'\,(\alpha_3,\ \delta,\ \epsilon),\\ 0&=\alpha_3\left[e^{\pi\frac{\rho}{\sigma}(1+\delta)}-e^{-\pi\frac{\rho}{\sigma}(1+\delta)}\right]+\epsilon\, Q_3'\,(\alpha_3,\ \delta,\ \epsilon).\end{aligned}\right\} \tag{92}$$

The determinant of the coefficients of the terms which are linear in α_3 and δ is

$$2\pi a_1\sqrt{-1}\left[e^{2\pi\frac{\rho}{\sigma}}-e^{-2\pi\frac{\rho}{\sigma}}\right],$$

which is not zero. Therefore equations (92) can be solved uniquely for α_3 and δ as power series in ϵ, vanishing with ϵ. Since, for a given value of ϵ, there is but one unrestricted orbit of this class, and since there is also one which is symmetrical with respect to the x-axis, it follows that all orbits of this class are symmetrical with respect to the x-axis.

The orbits of this class all re-enter after one revolution, for if we impose the conditions that they re-enter after ν revolutions, we find the solution is unique. Since it includes those which re-enter after one revolution, it follows that all orbits of this class re-enter after precisely one revolution.

93. Direct Construction of the Solutions for Class B.—It has been shown that in the periodic orbits of Class B the coördinates are uniquely developable in series of the form

$$u_i = \sum_{j=0}^{\infty} u_i^{(j)} \epsilon^j \qquad (i=1, \ldots, 4), \qquad \delta = \sum_{j=1}^{\infty} \delta_j \epsilon^j, \tag{93}$$

where the $u_i^{(j)}$ are periodic functions of τ with the period $2\pi/\sigma$, and where the δ_j are constants.

We have seen that without loss of generality we can put, at $\tau=0$,

$$u_1 = u_2 = a_1 = \frac{a}{2}, \qquad u_3 = u_4. \tag{94}$$

It follows from (33) and (93) that we have also

$$x = \sum_{i=0}^{\infty} x_i \epsilon^i, \qquad y = \sum_{i=0}^{\infty} y_i \epsilon^i. \tag{95}$$

Upon substituting (93) in (34), arranging as power series in ϵ, and equating coefficients of corresponding powers of ϵ, we obtain a series of sets of differential equations from which the $u_i^{(j)}$ can be determined.

The terms independent of ϵ are defined by

$$\frac{du_1^{(0)}}{d\tau} - \sigma\sqrt{-1}\, u_1^{(0)} = 0, \qquad \frac{du_3^{(0)}}{d\tau} - \rho\, u_3^{(0)} = 0,$$

$$\frac{du_2^{(0)}}{d\tau} + \sigma\sqrt{-1}\, u_2^{(0)} = 0, \qquad \frac{du_4^{(0)}}{d\tau} + \rho\, u_4^{(0)} = 0.$$

The solution of these equations which satisfies the periodicity conditions and the initial conditions is

$$u_1^{(0)} = \frac{a}{2} e^{\sigma\sqrt{-1}\tau}, \qquad u_2^{(0)} = \frac{a}{2} e^{-\sigma\sqrt{-1}\tau}, \qquad u_3^{(0)} = 0, \qquad u_4^{(0)} = 0. \tag{96}$$

From these results and (33) we get

$$x_0 = a \cos\sigma\tau, \qquad y_0 = -na \sin\sigma\tau. \tag{97}$$

The terms of the first degree in ϵ are defined by

$$\left.\begin{aligned}
\frac{du_1^{(1)}}{d\tau} - \sigma\sqrt{-1}\, u_1^{(1)} &= +\sigma\sqrt{-1}\,\delta_1 u_1^{(0)} + \frac{3mB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)\sqrt{-1}} - \frac{3Bx_0y_0}{2(m\rho+n\sigma)},\\
\frac{du_2^{(1)}}{d\tau} + \sigma\sqrt{-1}\, u_2^{(1)} &= -\sigma\sqrt{-1}\,\delta_1 u_2^{(0)} - \frac{3mB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)\sqrt{-1}} - \frac{3Bx_0y_0}{2(m\rho+n\sigma)},\\
\frac{du_3^{(1)}}{d\tau} - \rho\, u_3^{(1)} &= -\frac{3nB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)} + \frac{3Bx_0y_0}{2(m\rho+n\sigma)},\\
\frac{du_4^{(1)}}{d\tau} + \rho\, u_4^{(1)} &= +\frac{3nB[-2x_0^2+y_0^2]}{4(m\sigma-n\rho)} + \frac{3Bx_0y_0}{2(m\rho+n\sigma)}.
\end{aligned}\right\} \tag{98}$$

The conditions that the solutions of these equations shall be periodic with the period $2\pi/\sigma$ are that the right members shall be periodic with this period, that the right member of the first equation shall not contain the term $e^{\sigma\sqrt{-1}\tau}$, and that the right member of the second equation shall not contain the term $e^{-\sigma\sqrt{-1}\tau}$. The first condition is satisfied, and on referring to (97) we see that the second and third conditions can be satisfied only by $\delta_1=0$. Therefore we put δ_1 equal to zero.

Upon substituting from (97), equations (98) become in full

$$\left.\begin{aligned}
\frac{du_1^{(1)}}{d\tau}-\sigma\sqrt{-1}\,u_1^{(1)}&=+\frac{3mBa^2[(n^2-2)-(n^2+2)\cos 2\sigma\tau]}{8(m\sigma-n\rho)\sqrt{-1}}+\frac{3nBa^2\sin 2\sigma\tau}{4(m\rho+n\sigma)},\\
\frac{du_2^{(1)}}{d\tau}+\sigma\sqrt{-1}\,u_2^{(1)}&=-\frac{3mBa^2[(n^2-2)-(n^2+2)\cos 2\sigma\tau]}{8(m\sigma-n\rho)\sqrt{-1}}+\frac{3nBa^2\sin 2\sigma\tau}{4(m\rho+n\sigma)},\\
\frac{du_3^{(1)}}{d\tau}-\rho u_3^{(1)}&=-\frac{3nBa^2[(n^2-2)-(n^2+2)\cos 2\sigma\tau]}{8(m\sigma-n\rho)}-\frac{3nBa^2\sin 2\sigma\tau}{4(m\rho+n\sigma)},\\
\frac{du_4^{(1)}}{d\tau}+\rho u_4^{(1)}&=+\frac{3nBa^2[(n^2-2)-(n^2+2)\cos 2\sigma\tau]}{8(m\sigma-n\rho)}-\frac{3nBa^2\sin 2\sigma\tau}{4(m\rho+n\sigma)},
\end{aligned}\right\}\tag{99}$$

The solution of the first equation of (99) has the form

$$u_1^{(1)}=c_1^{(1)}\,e^{\sigma\sqrt{-1}\tau}+a_{10}^{(1)}+a_{12}^{(1)}\cos 2\sigma\tau-\sqrt{-1}\,b_{12}^{(1)}\sin 2\sigma\tau,\tag{100}$$

where $c_1^{(1)}$ is the arbitrary constant of integration. Upon substituting this expression in the first of (99) and equating coefficients of like functions of τ, we get

$$\left.\begin{aligned}
a_{10}^{(1)}&=+\frac{3mBa^2(n^2-2)}{8(m\sigma-n\rho)\sigma},\\
a_{12}^{(1)}&=+\frac{mBa^2(n^2+2)}{8(m\sigma-n\rho)\sigma}-\frac{nBa^2}{2(m\rho+n\sigma)\sigma},\\
b_{12}^{(1)}&=-\frac{mBa^2(n^2+2)}{4(m\sigma-n\rho)\sigma}+\frac{nBa^2}{4(m\rho+n\sigma)\sigma}.
\end{aligned}\right\}\tag{101}$$

It follows from the form of (99) that the solution of the second equation can be obtained from that of the first by changing the sign of $\sqrt{-1}$. Therefore

$$a_{10}^{(1)}=a_{20}^{(1)},\qquad a_{12}^{(1)}=a_{22}^{(1)},\qquad b_{12}^{(1)}=-b_{22}^{(1)}.\tag{102}$$

The solution of the third equation of (99) has the form

$$u_3^{(1)}=c_3^{(1)}e^{\rho\tau}+a_{30}^{(1)}+a_{32}^{(1)}\cos 2\sigma\tau+b_{32}^{(1)}\sin 2\sigma\tau,$$

where, because of the periodicity condition, $c_3^{(1)}=0$. We find by substitution in the differential equations that

$$\left.\begin{aligned} a_{30}^{(1)} &= +a_{40}^{(1)} = +\frac{3nBa^2(n^2-2)}{8(m\sigma-n\rho)\rho}, \\ a_{32}^{(1)} &= +a_{42}^{(1)} = -\frac{3nBa^2(n^2+2)\rho}{8(m\sigma-n\rho)(4\sigma^2+\rho^2)}+\frac{3nBa^2\sigma}{2(m\rho+n\sigma)(4\sigma^2+\rho^2)}, \\ b_{32}^{(1)} &= -b_{42}^{(1)} = +\frac{3nBa^2(n^2+2)\sigma}{4(m\sigma-n\rho)(4\sigma^2+\rho^2)}+\frac{3nBa^2\rho}{4(m\rho+n\sigma)(4\sigma^2+\rho^2)}. \end{aligned}\right\} \tag{103}$$

Since $x=a$ and $y=0$ at $\tau=0$ for all values of ϵ, we have $x_1(0)=y_1(0)=0$. Upon substituting $u_1^{(1)}, \ldots, u_4^{(1)}$ in (33) and applying these conditions, we find

$$c_1^{(1)}=c_2^{(1)}=-[a_{10}^{(1)}+a_{30}^{(1)}+a_{12}^{(1)}+a_{32}^{(1)}]. \tag{104}$$

Then the expressions for x_1 and y_1 become, by (33),

$$\left.\begin{aligned} x_1 &= 2[a_{10}^{(1)}+a_{30}^{(1)}]-2[a_{10}^{(1)}+a_{30}^{(1)}+a_{12}^{(1)}+a_{32}^{(1)}]\cos\sigma\tau+2[a_{12}^{(1)}+a_{32}^{(1)}]\cos 2\sigma\tau, \\ y_1 &= +2n[a_{10}^{(1)}+a_{30}^{(1)}+a_{12}^{(1)}+a_{32}^{(1)}]\sin\sigma\tau+2[nb_{12}^{(1)}+mb_{32}^{(1)}]\sin 2\sigma\tau, \\ \delta_1 &= 0. \end{aligned}\right\} \tag{105}$$

In order to see how the construction goes in general, the process must be continued one step further. The differential equations which define the next terms are

$$\left.\begin{aligned} \frac{du_1^{(2)}}{d\tau}-\sigma\sqrt{-1}u_1^{(2)} &= +\sigma\sqrt{-1}u_1^{(0)}\delta_2+\frac{3mB[-2x_0x_1+y_0y_1]}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3B\{x_0y_1+x_1y_0\}}{2(m\rho+n\sigma)} \\ &\quad +\frac{mC[2x_0^3-3x_0y_0^2]}{(m\sigma-n\rho)\sqrt{-1}}+\frac{3C\{4x_0^2y_0-y_0^3\}}{4(m\rho+n\sigma)}, \\ \frac{du_2^{(2)}}{d\tau}+\sigma\sqrt{-1}u_2^{(2)} &= -\sigma\sqrt{-1}u_2^{(0)}\delta_2-\frac{3mB[-2x_0x_1+y_0y_1]}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{3B\{x_0y_1+x_1y_0\}}{2(m\rho+n\sigma)} \\ &\quad -\frac{mC[2x_0^3-3x_0y_0^2]}{(m\sigma-n\rho)\sqrt{-1}}+\frac{3C\{4x_0^2y_0-y_0^3\}}{4(m\rho+n\sigma)}, \\ \frac{du_3^{(2)}}{d\tau}-\rho u_3^{(2)} &= -\frac{3nB[-2x_0x_1+y_0y_1]}{2(m\sigma-n\rho)}+\frac{3\{Bx_0y_1+x_1y_0\}}{2(m\rho+n\sigma)} \\ &\quad -\frac{nC[2x_0^3-3x_0y_0^2]}{m\sigma-n\rho}-\frac{3C\{4x_0^2y_0-y_0^3\}}{4(m\rho+n\sigma)}, \\ \frac{du_4^{(2)}}{d\tau}+\rho u_4^{(2)} &= +\frac{3nB[-2x_0x_1+y_0y_1]}{2(m\sigma-n\rho)}+\frac{3B\{x_0y_1+x_1y_0\}}{2(m\rho+n\sigma)} \\ &\quad +\frac{nC[2x_0^3-3x_0y_0^2]}{m\sigma-n\rho}-\frac{3C\{4x_0^2y_0-y_0^3\}}{4(m\rho+n\sigma)}. \end{aligned}\right\} \tag{106}$$

In order that the solutions of these equations shall be periodic with the period $2\pi/\sigma$, the right members must be periodic with this period, the right member of the first equation must not contain the term $e^{\sigma\sqrt{-1}\tau}$, and the right member of the second equation must not contain the term $e^{-\sigma\sqrt{-1}\tau}$. The first condition is satisfied as the equations stand. The expression $e^{\sigma\sqrt{-1}\tau}$ enters through $u_1^{(0)}$, $x_0 x_1$, $y_0 y_1$, $x_0 y_1$, and $x_1 y_0$. The sum of its coefficients must be put equal to zero; this condition determines δ_2 by the equation

$$\left.\begin{aligned}\delta_2 = &-\frac{3mB[4(a_{10}^{(1)}+a_{30}^{(1)})+2(a_{12}^{(1)}+a_{32}^{(1)})+n(nb_{12}^{(1)}+mb_{32}^{(1)})]}{2(m\sigma-n\rho)\rho}\\ &-\frac{3B[-2n(a_{10}^{(1)}+a_{30}^{(1)})+n(a_{12}^{(1)}+a_{32}^{(1)})+(nb_{12}^{(1)}+mb_{32}^{(1)})]}{2(m\rho+n\sigma)\sigma}\\ &+\frac{3ma^2C(2-n^2)}{8(m\sigma-n\rho)\sigma}-\frac{3na^2C(4-3n^2)}{32(m\rho+n\sigma)\sigma}.\end{aligned}\right\} \quad (107)$$

This disposes of all the arbitraries appearing in the equations, and the third condition remains to be satisfied. Upon comparing the first and second equations of (106), we see that the signs of the [ ] in the second are opposite the signs of the corresponding terms in the first, and that corresponding { } have the same sign in the two equations. It is observed that the [ ] are sums of cosines of multiples of $\sigma\tau$, while the { } are sums of sines of the same arguments. Since δ_2 enters in the second equation with the sign opposite to that in the first, it follows, as a consequence of the properties of [ ] and { }, that the condition on the second equation is satisfied by the same value of δ_2 as that which satisfies the condition on the first.

We now proceed to the general term. Suppose $x_1, \ldots, x_{\nu-1}$; $y_1, \ldots, y_{\nu-1}$ have been computed, and that they have been found to have the following properties:

1. The x_j are sums of cosines of multiples of $\sigma\tau$.
2. The y_j are sums of sines of multiples of $\sigma\tau$.
3. The highest multiple of $\sigma\tau$ in x_j and y_j is $j+1$.
4. The [ ] is an even function of y.
5. The { } is an odd function of y.

The equations defining the coefficients of ϵ^ν are

$$\left.\begin{aligned}&\frac{du_1^{(\nu)}}{d\tau}-\sigma\sqrt{-1}\,u_1^{(\nu)}=+\sigma\sqrt{-1}\,u_1^{(0)}\delta_\nu+\frac{m[\ ]}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{\{\ \}}{2(m\rho+n\sigma)},\\ &\frac{du_2^{(\nu)}}{d\tau}+\sigma\sqrt{-1}\,u_2^{(\nu)}=-\sigma\sqrt{-1}\,u_2^{(0)}\delta_\nu-\frac{m[\ ]}{2(m\sigma-n\rho)\sqrt{-1}}-\frac{\{\ \}}{2(m\rho+n\sigma)},\\ &\frac{du_3^{(\nu)}}{d\tau}-\rho u_3^{(\nu)}=-\frac{n[\ ]}{2(m\sigma-n\rho)}+\frac{\{\ \}}{2(m\rho+n\sigma)},\\ &\frac{du_4^{(\nu)}}{d\tau}+\rho u_4^{(\nu)}=+\frac{n[\ ]}{2(m\sigma-n\rho)}+\frac{\{\ \}}{2(m\rho+n\sigma)},\end{aligned}\right\} \quad (108)$$

where the $\delta_2, \ldots, \delta_{\nu-1}$ are included in [ ] and { }.

It follows from properties 1, 2, 4, and 5 that [ ] is a sum of cosines of multiples of $\sigma\tau$, and that { } is a sum of sines of multiples of $\sigma\tau$. The general term of [ ] is

$$T_\nu = x_{\lambda_1}^{\mu_1} \cdot \cdot \cdot x_{\lambda_\kappa}^{\mu_\kappa} \cdot y_{\lambda_1'}^{\mu_1'} \cdot \cdot \cdot y_{\lambda_{\kappa'}'}^{\mu_{\kappa'}'} \cdot \delta_p^q . \tag{109}$$

The exponents and subscripts of this expression satisfy the conditions:

(a) $\mu_1' + \cdot \cdot \cdot + \mu_{\kappa'}'$ is an even integer because of 4.

(b) q is 0 or 1, since δ enters (34) linearly.

(c) $\mu_1\lambda_1 + \cdot \cdot \cdot + \mu_\kappa\lambda_\kappa + \mu_1'\lambda_1' + \cdot \cdot \cdot + \mu_{\kappa'}'\lambda_{\kappa'}' + pq$
$+ \mu_1 + \cdot \cdot \cdot + \mu_\kappa + \mu_1' + \cdot \cdot \cdot + \mu_{\kappa'}' - 1 = \nu$.

The product $x_{\lambda_1}^{\mu_1} \cdot \cdot \cdot x_{\lambda_\kappa}^{\mu_\kappa}$ is a sum of cosines of multiples of $\sigma\tau$, by 1. There is an even number of odd $\mu_1', \ldots, \mu_{\kappa'}'$, by (a). Those factors $y_{\lambda_j'}^{\mu_j'}$ for which μ_j' are odd are sums of sines of multiples of $\sigma\tau$. The product of an even number of such factors is a sum of cosines of $\sigma\tau$. It follows that T_ν is a sum of cosines of multiples of $\sigma\tau$.

The highest multiple of $\sigma\tau$ in T_ν is

$$N = \mu_1(\lambda_1+1) + \cdot \cdot \cdot + \mu_\kappa(\lambda_\kappa+1) + \mu_1'(\lambda_1'+1) + \cdot \cdot \cdot + \mu_{\kappa'}'(\lambda_{\kappa'}'+1).$$

It follows from (c) that

$$N = \nu + 1 - pq = \nu + 1 \text{ when } q = 0. \tag{110}$$

Therefore [ ] is a sum of cosines of multiples of $\sigma\tau$, the highest multiple being $\nu+1$.

It can be shown in a similar way that { } is a sum of sines of multiples of $\sigma\tau$, the highest multiple being $\nu+1$.

Equations (108) can be written, therefore, in the form

$$\left.\begin{aligned}
\frac{du_1^{(\nu)}}{d\tau} - \sigma\sqrt{-1}\,u_1^{(\nu)} &= + \frac{a\sigma\sqrt{-1}}{2} e^{\sigma\sqrt{-1}\,\tau}\,\delta_\nu \\
&\quad + m\sqrt{-1}\left[\sum_{i=0}^{\nu+1} A_i^{(\nu)} \cos i\sigma\tau\right] - \left\{\sum_{i=1}^{\nu+1} B_i^{(\nu)} \sin i\sigma\tau\right\}, \\
\frac{du_2^{(\nu)}}{d\tau} + \sigma\sqrt{-1}\,u_2^{(\nu)} &= - \frac{a\sigma\sqrt{-1}}{2} e^{-\sigma\sqrt{-1}\,\tau}\,\delta_\nu \\
&\quad - m\sqrt{-1}\left[\sum_{i=0}^{\nu+1} A_i^{(\nu)} \cos i\sigma\tau\right] - \left\{\sum_{i=1}^{\nu+1} B_i^{(\nu)} \sin i\sigma\tau\right\}, \\
\frac{du_3^{(\nu)}}{d\tau} - \rho u_3^{(\nu)} &= + n\left[\sum_{i=0}^{\nu+1} A_i^{(\nu)} \cos i\sigma\tau\right] + \left\{\sum_{i=1}^{\nu+1} B_i^{(\nu)} \sin i\sigma\tau\right\}, \\
\frac{du_4^{(\nu)}}{d\tau} + \rho u_4^{(\nu)} &= - n\left[\sum_{i=0}^{\nu+1} A_i^{(\nu)} \cos i\sigma\tau\right] + \left\{\sum_{i=1}^{\nu+1} B_i^{(\nu)} \sin i\sigma\tau\right\},
\end{aligned}\right\} \tag{111}$$

where the $A_j^{(\nu)}$ and $B_j^{(\nu)}$ are all known real constants.

In order that the solution of equations (111) shall be periodic, we must impose the conditions that the coefficient of $e^{\sigma\sqrt{-1}\tau}$ in the first equation, and of $e^{-\sigma\sqrt{-1}\tau}$ in the second equation, shall be zero. It is easily seen that the two conditions are identical, and they uniquely determine δ_ν by the equation

$$a\delta_\nu = -\frac{m}{\sigma}A_1^{(\nu)} - \frac{1}{\sigma}B_1^{(\nu)}. \tag{112}$$

The periodic solutions of (111) are of the form

$$\left.\begin{aligned}
u_1^{(\nu)} &= a_{10}^{(\nu)} + c_1^{(\nu)}\, e^{\sigma\sqrt{-1}\tau} + a_{11}^{(\nu)}\, e^{-\sigma\sqrt{-1}\tau} + \sum_{j=2}^{\nu+1} a_{1j}^{(\nu)} \cos j\sigma\tau - \sqrt{-1}\sum_{j=2}^{\nu+1} b_{1j}^{(\nu)} \sin j\sigma\tau,\\
u_2^{(\nu)} &= a_{20}^{(\nu)} + a_{21}^{(\nu)}\, e^{\sigma\sqrt{-1}\tau} + c_2^{(\nu)}\, e^{-\sigma\sqrt{-1}\tau} + \sum_{j=2}^{\nu+1} a_{2j}^{(\nu)} \cos j\sigma\tau - \sqrt{-1}\sum_{j=2}^{\nu+1} b_{2j}^{(\nu)} \sin j\sigma\tau,\\
u_3^{(\nu)} &= a_{30}^{(\nu)} + \sum_{j=1}^{\nu+1} a_{3j}^{(\nu)} \cos j\sigma\tau + \sum_{j=1}^{\nu+1} b_{3j}^{(\nu)} \sin j\sigma\tau,\\
u_4^{(\nu)} &= a_{40}^{(\nu)} + \sum_{j=1}^{\nu+1} a_{4j}^{(\nu)} \cos j\sigma\tau + \sum_{j=1}^{\nu+1} b_{4j}^{(\nu)} \sin j\sigma\tau,
\end{aligned}\right\} \tag{113}$$

where $c_1^{(\nu)}$ and $c_2^{(\nu)}$ are arbitrary constants of integration.

Upon substituting (113) in (111) and equating coefficients of corresponding functions of τ, we find

$$\left.\begin{aligned}
&a_{10}^{(\nu)} = +a_{20}^{(\nu)} = -\frac{m}{\sigma}A_0^{(\nu)}, \qquad a_{30}^{(\nu)} = +a_{40}^{(\nu)} = +\frac{n}{\rho}A_0^{(\nu)},\\
&a_{11}^{(\nu)} = +a_{21}^{(\nu)} = -\frac{1}{4\sigma}[mA_1^{(\nu)} - B_1^{(\nu)}],\\
&a_{1j}^{(\nu)} = +a_{2j}^{(\nu)} = +\frac{mA_j^{(\nu)} + jB_j^{(\nu)}}{\sigma(j^2-1)} \qquad (j=2,\ldots,\nu+1),\\
&b_{1j}^{(\nu)} = -b_{2j}^{(\nu)} = -\frac{(jA_j^{(\nu)} + B_j^{(\nu)})}{\sigma(j^2-1)} \qquad (j=2,\ldots,\nu+1),\\
&a_{3j}^{(\nu)} = +a_{4j}^{(\nu)} = -\frac{n\rho A_j^{(\nu)} - j\sigma B_j^{(\nu)}}{j^2\sigma^2+\rho^2} \qquad (j=1,\ldots,\nu+1),\\
&b_{3j}^{(\nu)} = -b_{4j}^{(\nu)} = +\frac{nj\sigma A_j^{(\nu)} - \rho B_j^{(\nu)}}{j^2\sigma^2+\rho^2} \qquad (j=1,\ldots,\nu+1).
\end{aligned}\right\} \tag{114}$$

Then equations (33) give

$$\left.\begin{aligned}
x_\nu &= 2\,(a_{10}^{(\nu)} + a_{30}^{(\nu)}) + [c_1^{(\nu)}e^{\sigma\sqrt{-1}\tau} + c_2^{(\nu)}e^{-\sigma\sqrt{-1}\tau}] + 2\sum_{j=1}^{\nu+1}[a_{1j}^{(\nu)} + a_{3j}^{(\nu)}]\cos j\sigma\tau,\\
y_\nu &= n\sqrt{-1}\,[c_1^{(\nu)}e^{\sigma\sqrt{-1}\tau} - c_2^{(\nu)}e^{-\sigma\sqrt{-1}\tau}] + 2\sum_{j=1}^{\nu+1}[nb_{1j}^{(\nu)} + mb_{3j}^{(\nu)}]\sin j\sigma\tau.
\end{aligned}\right\} \tag{115}$$

The arbitraries $c_1^{(\nu)}$ and $c_2^{(\nu)}$ are determined by the conditions that $x_\nu=0$ and $y_\nu=0$ at $\tau=0$. Upon applying these conditions, the final results are

$$\left.\begin{aligned} c_1^{(\nu)}&=c_2^{(\nu)}=-\sum_{j=0}^{\nu+1}[a_{1j}^{(\nu)}+a_{3j}^{(\nu)}],\\ x_\nu&=2[a_{10}^{(\nu)}+a_{30}^{(\nu)}]-2\sum_{j=0}^{\nu+1}[a_{1j}^{(\nu)}+a_{3j}^{(\nu)}]\cos\sigma\tau+2\sum_{j=1}^{\nu+1}[a_{1j}^{(\nu)}+a_{3j}^{(\nu)}]\cos j\sigma\tau,\\ y_\nu&=2n\sum_{j=0}^{\nu+1}[a_{1j}^{(\nu)}+a_{3j}^{(\nu)}]\sin\sigma\tau+2\sum_{j=1}^{\nu+1}[n\,b_{1j}^{(\nu)}+m\,b_{3j}^{(\nu)}]\sin j\sigma\tau.\end{aligned}\right\}\quad(116)$$

These expressions have the properties 1 and 2. Since x_0, y_0, x_1, y_1 also have these properties, the induction is complete and x depends only upon cosines of multiples of $\sigma\tau$, and y upon sines of the same argument. The orbits are therefore symmetrical with respect to the x-axis.

94. Additional Properties of the Orbits of Class B.—It is observed that x_0 and y_0 are homogeneous of the first degree in a, and that x_1, y_1, and δ_2 are homogeneous of the second degree in a. It is easily proved by induction, making use of the general term (109), that x_ν and y_ν are homogeneous of degree $\nu+1$ in a, and that $\delta_{\nu+1}$ is homogeneous of degree $\nu+1$ in a. Consequently, it follows from (10) that the actual coördinates, x' and y', carry in each term of their expansions ϵ' and a to the same degree. That is, ϵ' and a are equivalent to a single parameter, and we may put one of them, say a, equal to unity without any loss of generality. Then the explicit expressions for the coördinates become

$$\left.\begin{aligned} x'&=[\cos\sigma\tau]\epsilon'+2[(a_{10}^{(1)}+a_{30}^{(1)})-(a_{10}^{(1)}+a_{30}^{(1)}+a_{12}^{(1)}+a_{32}^{(1)})\cos\sigma\tau\\ &\qquad+(a_{12}^{(1)}+a_{32}^{(1)})\cos 2\sigma\tau]\,\epsilon'^2+\cdots,\\ y'&=[-n\sin\sigma\tau]\epsilon'+2[+n\,(a_{10}^{(1)}+a_{30}^{(1)}+a_{12}^{(1)}+a_{32}^{(1)})\sin\sigma\tau\\ &\qquad+(nb_{12}^{(1)}+mb_{32}^{(1)})\sin 2\sigma\tau]\,\epsilon'^2+\cdots.\end{aligned}\right\}\quad(117)$$

Since n is a positive constant, it follows that in all cases the motion in these orbits is retrograde. For small values of ϵ' the orbits are approximately elliptical in form, the axes of the ellipses coinciding with the x' and y'-axes.

The integral can be applied as before to check the computation, for it has the form

$$F\left(x,\,y,\,\frac{dx}{d\tau},\,\frac{dy}{d\tau}\right)=F_0+F_1\,\epsilon+F_2\,\epsilon^2+\cdots=\text{constant}.$$

Since this equation is an identity in ϵ, each F_j separately is a constant. It is seen that F_j is a sum of cosines of multiples of $\sigma\tau$, the highest multiple being $j+2$. Since the equation is an identity in τ, the coefficient of each $\cos j\sigma\tau\,(j=1,\ldots,\nu+2)$ separately is zero. These coefficients involve the $a_{ik}^{(j)}$ and $b_{ik}^{(j)}$ linearly, and their vanishing constitutes a check on all the $a_{ik}^{(j)}$, $b_{ik}^{(j)}$ from the beginning of the computation to the step under consideration.

95. Numerical Example of Orbits of Class B.—In Darwin's memoir, cited at the beginning of this chapter, there are a few examples of orbits of this class in the vicinity of the equilibrium points (a) and (b). In all his computations Darwin took one finite mass ten times that of the other. To be able to compare the results of this analysis with his orbits, we shall apply the formulas for the same ratio of the masses. This was the ratio used in the computation of §90, and the first part of the table given there can be used here.

Upon making use of the preceding computations and (36), (101), (103), (104), (107), and (117), we get the following table of results:

Coefficient.		Point (a).	Point (b).	Point (c).
$m\rho+n\sigma$	(36)	$+3.085$	$+9.081$	$+0.685$
$m\sigma-n\rho$	(36)	-6.121	-14.467	-4.263
n^2	(28)	$+7.061$	$+15.920$	$+4.058$
$a_{10}^{(1)}$	(101)	$+0.905$	-0.605	-0.595
$\dfrac{mB(n^2+2)}{8(m\sigma-n\rho)\sigma}$	(101)	$+0.540$	-0.259	-0.583
$\dfrac{n\,B}{2(m\rho+n\sigma)\sigma}$	(101)	$+1.682$	-0.922	-1.562
$a_{12}^{(1)}$	(101)	-1.142	$+0.663$	$+0.979$
$b_{12}^{(1)}$	(101)	-0.239	$+0.057$	$+0.385$
$a_{30}^{(1)}$	(103)	-2.944	$+4.691$	$+0.872$
$\dfrac{-3nB(n^2+2)\rho}{8(m\sigma-n\rho)(4\sigma^2+\rho^2)}$		$+1.212$	-1.772	-0.121
$\dfrac{+3nB\sigma}{2(m\rho+n\sigma)(4\sigma^2+\rho^2)}$		$+0.971$	-0.489	-1.116
$a_{32}^{(1)}$	(103)	$+2.183$	-2.261	-1.237
$b_{32}^{(1)}$	(103)	-1.687	$+2.433$	$+0.295$
$c_{1}^{(1)}$	(104)	$+0.998$	-2.488	-0.019
δ_2	(107)	$+3.955$	$+8.553$	-1.407

We find from this table that when the ratio of the finite masses is ten to one, equations (117) for the three equilibrium points (a), (b), and (c) are

$$\left.\begin{array}{ll}
(a) & \begin{cases} x'=[\cos\sigma\tau]\epsilon'+[-4.078+1.996\cos\sigma\tau+2.082\cos 2\sigma\tau]\epsilon'^2+\cdots, \\ y'=[-2.657\sin\sigma\tau]\epsilon'+[-5.305\sin\sigma\tau+1.250\sin 2\sigma\tau]\epsilon'^2+\cdots; \end{cases} \\[2ex]
(b) & \begin{cases} x'=[\cos\sigma\tau]\epsilon'+[+8.172-4.976\cos\sigma\tau-3.196\cos 2\sigma\tau]\epsilon'^2+\cdots, \\ y'=[-3.990\sin\sigma\tau]\epsilon'+[+19.855\sin\sigma\tau-1.478\sin 2\sigma\tau]\epsilon'^2+\cdots; \end{cases} \\[2ex]
(c) & \begin{cases} x'=[\cos\sigma\tau]\epsilon'+[+0.554-0.038\cos\sigma\tau-0.516\cos 2\sigma\tau]\epsilon'^2+\cdots, \\ y'=[-2.014\sin\sigma\tau]\epsilon'+[+0.077\sin\sigma\tau-0.276\sin 2\sigma\tau]\epsilon'^2+\cdots. \end{cases}
\end{array}\right\} \quad (118)$$

96. On the Existence of Orbits of Class C.—For $\epsilon=0$ the equations of these orbits are given in (32). It follows from these equations and (33) that in this case

$$a_1=\tfrac{1}{2}a+\frac{b}{2\sqrt{-1}}, \qquad a_2=\tfrac{1}{2}a-\frac{b}{2\sqrt{-1}}, \qquad a_3=a_4=0. \tag{119}$$

If the initial conditions are

$$u_1=a_1+\alpha_1, \quad u_2=a_2+\alpha_2, \quad u_3=\alpha_3, \quad u_4=\alpha_4, \quad z=0, \quad z'=c+\gamma, \tag{120}$$

the solutions of (34) may be written in the form

$$\left.\begin{aligned} u_i&=P_i(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \tau) \qquad (i=1, \ldots, 4),\\ z&=P_5(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \tau),\\ z'&=P_6(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon; \tau), \end{aligned}\right\} \tag{121}$$

where the P_i, P_5, and P_6 are power series in $\alpha_1, \ldots, \alpha_4, \lambda, \delta$, and ϵ. The conditions for periodic solutions with the period $\dfrac{2\pi q}{\sqrt{A}}=\dfrac{2\pi p}{\sigma}$ are

$$\left.\begin{aligned} 0&=(a_1+\alpha_1)[e^{+(1+\delta)\sqrt{-1}2\pi p}-1]+\epsilon\, Q_1(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon),\\ 0&=(a_2+\alpha_3)[e^{-(1+\delta)\sqrt{-1}2\pi p}-1]+\epsilon\, Q_2(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon),\\ 0&=\alpha_3[e^{+\rho(1+\delta)\frac{2\pi p}{\sigma}}-1]+\epsilon\, Q_3(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon),\\ 0&=\alpha_4[e^{-\rho(1+\delta)\frac{2\pi p}{\sigma}}-1]+\epsilon\, Q_4(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon),\\ 0&=\frac{c+\gamma}{\sqrt{A}(1+\delta)}\sin 2\pi q(1+\delta)+\epsilon\, Q_5(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon),\\ 0&=+(c+\gamma)\cos 2\pi q(1+\delta)+\epsilon\, Q_6(\alpha_1, \ldots, \alpha_4, \gamma, \delta, \epsilon). \end{aligned}\right\} \tag{122}$$

Since the last equation of (34) carries z as a factor, it follows that the last two equations of (122) are divisible by $c+\gamma$. It will be assumed that this factor is distinct from zero and is divided out. We shall let the undetermined constant c absorb the arbitrary γ. Since we assume that c is distinct from zero, it follows from the integral (13) that the last equation of (122) is redundant and can be suppressed. There remain five equations whose solutions for $\alpha_1, \ldots, \alpha_4$, and δ as power series in ϵ, vanishing with ϵ, will now be considered.

It will appear in the course of the work that we shall need all of the terms of the first degree in ϵ, and all of those of the second degree which are not periodic. We integrate equations (34) as power series in ϵ, introducing δ in the combination $(1+\delta)\tau$, and $\alpha_1, \ldots, \alpha_4$ by means of the initial conditions (120). It will be found that only the first power of δ is needed, and then only in terms independent of ϵ; elsewhere it will be omitted. Likewise only the first powers of α_3 and α_4 will be needed, and therefore the higher powers will be omitted. Since α_1 and α_2 enter only in the combinations $a_1+\alpha_1$ and $a_2+\alpha_2$, we may omit them for brevity until the end, using simply a_1 and a_2, and then restore them where they are needed.

The terms of degree zero in ϵ satisfying the initial conditions (120) are

$$\left.\begin{aligned} &u_1^{(0)}=(a_1+\alpha_1)e^{+\sigma i(1+\delta)\tau}, \qquad && u_3^{(0)}=\alpha_3 e^{+\rho(1+\delta)\tau}, \\ &u_2^{(0)}=(a_2+\alpha_2)e^{-\sigma i(1+\delta)\tau}, && u_4^{(0)}=\alpha_4 e^{-\rho(1+\delta)\tau}, \\ &\qquad z_0=\frac{c}{\sqrt{A}(1+\delta)}\sin\sqrt{A}(1+\delta)\tau, \end{aligned}\right\}\tag{123}$$

where $i=\sqrt{-1}$.

The terms of the first degree in ϵ are defined by the equations

$$\left.\begin{aligned} &\frac{du_1^{(1)}}{d\tau}-\sigma i u_1^{(1)}=-mEi[\ ]-F\{\ \}, \qquad && \frac{du_3^{(1)}}{d\tau}-\rho u_3^{(1)}=-nE[\ ]+F\{\ \}, \\ &\frac{du_2^{(1)}}{d\tau}+\sigma i u_2^{(1)}=+mEi[\ ]-F\{\ \}, && \frac{du_4^{(1)}}{d\tau}+\rho u_4^{(1)}=+nE[\ ]+F\{\ \}, \\ &\qquad \frac{d^2 z_1}{d\tau^2}+Az_1=+3Bx_0z_0, \end{aligned}\right\}\tag{124}$$

where

$$\left.\begin{aligned} &E=\frac{+3B}{4(m\sigma-n\rho)}, \quad F=\frac{+3B}{2(m\rho+n\sigma)}, \\ &[\ ]=-\Big[2(2-n^2)a_1a_2-\frac{c^2}{2A}+(2+n^2)a_1^2e^{2\sigma i\tau}+(2+n^2)a_2^2e^{-2\sigma i\tau}+\frac{c^2}{2A}\cos 2\sqrt{A}\tau \\ &\quad +2(2-mni)a_1\alpha_3e^{+(\sigma i+\rho)\tau}+2(2+mni)a_1\alpha_4e^{+(\sigma i-\rho)\tau} \\ &\quad +2(2+mni)a_2\alpha_3e^{(-\sigma i+\rho)\tau}+2(2-mni)a_2\alpha_4e^{(-\sigma i-\rho)\tau}\Big], \\ &\{\ \}=\{nia_1^2e^{2\sigma i\tau}-nia_2^2e^{-2\sigma i\tau}+(ni+m)a_1\alpha_3e^{(\sigma i+\rho)\tau}+(ni-m)a_1\alpha_4e^{(\sigma i-\rho)\tau} \\ &\quad -(ni-m)a_2\alpha_3e^{(-\sigma i+\rho)\tau}-(ni+m)a_2\alpha_4e^{(-\sigma i-\rho)\tau}\}. \end{aligned}\right\}\text{125)}$$

The solutions of (124) are the respective complementary functions, $K_1^{(1)}e^{\sigma i\tau}$, $K_2^{(1)}e^{-\sigma i\tau}$, $K_3^{(1)}e^{\rho\tau}$, $K_4^{(1)}c^{-\rho\tau}$, plus terms of the same character as their right members. In the solution of the first equation, the coefficients of these terms are respectively the coefficients of the right members written in the order given in (125), omitting the term $\cos 2\sqrt{A}\tau$, divided by

$$-\sigma i, \quad +\sigma i, \quad -3\sigma i, \quad +\rho, \quad -\rho, \quad -2\sigma i+\rho, \quad -2\sigma i-\rho.$$

The term $+(mEic^2/2A)\cos 2\sqrt{A}\tau$ gives rise to

$$+\frac{mE\sigma c^2}{2(4A-\sigma^2)A}\cos 2\sqrt{A}\tau+\frac{mEic^2}{(4A-\sigma^2)\sqrt{A}}\sin 2\sqrt{A}\tau.$$

The corresponding divisors in the solution of the second equation are respectively

$$+\sigma i, \quad +3\sigma i, \quad -\sigma i, \quad 2\sigma i+\rho, \quad 2\sigma i-\rho, \quad +\rho, \quad -\rho,$$

and the terms coming from $(+mEic^2/2A)\cos 2\sqrt{A}\tau$ are

$$+\frac{mE\sigma c^2}{2(4A-\sigma^2)A}\cos 2\sqrt{A}\tau-\frac{mEic^2}{(4A-\sigma^2)\sqrt{A}}\sin 2\sqrt{A}\tau.$$

In the solution of the third and fourth equations it is unnecessary to compute the terms which carry α_3 and α_4 as factors. Omitting these terms, the divisors for the third equation are respectively $-\rho$, $2\sigma i-\rho$, $-2\sigma i-\rho$, and the terms coming from $(-nEc^2/2A)\cos 2\sqrt{A}\tau$ are

$$-\frac{nE\rho c^2}{2(4A+\rho^2)A}\cos 2\sqrt{A}\tau+\frac{nEc^2}{(4A+\rho^2)\sqrt{A}}\sin 2\sqrt{A}\tau.$$

For the fourth equation the corresponding quantities are

$$\rho,\quad 2\sigma i+\rho,\quad -2\sigma i+\rho,\quad -\frac{nE\rho c^2}{2(4A+\rho^2)A}\cos 2\sqrt{A}\tau-\frac{nEc^2}{(4A+\rho^2)\sqrt{A}}\sin 2\sqrt{A}\tau.$$

The solution of the last equation of (124) is

$$\left.\begin{aligned}
z_1=&+L_1^{(1)}\cos\sqrt{A}\tau+L_2^{(1)}\sin\sqrt{A}\tau+\frac{3B\alpha_1 ci}{2(\sigma^2+2\sigma\sqrt{A})\sqrt{A}}e^{(\sigma+\sqrt{A})i\tau}\\
&+\frac{3B\alpha_2 ci}{2(\sigma^2-2\sigma\sqrt{A})\sqrt{A}}e^{(-\sigma+\sqrt{A})i\tau}-\frac{3Bc\alpha_3 i}{2(\rho^2+2i\rho\sqrt{A})\sqrt{A}}e^{(\rho+\sqrt{A}i)\tau}\\
&-\frac{3Bc\alpha_4 i}{2(\rho^2-2i\rho\sqrt{A})\sqrt{A}}e^{(-\rho+\sqrt{A}i)\tau}-\frac{3B\alpha_1 ci}{2(\sigma^2-2\sigma\sqrt{A})\sqrt{A}}e^{(\sigma-\sqrt{A})i\tau}\\
&-\frac{3B\alpha_2 ci}{2(\sigma^2+2\sigma\sqrt{A})\sqrt{A}}e^{(-\sigma-\sqrt{A})i\tau}+\frac{3Bc\alpha_3 i}{2(\rho^2-2i\rho\sqrt{A})\sqrt{A}}e^{(\rho-\sqrt{A}i)\tau}\\
&+\frac{3Bc\alpha_4 i}{2(\rho^2+2i\rho\sqrt{A})\sqrt{A}}e^{(-\rho-\sqrt{A}i)\tau}.
\end{aligned}\right\}\quad(126)$$

The constants of integration $K_1^{(1)},\ldots,K_4^{(1)}$ are determined by the conditions that $u_1^{(1)},\ldots,u_4^{(1)}$ shall vanish at $\tau=0$, and the constants $L_1^{(1)}$ and $L_2^{(1)}$ by the conditions $z_1(0)=z_1'(0)=0$.

It is necessary to compute all non-periodic terms of u_1, u_2, and z which are of the second degree in ϵ and which are independent of $\alpha_1,\ldots,\alpha_4$, and δ. The right members of the differential equations involve

$$\left.\begin{aligned}
[\;]^{(2)}&=\tfrac{3}{2}B[-2x_0x_1+y_0y_1+z_0z_1]+2C[2x_0^3-3x_0y_0^2-3x_0z_0^2],\\
\{\;\}^{(2)}&=3B\{x_0y_1+x_1y_0\}\qquad\qquad+\tfrac{3}{2}C\{-4x_0^2y_0+y_0^3+y_0z_0^2\}.
\end{aligned}\right\}\quad(127)$$

The quantities x_1 and y_1 are defined by

$$x_1=u_1^{(1)}+u_2^{(1)}+u_3^{(1)}+u_4^{(1)},\qquad y_1=ni(u_1^{(1)}-u_2^{(1)})+m(u_3^{(1)}-u_4^{(1)}).$$

In order to get all the non-periodic parts of the solutions at this step, the terms of the differential equations which are non-periodic, that is, which carry ρ in the exponential, must be retained; in the first and second equations the terms in $e^{\sigma i\tau}$ and $e^{-\sigma i\tau}$ respectively must be retained; and in the z-equation the terms in $\cos\sqrt{A}\tau$ and $\sin\sqrt{A}\tau$ must be retained, for these periodic terms give rise to terms in the solution which are multiplied by τ, and which therefore are not periodic.

The conditions for a periodic solution with the period $\dfrac{2\pi p}{\sigma} = \dfrac{2\pi q}{\sqrt{A}} = T$ are

$$
\left.\begin{aligned}
0 = u_i(T) - u_i(0) &= [u_i^{(0)}(T) - u_i^{(0)}(0)] + [u_i^{(1)}(T) - u_i^{(1)}(0)]\,\epsilon \\
&\quad + [u_i^{(2)}(T) - u_i^{(2)}(0)]\,\epsilon^2 + \cdots \qquad (i = 1, \ldots, 4), \\
0 = z(T) - z(0) &= [z_0(T) - z_0(0)] + [z_1(T) - z_1(0)]\,\epsilon \\
&\quad + [z_2(T) - z_2(0)]\,\epsilon^2 + \cdots.
\end{aligned}\right\} \qquad (128)
$$

By means of the steps explained on pages 191 to 193, we find explicitly

$$
\left.\begin{aligned}
&u_1^{(0)}(T) - u_1^{(0)}(0) = (a_1 + \alpha_1)[e^{+2\pi p(1+\delta)i} - 1], \quad u_3^{(0)}(T) - u_3^{(0)}(0) = \alpha_3[e^{+2\pi p\frac{\rho}{\sigma}} - 1], \\
&u_2^{(0)}(T) - u_2^{(0)}(0) = (a_2 + \alpha_2)[e^{-2\pi p(1+\delta)i} - 1], \quad u_4^{(0)}(T) - u_4^{(0)}(0) = \alpha_4[e^{-2\pi p\frac{\rho}{\sigma}} - 1], \\
&z_0(T) - z_0(0) = \frac{c}{\sqrt{A}(1+\delta)} \sin 2\pi(1+\delta);
\end{aligned}\right\} (129)
$$

$$
\begin{aligned}
u_1^{(1)}(T) - u_1^{(1)}(0) = &\left\{ \frac{[2mEi(2 - mni) - F(ni + m)]a_1\alpha_3}{\rho} \right. \\
&\left. + \frac{[-2mEi(2 + mni) - F(ni - m)]a_2\alpha_3}{2\sigma i - \rho} \right\} \left[e^{+2\pi p\frac{\rho}{\sigma}} - 1\right] \\
&+ \left\{ \frac{[-2mEi(2 + mni) + F(ni - m)]a_1\alpha_4}{\rho} \right. \\
&\left. + \frac{[-2mEi(2 - mni) - F(ni + m)]a_2\alpha_4}{2\sigma i + \rho} \right\} \left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right], \\
u_2^{(1)}(T) - u_2^{(1)}(0) = &\left\{ \frac{[-2mEi(2 - mni) - F(ni + m)]a_1\alpha_3}{2\sigma i + \rho} \right. \\
&\left. + \frac{[-2mEi(2 + mni) + F(ni - m)]a_2\alpha_3}{\rho} \right\} \left[e^{+2\pi p\frac{\rho}{\sigma}} - 1\right] \\
&+ \left\{ \frac{[-2mEi(2 + mni) - F(ni - m)]a_1\alpha_4}{2\sigma i - \rho} \right. \\
&\left. + \frac{[+2mEi(2 - mni) - F(ni + m)]a_2\alpha_4}{\rho} \right\} \left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right], \\
u_3^{(1)}(T) - u_3^{(1)}(0) = &\left\{ \frac{[-nE(2 + n^2) - nFi]a_1^2}{2\sigma i - \rho} + \frac{2nE(2 - n^2)a_1a_2}{\rho} \right. \\
&\left. + \frac{[nE(2 + n^2) - nFi]a_2^2}{2\sigma i + \rho} - \frac{2nEc^2}{(4A + \rho^2)\rho} \right\} \left[e^{+2\pi p\frac{\rho}{\sigma}} - 1\right], \\
u_4^{(1)}(T) - u_4^{(1)}(0) = &\left\{ \frac{[nE(2 + n^2) - nFi]a_1^2}{2\sigma i + \rho} + \frac{2nE(2 - n^2)a_1a_2}{\rho} \right. \\
&\left. + \frac{[-nE(2 + n^2) - nFi]a_2^2}{2\sigma i - \rho} - \frac{2nEc^2}{(4A + \rho^2)\rho} \right\} \left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right], \\
z_1(T) - z_1(0) = &- \frac{6Bc\alpha_3}{(4A + \rho^2)\rho} \left[e^{+2\pi p\frac{\rho}{\sigma}} - 1\right] + \frac{6Bc\alpha_4}{(4A + \rho^2)\rho} \left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right];
\end{aligned}
$$

$$
\begin{aligned}
u_1^{(2)}(T) - u_1^{(2)}(0) = {} & a_1[a_1 a_2 L_1 + c^2 M_1] + 4mEi\Bigg\{\frac{[-nE(2+n^2)-nFi]\,a_1^2}{2\sigma i-\rho} \\
& + \frac{2nE(2-n^2)a_1a_2}{\rho} + \frac{[nE(2+n^2)-nFi]a_2^2}{2\sigma i+\rho} \\
& - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} - \frac{a_2}{2\sigma i-\rho}\right\}\left[e^{2\pi p\frac{\rho}{\sigma}} - 1\right] \\
& - 4mEi\Bigg\{\frac{[nE(2+n^2)-nFi]a_1^2}{2\sigma i+\rho} + \frac{2nE(2-n^2)a_1a_2}{\rho} \\
& + \frac{[-nE(2+n^2)-nFi]a_2^2}{2\sigma i-\rho} - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} + \frac{a_2}{2\sigma i+\rho}\right\} \\
& \times\left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right] + 2m^2nE\Bigg\{\frac{[-nE(2+n^2)-nFi]a_1^2}{2\sigma i-\rho} \\
& + \frac{2nE(2-n^2)a_1a_2}{\rho} + \frac{[nE(2+n^2)-nFi]a_2^2}{2\sigma i+\rho} \\
& - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} + \frac{a_2}{2\sigma i-\rho}\right\}\left[e^{2\pi p\frac{\rho}{\sigma}} - 1\right] \\
& + 2m^2nE\Bigg\{\frac{[nE(2+n^2)-nFi]a_1^2}{2\sigma i+\rho} + \frac{2nE(2-n^2)a_1a_2}{\rho} \\
& + \frac{[-nE(2+n^2)-nFi]a_2^2}{2\sigma i-\rho} - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} - \frac{a_2}{2\sigma i+\rho}\right\} \\
& \times\left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right] - mF\Bigg\{\frac{[-nE(2+n^2)-nFi]a_1^2}{2\sigma i-\rho} \\
& + \frac{2nE(2-n^2)a_1a_2}{\rho} + \frac{[nE(2+n^2)-nFi]a_2^2}{2\sigma i+\rho} \\
& - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} - \frac{a_2}{2\sigma i-\rho}\right\}\left[e^{2\pi p\frac{\rho}{\sigma}} - 1\right] \\
& - mF\Bigg\{\frac{[nE(2+n^2)-nFi]a_1^2}{2\sigma i+\rho} + \frac{2nE(2-n^2)a_1a_2}{\rho} \\
& + \frac{[-nE(2+n^2)-nFi]a_2^2}{2\sigma i-\rho} - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} + \frac{a_2}{2\sigma i+\rho}\right\} \\
& \times\left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right] - nFi\Bigg\{\frac{[-nE(2+n^2)-nFi]a_1^2}{2\sigma i-\rho} \\
& + \frac{2nE(2-n^2)a_1a_2}{\rho} + \frac{[nE(2+n^2)-nFi]a_2^2}{2\sigma i+\rho} - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\} \\
& \left\{\frac{a_1}{\rho} + \frac{a_2}{2\sigma i-\rho}\right\}\left[e^{2\pi p\frac{\rho}{\sigma}} - 1\right] + nFi\Bigg\{\frac{[nE(2+n^2)-nFi]a_1^2}{2\sigma i+\rho} \\
& + \frac{2nE(2-n^2)a_1a_2}{\rho} + \frac{[-nE(2+n^2)-nFi]a_2^2}{2\sigma i-\rho} \\
& - \frac{2nEc^2}{(4A+\rho^2)\rho}\Bigg\}\left\{\frac{a_1}{\rho} - \frac{a_2}{2\sigma i+\rho}\right\}\left[e^{-2\pi p\frac{\rho}{\sigma}} - 1\right],
\end{aligned}
$$

where

$$\left.\begin{aligned}
L_1 &= 4mETi\left\{\frac{[2mE(2+n^2)-4nF]}{3\sigma}+\frac{[-2n\rho E(2+n^2)+4n\sigma F]}{4\sigma^2+\rho^2}\right.\\
&\quad\left.-\frac{4E(2-n^2)(m\rho+n\sigma)}{\sigma\rho}\right\}+2mn^2ETi\left\{\frac{[-4mE(2+n^2)+2nF]}{3\sigma}\right.\\
&\quad\left.+\frac{2mn[2\sigma E(2+n^2)+\rho F]}{4\sigma^2+\rho^2}\right\}-n^2FTi\left\{\frac{[2mE(2+n^2)+2nF]}{3\sigma}\right.\\
&\quad\left.-\frac{2m[2\sigma E(2+n^2)+\rho F]}{4\sigma^2+\rho^2}+\frac{2n[\rho E(2+n^2)-\sigma F]}{4\sigma^2+\rho^2}\right\}\\
&\quad+nFTi\left\{\frac{4E(2-n^2)(m\rho+n\sigma)}{\sigma\rho}\right\}+\frac{4mECTi(2-n^2)}{B}\\
&\quad+\frac{2nFCTi}{B}-\frac{3n^2FCTi}{2B},\\
M_1 &= \frac{4mE^2Ti(m\rho+n\sigma)}{\sigma\rho A}-\frac{3mBETi}{(4A-\sigma^2)A}-\frac{nEFTi(m\rho+n\sigma)}{\sigma\rho A}\\
&\quad-\frac{mECTi}{AB}-\frac{nFCTi}{4AB}.
\end{aligned}\right\}\tag{130}$$

There are corresponding equations for $u_2^{(2)}(T)-u_2^{(2)}(0)$ and $z_2(T)-z_2(0)$.

The third and fourth equations of (128) can be solved for α_3 and α_4 as power series in ϵ, α_1, α_2, and δ, vanishing with ϵ. The explicit results are

$$\left.\begin{aligned}
\alpha_3 &= \left\{\frac{[+nE(2+n^2)+nFi]a_1^2}{2\sigma i-\rho}-\frac{2nE(2-n^2)a_1a_2}{\rho}\right.\\
&\qquad\left.+\frac{[-nE(2+n^2)+nFi]a_2^2}{2\sigma i+\rho}+\frac{2nEc^2}{(4A+\rho^2)\rho}\right\}\epsilon+\cdots,\\
\alpha_4 &= \left\{\frac{[-nE(2+n^2)+nFi]a_1^2}{2\sigma i+\rho}-\frac{2nE(2-n^2)a_1a_2}{\rho}\right.\\
&\qquad\left.+\frac{[+nE(2+n^2)+nFi]a_2^2}{2\sigma i-\rho}+\frac{2nEc^2}{(4A+\rho^2)\rho}\right\}\epsilon+\cdots.
\end{aligned}\right\}\tag{131}$$

When these expressions for α_3 and α_4 are substituted in the first, second, and fifth equations of (128), we obtain

$$\left.\begin{aligned}
0 &= (a_1+\alpha_1)\left[e^{+2\pi\rho\delta i}-1\right]+(a_1+\alpha_1)\left[(a_1+\alpha_1)(a_2+\alpha_2)L_1+c^2M_1\right]\epsilon^2+\cdots,\\
0 &= (a_2+\alpha_2)\left[e^{-2\pi\rho\delta i}-1\right]-(a_2+\alpha_2)\left[(a_1+\alpha_1)(a_2+\alpha_2)L_1+c^2M_1\right]\epsilon^2+\cdots,\\
0 &= \frac{c}{\sqrt{A}(1+\delta)}\sin 2\pi q\delta-\frac{3BT}{2A}\left\{\frac{Ec^3}{A}\left[\frac{m\rho+n\sigma}{\sigma\rho}+\frac{1}{2}\left(\frac{-m\sigma}{4A-\sigma^2}+\frac{n\rho}{4A+\rho^2}\right)\right]\right.\\
&\qquad\left.-\frac{6B(a_1+\alpha_1)(a_2+\alpha_2)c}{4A-\sigma^2}\right\}\epsilon^2\\
&\qquad+\frac{3C}{4A}\left\{2(4-n^2)(a_1+\alpha_1)(a_2+\alpha_2)c-\frac{3c^3}{4A}\right\}\epsilon^2+\cdots.
\end{aligned}\right\}\tag{132}$$

After removing the factor c from the last equation, solving for δ, and substituting the result in the first two equations, we have

$$\left.\begin{aligned} 0&=(a_1+\alpha_1)[(a_1+\alpha_1)(a_2+\alpha_2)L+c^2M]\,\epsilon^2+\cdots,\\ 0&=(a_2+\alpha_2)[(a_1+\alpha_1)(a_2+\alpha_2)L+c^2M]\,\epsilon^2+\cdots,\end{aligned}\right\}\tag{133}$$

where

$$\left.\begin{aligned} L&=\frac{L_1}{2\pi pi}-\frac{9B^2}{(4A-\sigma^2)\sqrt{A}}+\frac{3C(4-n^2)}{2\sqrt{A}},\\ M&=\frac{M_1}{2\pi pi}+\frac{3BE}{2A^{\frac{3}{2}}}\left[\frac{m\rho+n\sigma}{\sigma\rho}+\frac{1}{2}\left(\frac{-m\sigma}{4A-\sigma^2}+\frac{n\rho}{4A+\rho^2}\right)\right]-\frac{9C}{16A^{\frac{3}{2}}}.\end{aligned}\right\}\tag{134}$$

Equations (133) can not be solved for α_1 and α_2 as power series in ϵ, vanishing with ϵ, unless

$$a_1[a_1a_2L+c^2M]=0, \qquad a_2[a_1a_2L+c^2M]=0. \tag{135}$$

One solution of these equations is $a_1=a_2=0$, and with this determination of a_1 and a_2, equations (133) are uniquely solvable for α_1 and α_2 as power series in ϵ, vanishing with ϵ. In this case the generating solution reduces to the form of that of Class A. But the orbits of Class A heretofore treated were those for which $\sqrt{A}$ and σ are incommensurable. This restriction was not necessary in order to prove that orbits exist which re-enter after *one* revolution, but it was not certain that there are not others re-entering only after many revolutions. The uniqueness of the solution of (133), for $a_1=a_2=0$, proves that *all of the orbits of Class* A, *of the analytic type under consideration here, re-enter after a single revolution.*

At the beginning of the present discussion the assumption was made that c is distinct from zero, and this permitted the suppression of the last equation of (122). If we had assumed that a_1 is distinct from zero, we could have suppressed the first equation of (122); and solving in a different order, we should finally have arrived at two equations corresponding to (133) containing $(c+\gamma)$ as a factor. The equations would have been found solvable after imposing the condition $c=0$, and we should have arrived at the conclusion that *all orbits* of *Class* B *re-enter after one revolution.*

Equations (135) also have the solution

$$a_1a_2L+c^2M=0. \tag{136}$$

This equation defines c when a_1 and a_2 have been given arbitrary values. If the orbits are to be real, a_1 and a_2 must be conjugate complex quantities. Under these circumstances their product is positive, and L and M must be opposite in sign in order that c shall be real.

After the condition (136) has been applied, equations (133) become

$$\left.\begin{aligned} 0&=(a_1+\alpha_1)[a_1\alpha_2+a_2\alpha_1+\alpha_1\alpha_2]\,L+\epsilon\,[\alpha_1\,,\ \alpha_2\,,\ \epsilon]+\cdots,\\ 0&=(a_2+\alpha_2)[a_1\alpha_2+a_2\alpha_1+\alpha_1\alpha_2]\,L+\epsilon\,[\alpha_1\,,\ \alpha_2\,,\ \epsilon]+\cdots.\end{aligned}\right\}\tag{137}$$

Since the terms of these equations which are independent of ϵ are identical, except for the non-vanishing factors $a_1+\alpha_1$ and $a_2+\alpha_2$, it follows that if one is solved for α_1 and the result substituted in the other, the latter becomes divisible by ϵ. After dividing out ϵ, there is a term independent of α_2 and ϵ which must be equal to zero in order that the equation may be solved for α_2 as a power series in ϵ, vanishing with ϵ. This term involves the coefficients of ϵ^3 in the original solutions (122), since ϵ^3 has been divided out. Likewise, terms enter from lower powers of ϵ through the elimination of α_3, α_4, and δ. It is not possible to construct these terms without an unreasonable amount of work. But we see from the way in which they originate that they are homogeneous of the fourth degree in a_1 and a_2. Unless one or the other of these constants is absent, their ratio is determined by this constant term set equal to zero. If one is absent, the only solution is the other set equal to zero, which throws us back on Class A, which has been already completely treated.

Suppose both constants are present and that their ratio is determined. Since they must be conjugate in order that the orbit may be real, the solution for the ratio has the form

$$\frac{a_1}{a_2}=\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}=\frac{a^2-b^2+2ab\sqrt{-1}}{a^2+b^2}\,.$$

It is clear that only for special values of the coefficients, which might never be possible in the problem, could the solution for the ratio have this form. The complexity of the problem is such that no further attempt will be made here to determine whether there exist solutions of Class C which are distinct from those of Class A and Class B.

If the attempt is made to construct the periodic solution of which (32) are the terms independent of ϵ, no difficulty will be encountered until the terms in ϵ^2 are reached. Then it will be found that equations (135) must be satisfied in order that the solutions at this step shall be periodic. That is, step by step, the construction agrees with the existence, though the computation is somewhat less laborious.

CHAPTER VI.

OSCILLATING SATELLITES.

SECOND METHOD.*

97. Outline of Method.—The problem treated in this chapter is the same as that considered in the preceding, but the method employed is quite different. In this particular question the preceding method is somewhat more convenient, but in other problems where the same general style of analysis can be used it is much less so.

There is a definite physical situation for which the analysis is to be developed. One of its principal features is that the periodic orbits form a continuous series from those of zero dimensions at the points of equilibrium, and as they vary in dimensions the periods undergo corresponding changes. In the analysis of Chapter V the dimensions were controlled by means of the scale factor ϵ', and the varying periods were properly secured by the introduction of δ and its subsequent determination in terms of ϵ'. As ϵ' approached zero the orbits approached zero dimensions and the period approached the value which corresponds to $\delta=0$.

In the present treatment no parameters corresponding to ϵ' and δ are employed. Instead, we introduce a parameter λ by means of $\mu=\mu_0+\lambda$, where μ_0 is kept fixed in numerical value while λ is a parameter in terms of which the solutions are expressed. Periodic solutions are found for all λ whose moduli are sufficiently small, but only those solutions belong to the physical problem for which $\lambda=\mu-\mu_0$. The dimensions of the physical orbit depends upon this value of λ, and its period depends upon μ_0. That is, we find a family of periodic solutions having a constant period depending upon μ_0, but only one of them belongs to the physical problem. It is because of this fact that it is not necessary to make the period variable and dependent upon the parameter in terms of which the solutions are developed.

98. The Differential Equations.—We shall start from equations (6) of Chapter V, omitting the accents which will not be needed. The right members of these equations involve the parameter μ explicitly in the last two terms of U, and implicitly through r_1 and r_2 which depend upon $r_2^{(0)}$ defined in (4). We shall make the transformation

$$\mu=\mu_0+\lambda, \tag{1}$$

but it is not necessary to do so in all places, both explicit and implicit, in which this parameter occurs in the differential equations.

*The problem of oscillating satellites was first treated by the author by the methods of this chapter. However, the two methods were reported on simultaneously in the paper referred to at the beginning of Chapter V.

For simplicity the transformation will be made where it appears explicitly in U, and elsewhere μ will be supposed to retain its original given value, which is regarded as a fixed constant. This particular generalization of the parameter μ is not the only possible one, and the series obtained differ according to the particular generalization made, but when the conditions for convergence are satisfied their sums are identical in t.

After the transformation (1), equations (6) of Chapter V become

$$\left.\begin{aligned} x''-2y'-(1+2A_0)x &= P_1(x,\ y^2,\ z^2,\ \lambda), \\ y''+2x'-(1-\ A_0)y &= yP_2(x,\ y^2,\ z^2,\ \lambda), \\ z''+A_0\ z &= z\,P_2(x,\ y^2,\ z^2,\ \lambda), \end{aligned}\right\} \quad (2)$$

where

$$\left.\begin{aligned} &A_0=\frac{1-\mu_0}{r_1^{(0)3}}+\frac{\mu_0}{r_2^{(0)3}}, \\ &P_1=2A'x\lambda+\tfrac{3}{2}B_0[-2x^2+y^2+z^2]+2C_0[2x^3-3xy^2-3xz^2]+\cdots, \\ &P_2=-A'\lambda+3B_0x+\tfrac{3}{2}C_0[-4x^2+y^2+z^2]+\cdots, \\ &A'=-\frac{1}{r_1^{(0)3}}+\frac{1}{r_2^{(0)3}}, \qquad B_0=\begin{smallmatrix}+\\ \pm\\ -\end{smallmatrix}\frac{1-\mu_0}{r_1^{(0)4}}\begin{smallmatrix}+\\ \mp\\ -\end{smallmatrix}\frac{\mu_0}{r_2^{(0)4}}, \qquad C_0=+\frac{1-\mu_0}{r_2^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}, \end{aligned}\right\} \quad (3)$$

the signs in B_0 being the first, second, or third according as orbits in the vicinity of (a), (b), or (c) are in question. The regions of convergence of P_1 and P_2 are precisely the same as those found in §77.

We shall need the differential equations in the normal form so far as the linear terms are concerned. When the right members of (2) are put equal to zero, their solutions are

$$\left.\begin{aligned} &x=K_1e^{\sigma_0\sqrt{-1}t}+K_2e^{-\sigma_0\sqrt{-1}t}+K_3e^{\rho_0t}+K_4e^{-\rho_0t}, \\ &y=n_0\sqrt{-1}(K_1e^{\sigma_0\sqrt{-1}t}-K_2e^{-\sigma_0\sqrt{-1}t})+m_0(K_3e^{\rho_0t}-K_4e^{-\rho_0t}), \\ &z=c_1\cos\sqrt{A_0}t+c_2\sin\sqrt{A_0}t, \end{aligned}\right\} \quad (4)$$

where $K_1, \ldots, K_4$, c_1, and c_2 are arbitrary constants of integration, and where $+\sigma_0\sqrt{-1}$, $-\sigma_0\sqrt{-1}$, $+\rho_0$, and $-\rho_0$ are the four roots of

$$\omega^4+(2-A_0)\omega^2+(1+2A_0)(1-A_0)=0;$$

and where also

$$n_0=\frac{\sigma_0^2+1+2A_0}{2\sigma_0}, \qquad m_0=\frac{\rho_0^2-1-2A_0}{2\rho_0}. \quad (5)$$

Consequently the normal form is secured by the transformation

$$\left.\begin{aligned} x &= +(u_1+u_2)+(u_3+u_4), \\ x' &= +\sigma_0\sqrt{-1}(u_1-u_2)+\rho_0(u_3-u_4), \\ y &= +n_0\sqrt{-1}(u_1-u_2)+m_0(u_3-u_4), \\ y' &= -n_0\sigma_0(u_1+u_2)+m_0\rho_0(u_3+u_4), \end{aligned}\right\} \quad (6)$$

which reduces equations (2) to

$$
\left.\begin{aligned}
u_1'-\sigma_0 i u_1 &= +\frac{m_0 P_1(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\sigma_0-n_0\rho_0)\sqrt{-1}}-\frac{yP_2(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\rho_0+n_0\sigma_0)},\\
u_2'+\sigma_0 i u_2 &= -\frac{m_0 P_1(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\sigma_0-n_0\rho_0)\sqrt{-1}}-\frac{yP_2(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\rho_0+n_0\sigma_0)},\\
u_3'-\rho_0 u_3 &= -\frac{n_0 P_1(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\sigma_0-n_0\rho_0)}+\frac{yP_2(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\rho_0+n_0\sigma_0)},\\
u_4'+\rho_0 u_4 &= +\frac{n_0 P_1(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\sigma_0-n_0\rho_0)}+\frac{yP_2(x,\, y^2,\, z^2,\, \lambda)}{2(m_0\rho_0+n_0\sigma_0)},\\
z''+A_0 z &= +zP_2(x,\, y^2,\, z^2,\, \lambda),
\end{aligned}\right\} \tag{7}
$$

where $i=\sqrt{-1}$ and where P_1 and P_2 have the values given in (3).

99. Integration of the Differential Equations.—Equations (2) admit the integral

$$
\left.\begin{aligned}
F\equiv x'^2+y'^2+z'^2-\Big\{&x^2+y^2+A_0(2x^2-y^2-z^2)+A'(2x^2-y^2-z^2)\lambda\\
&+B_0(-2x^3+3xy^2+3xz^2)+\cdots\Big\}=\text{constant},
\end{aligned}\right\} \tag{8}
$$

which holds for x, y, and z within the region for which the series converge.

Since there is always a component of acceleration toward the xy-plane, there can be no closed orbit entirely on one side of this plane. Therefore, in all cases we can take the origin of time so that $z=0$ at $t=0$. Suppose

$$
u_j=\alpha_j, \qquad z=0, \qquad z'=\gamma \quad \text{at } t=0. \tag{9}
$$

We now integrate equations (7) as power series in the α_j, γ, and λ. The solutions are

$$
\left.\begin{aligned}
u_1 &= \alpha_1 e^{+\sigma_0 it}+p_1(\alpha_1,\, \ldots,\, \alpha_4,\, \gamma,\, \lambda;\, t),\\
u_2 &= \alpha_2 e^{-\sigma_0 it}+p_2(\alpha_1,\, \ldots,\, \alpha_4,\, \gamma,\, \lambda;\, t),\\
u_3 &= \alpha_3 e^{+\rho_0 t}+p_3(\alpha_1,\, \ldots,\, \alpha_4,\, \gamma,\, \lambda;\, t),\\
u_4 &= \alpha_4 e^{-\rho_0 t}+p_4(\alpha_1,\, \ldots,\, \alpha_4,\, \gamma,\, \lambda;\, t),\\
z &= \frac{\gamma}{\sqrt{A_0}}\sin\sqrt{A_0}\,t+p_5(\alpha_1,\, \ldots,\, \alpha_4,\, \gamma,\, \lambda;\, t),\\
z' &= \gamma\cos\sqrt{A_0}\,t+p_6(\alpha_1,\, \ldots,\, \alpha_4,\, \gamma,\, \lambda;\, t),
\end{aligned}\right\} \tag{10}
$$

where $p_1, \ldots, p_6$ are power series in $\alpha_1, \ldots, \alpha_4$, γ, and λ. The moduli of these parameters can be taken so small that the series converge for all $0\leqq t\leqq \mathrm{T}$, where T (finite) is taken arbitrarily in advance (§16). The p_j are of the second and higher degrees in the α_j, γ, and λ. It follows from the way in which λ was introduced that the p_j identically vanish for $\alpha_1=\cdots=\alpha_4=\gamma=0$. Since the last equation of (7) contains z as a factor, $p_5=p_6=0$ for $\gamma=0$, whatever the other initial conditions may be.

100. Existence of Periodic Solutions.—Since the right members of equation (7) do not contain t explicitly, sufficient conditions that (10) shall be a periodic solution with the period T are

$$\left.\begin{aligned}
0 &= u_1(\mathrm{T}) - u_1(0) = \alpha_1\left[e^{+\sigma_0\sqrt{-1}\mathrm{T}} - 1\right] + p_1(\mathrm{T}) - p_1(0),\\
0 &= u_2(\mathrm{T}) - u_2(0) = \alpha_2\left[e^{-\sigma_0\sqrt{-1}\mathrm{T}} - 1\right] + p_2(\mathrm{T}) - p_2(0),\\
0 &= u_3(\mathrm{T}) - u_3(0) = \alpha_3\left[e^{+\rho_0\mathrm{T}} - 1\right] + p_3(\mathrm{T}) - p_3(0),\\
0 &= u_4(\mathrm{T}) - u_4(0) = \alpha_4\left[e^{-\rho_0\mathrm{T}} - 1\right] + p_4(\mathrm{T}) - p_4(0),\\
0 &= z(\mathrm{T}) - z(0) = \frac{\gamma}{\sqrt{A_0}}\sin\sqrt{A_0}\,\mathrm{T} + p_5(\mathrm{T}) - p_5(0),\\
0 &= z'(\mathrm{T}) - z'(0) = \gamma\left[\cos\sqrt{A_0}\,\mathrm{T} - 1\right] + p_6(\mathrm{T}) - p_6(0).
\end{aligned}\right\} \quad (11)$$

The last two equations of (11) are satisfied by $\gamma = 0$. Suppose $\gamma \neq 0$. Then it follows from the form of the integral (8) that unless $\sqrt{A_0}\mathrm{T} = (2n+1)\pi/2$, where n is an integer, the last equation is a consequence of the first five. We shall suppose T does not have one of these special values, and we shall suppress the last equation since it is a redundant condition. The first five equations are to be solved for $\alpha_1, \ldots, \alpha_4$, and γ in terms of λ, and we can use only those solutions which vanish with λ. These equations are satisfied by $\alpha_1 = \cdots = \alpha_4 = \gamma = 0$. In order that this may be not the only solution vanishing with λ, the determinant of the coefficients of the linear terms in $\alpha_1, \ldots, \alpha_4$, and γ must be zero. This condition is explicitly

$$\left[e^{\sigma_0\sqrt{-1}\,\mathrm{T}} - 1\right]\left[e^{-\sigma_0\sqrt{-1}\,\mathrm{T}} - 1\right]\left[e^{\rho_0\mathrm{T}} - 1\right]\left[e^{-\rho_0\mathrm{T}} - 1\right]\sin\sqrt{A_0}\mathrm{T} = 0. \quad (12)$$

This equation has the solutions

$$\mathrm{T}_1 = \frac{\nu\pi}{\sqrt{A_0}}, \qquad \mathrm{T}_2 = \frac{2\nu\pi}{\sigma_0} \qquad (\nu \text{ an integer}). \quad (13)$$

Consider first the solution $\mathrm{T} = \mathrm{T}_1$. For this value of T the determinant of the linear terms in $\alpha_1, \ldots, \alpha_4$ of the first four equations of (11) is distinct from zero unless $\nu\sigma_0/2\sqrt{A_0}$ is an integer. This condition can not be fulfilled for all ν unless σ_0 is an integral multiple of $2\sqrt{A_0}$. Now since $\omega = \sigma_0\sqrt{-1}$ satisfies

$$\omega^4 + (2 - A_0)\omega^2 + (1 + 2A_0)(1 - A_0) = 0,$$

this condition can not be fulfilled unless A_0 is negative, but from its definition in (3), A_0 is always positive. Therefore there are values of ν for which $\nu\sigma_0/2\sqrt{A_0}$ is not an integer, and one of these values of ν is necessarily unity. It follows that the first four equations of (11) can be solved for $\alpha_1, \ldots, \alpha_4$ as power series in γ and λ. Since $u_1, \ldots, u_4$, and z vanish with $\alpha_1 = \cdots = \alpha_4 = \gamma = 0$, these solutions vanish with γ, and since the first four equations of (7) are functions of z^2, they carry γ^2 as a factor. On substituting the solutions of the first four equations of (11) in the fifth, it becomes a power series in γ and λ alone, and is divisible by γ.

In order to prove the possibility of the solution of the fifth equation for γ in terms of λ, and to determine the character of the solution, we must work out the first terms of the series. Terms in λ alone can not be introduced from the solutions of the first four equations of (10) unless the fifth equation is divisible by γ^2, for the former carry γ^2 as a factor. We shall show first that the fifth equation of (11) has a term in $\gamma\lambda$.

It is seen from (7) and (3) that the coefficient of $\gamma\lambda$ in the expression for z is defined by

$$z''_{1,1}+A_0 z_{1,1}=-A' z_{0,0}=-\frac{A'}{\sqrt{A_0}}\sin\sqrt{A_0}\,t. \tag{14}$$

The solution of this equation satisfying the conditions that $z_{11}(0)=0$ and $z'_{1,1}(0)=0$ is

$$z_{1,1}=-\frac{A'}{2A_0^{\frac{3}{2}}}\sin\sqrt{A_0}\,t+\frac{A't}{2A_0}\cos\sqrt{A_0}\,t. \tag{15}$$

It follows from the non-periodic term of this equation that the fifth equation of (11) has a term in $\gamma\lambda$, and therefore that the solutions exist. To get their character we must find the terms of lowest degree in γ alone.

The coefficients of γ^2 are defined by

$$\left.\begin{aligned}
\frac{du_1^{(2,0)}}{dt}-\sigma_0 i u_1^{(2,0)}&=+\frac{m_0[\ \ ]^{(2,0)}\sqrt{-1}}{2(m_0\sigma_0-n_0\rho_0)}-\frac{\{\ \ \}^{(2,0)}}{2(m_0\rho_0+n_0\sigma_0)},\\
\frac{du_2^{(2,0)}}{dt}+\sigma_0 i u_2^{(2,0)}&=-\frac{m_0[\ \ ]^{(2,0)}\sqrt{-1}}{2(m_0\sigma_0-n_0\rho_0)}-\frac{\{\ \ \}^{(2,0)}}{2(m_0\rho_0+n_0\sigma_0)},\\
\frac{du_3^{(2,0)}}{dt}-\rho_0 u_3^{(2,0)}&=-\frac{n_0[\ \ ]^{(2,0)}}{2(m_0\sigma_0-n_0\rho_0)}+\frac{\{\ \ \}^{(2,0)}}{2(m_0\rho_0+n_0\sigma_0)},\\
\frac{du_4^{(2,0)}}{dt}+\rho_0 u_4^{(2,0)}&=+\frac{n_0[\ \ ]^{(2,0)}}{2(m_0\sigma_0-n_0\rho_0)}+\frac{\{\ \ \}^{(2,0)}}{2(m_0\rho_0+n_0\sigma_0)},\\
z''_{2,0}+A_0 z_{2,0}&=+3B_0 x_{1,0} z_{1,0},
\end{aligned}\right\} \tag{16}$$

where

$$[\ \ ]^{(2,0)}=-\tfrac{3}{2}B_0[-2x_{1,0}^2+y_{1,0}^2+z_{1,0}^2],\qquad \{\ \ \}^{(2,0)}=3B_0 x_{1,0}y_{1,0}. \tag{17}$$

We shall need only the terms $u_1^{(2,0)},\ \ldots,\ u_4^{(2,0)}$ carrying γ^2 as a factor. Hence in the first four equations we may omit $x_{1,0}$ and $y_{1,0}$. From equations (6) and (10) we have

$$\left.\begin{aligned}
z_{1,0}^2&=\frac{\gamma^2}{2A_0}-\frac{\gamma^2}{2A_0}\cos 2\sqrt{A_0}\,t,\\
x_{1,0}z_{1,0}&=+\frac{a_1\gamma}{2\sqrt{A_0}\sqrt{-1}}\left[e^{(+\sigma_0+\sqrt{A_0})\sqrt{-1}t}-e^{(+\sigma_0-\sqrt{A_0})\sqrt{-1}t}\right]\\
&\quad+\frac{a_2\gamma}{2\sqrt{A_0}\sqrt{-1}}\left[e^{(-\sigma_0+\sqrt{A_0})\sqrt{-1}t}-e^{(-\sigma_0-\sqrt{A_0})\sqrt{-1}t}\right]\\
&\quad+\frac{a_3\gamma}{2\sqrt{A_0}\sqrt{-1}}\left[e^{(+\rho_0+\sqrt{A_0}\sqrt{-1})t}-e^{(+\rho_0-\sqrt{A_0}\sqrt{-1})t}\right]\\
&\quad+\frac{a_4\gamma}{2\sqrt{A_0}\sqrt{-1}}\left[e^{(-\rho_0+\sqrt{A_0}\sqrt{-1})t}-e^{(-\rho_0-\sqrt{A_0}\sqrt{-1})t}\right].
\end{aligned}\right\} \tag{18}$$

Therefore, integrating (16) so far as the first four equations depend upon terms involving γ^2 as a factor, and determining the constants of integration so that $u_i^{(2,0)}$, $z_{2,0}$, and $z'_{2,0}$ are zero at $t=0$, we get

$$
\left.
\begin{aligned}
u_1^{(2,0)} &= a_{10}^{(2,0)} + a_{11}^{(2,0)}\, e^{+\sigma_0\sqrt{-1}t} + a_{12}^{(2,0)} \cos 2\sqrt{A_0}t + b_{12}^{(2,0)} \sin 2\sqrt{A_0}t,\\
u_2^{(2,0)} &= a_{10}^{(2,0)} + a_{11}^{(2,0)}\, e^{-\sigma_0\sqrt{-1}t} + a_{12}^{(2,0)} \cos 2\sqrt{A_0}t - b_{12}^{(2,0)} \sin 2\sqrt{A_0}t,\\
u_3^{(2,0)} &= a_{30}^{(2,0)} + a_{31}^{(2,0)}\, e^{+\rho_0 t} \quad + a_{32}^{(2,0)} \cos 2\sqrt{A_0}t + b_{32}^{(2,0)} \sin 2\sqrt{A_0}t,\\
u_4^{(2,0)} &= a_{30}^{(2,0)} + a_{31}^{(2,0)}\, e^{-\rho_0 t} \quad + a_{32}^{(2,0)} \cos 2\sqrt{A_0}t - b_{32}^{(2,0)} \sin 2\sqrt{A_0}t,\\
z_{2,0} &= c_1^{(2,0)} \cos\sqrt{A_0}t + c_2^{(2,0)} \sin\sqrt{A_0}t + 3B_0 \left\{ \frac{-\alpha_1\gamma\, e^{(+\sigma_0+\sqrt{A_0})\sqrt{-1}t}}{2(\sigma_0+2\sqrt{A_0})\sigma_0\sqrt{A_0}\sqrt{-1}} \right.\\
&\quad + \frac{\alpha_1\gamma\, e^{(+\sigma_0-\sqrt{A_0})\sqrt{-1}t}}{2(\sigma_0-2\sqrt{A_0})\sigma_0\sqrt{A_0}\sqrt{-1}} - \frac{\alpha_2\gamma\, e^{(-\sigma_0+\sqrt{A_0})\sqrt{-1}t}}{2(\sigma_0-2\sqrt{A_0})\sigma_0\sqrt{A_0}\sqrt{-1}}\\
&\quad + \frac{\alpha_2\gamma\, e^{(-\sigma_0-\sqrt{A_0})\sqrt{-1}t}}{2(\sigma_0+2\sqrt{A_0})\sigma_0\sqrt{A_0}\sqrt{-1}} + \frac{\alpha_3\gamma\, e^{(+\rho_0+\sqrt{A_0}\sqrt{-1})t}}{2(\rho_0+2\sqrt{A_0}\sqrt{-1})\rho_0\sqrt{A_0}\sqrt{-1}}\\
&\quad - \frac{\alpha_3\gamma\, e^{(+\rho_0-\sqrt{A_0}\sqrt{-1})t}}{2(\rho_0-2\sqrt{A_0}\sqrt{-1})\rho_0\sqrt{A_0}\sqrt{-1}} + \frac{\alpha_4\gamma\, e^{(-\rho_0+\sqrt{A_0}\sqrt{-1})t}}{2(\rho_0-2\sqrt{A_0}\sqrt{-1})\rho_0\sqrt{A_0}\sqrt{-1}}\\
&\quad \left. - \frac{\alpha_4\gamma\, e^{(-\rho_0-\sqrt{A_0}\sqrt{-1})t}}{2(\rho_0+2\sqrt{A_0}\sqrt{-1})\rho_0\sqrt{A_0}\sqrt{-1}} \right\},
\end{aligned}
\right\} \quad (19)
$$

where

$$
\left.
\begin{aligned}
a_{10}^{(2,0)} &= + \frac{3m_0B_0\gamma^2}{8(m_0\sigma_0-n_0\rho_0)A_0\sigma_0}, \quad a_{30}^{(2,0)} = + \frac{3n_0B_0\gamma^2}{8(m_0\sigma_0-n_0\rho_0)A_0\rho_0},\\
a_{12}^{(2,0)} &= + \frac{3m_0B_0\sigma_0\gamma^2}{8(m_0\sigma_0-n_0\rho_0)(4A_0-\sigma_0^2)A_0}, \quad a_{32}^{(2,0)} = - \frac{3n_0B_0\rho_0\gamma^2}{8(m_0\sigma_0-n_0\rho_0)(4A_0+\rho_0^2)A_0},\\
a_{11}^{(2,0)} &= - \frac{3m_0B_0\gamma^2}{2(m_0\sigma_0-n_0\rho_0)(4A_0-\sigma_0^2)\sigma_0}, \quad a_{31}^{(2,0)} = - \frac{3n_0B_0\gamma^2}{2(m_0\sigma_0-n_0\rho_0)(4A_0+\rho_0^2)\rho_0},\\
b_{12}^{(2,0)} &= - \frac{3m_0B_0\gamma^2}{4(m_0\sigma_0-n_0\rho_0)(4A_0-\sigma_0^2)\sqrt{A_0}\sqrt{-1}},\\
b_{32}^{(2,0)} &= + \frac{3n_0B_0\gamma^2}{4(m_0\sigma_0-n_0\rho_0)(4A_0+\rho_0^2)\sqrt{A_0}},\\
c_1^{(2,0)} &= + (\alpha_1-\alpha_2)\frac{6B_0\gamma}{(4A_0-\sigma_0^2)\sigma_0\sqrt{-1}} + (\alpha_3-\alpha_4)\frac{6B_0\gamma}{(4A_0+\rho_0^2)\rho_0},\\
c_2^{(2,0)} &= + (\alpha_1+\alpha_2)\frac{3B_0\gamma}{(4A_0-\sigma_0^2)\sqrt{A_0}} + (\alpha_3+\alpha_4)\frac{3B_0\gamma}{(4A_0+\rho_0^2)\sqrt{A_0}}.
\end{aligned}
\right\} \quad (20)
$$

The terms of z of the third degree in γ are defined by

$$z''_{3,0} + A_0 z_{3,0} = 3B_0 x_{2,0} z_{1,0} + \tfrac{3}{2} C_0 z_{1,0}^3. \quad (21)$$

We shall need only the non-periodic terms and those whose period is not $2\pi/\sqrt{A_0}$. The former come from terms in $\sin\sqrt{A_0}t$ and $e^{\pm\rho_0 t}$, and the latter from terms involving $e^{\pm\sigma_0\sqrt{-1}t}$, which together with $e^{\pm\rho_0 t}$ are introduced by $x_{2,0}$. Since

$$x_{2,0} = u_1^{(2,0)} + u_2^{(2,0)} + u_3^{(2,0)} + u_4^{(2,0)},$$

we find for the required terms

$$\left.\begin{aligned}
3B_0 x_{2,0} z_{1,0} &= 3B_0 \frac{\gamma}{\sqrt{A_0}}\left[2(a_{10}^{(2,0)}+a_{30}^{(2,0)})-(a_{12}^{(2,0)}+a_{32}^{(2,0)})\right]\sin\sqrt{A_0}t \\
&\quad+\frac{3B_0 a_{11}^{(2,0)}\gamma}{\sqrt{A_0}}\left(e^{\sigma_0\sqrt{-1}t}+e^{-\sigma_0\sqrt{-1}t}\right)\sin\sqrt{A_0}t \\
&\quad+\frac{3B_0 a_{31}^{(2,0)}\gamma}{\sqrt{A_0}}\left(e^{\rho_0 t}+e^{-\rho_0 t}\right)\sin\sqrt{A_0}t, \\
\frac{3}{2}C_0 z_{1,0}^3 &= \frac{9C_0\gamma^3}{8A_0^{\frac{3}{2}}}\sin\sqrt{A_0}t.
\end{aligned}\right\}\tag{22}$$

Therefore the part of the solution of (21) not having the period T_1 is

$$\left.\begin{aligned}
z_{3,0} = &-\left\{\frac{3B_0\gamma}{2A_0}\left[2(a_{10}^{(2,0)}+a_{30}^{(2,0)})-(a_{12}^{(2,0)}+a_{32}^{(2,0)})\right]-\frac{9C_0\gamma^3}{16A_0^2}\right\}t\cos\sqrt{A_0}t \\
&-\frac{3B_0 a_{11}^{(2,0)}\gamma}{(\sigma_0+2\sqrt{A_0})\sqrt{A_0}\,\sigma_0}\sin(\sigma_0+\sqrt{A_0})t+\frac{3B_0 a_{11}^{(2,0)}\gamma}{(\sigma_0-2\sqrt{A_0})\sqrt{A_0}\,\sigma_0}\sin(\sigma_0-\sqrt{A_0})t \\
&+\frac{3B_0 a_{31}^{(2,0)}\gamma}{(\rho_0-2\sqrt{A_0}\sqrt{-1})\sqrt{A_0}\,\rho_0}\sin(\rho_0\sqrt{-1}+\sqrt{A_0})t \\
&-\frac{3B_0 a_{31}^{(2,0)}\gamma}{(\rho_0+2\sqrt{A_0}\sqrt{-1})\sqrt{A_0}\,\rho_0}\sin(\rho_0\sqrt{-1}-\sqrt{A_0})t.
\end{aligned}\right\}\tag{23}$$

We can now write the conditions for the existence of periodic solutions. Upon using the results just obtained, we find for the values of (11)

$$\left.\begin{aligned}
0 &= \alpha_1\left[e^{+\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right]-\frac{3m_0B_0\gamma^2}{2(m_0\sigma_0-n_0\rho_0)(4A_0-\sigma_0^2)\sigma_0}\left[e^{+\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right]+\cdots, \\
0 &= \alpha_2\left[e^{-\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right]-\frac{3m_0B_0\gamma^2}{2(m_0\sigma_0-n_0\rho_0)(4A_0-\sigma_0^2)\sigma_0}\left[e^{-\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right]+\cdots, \\
0 &= \alpha_3\left[e^{+\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right]-\frac{3n_0B_0\gamma^2}{2(m_0\sigma_0-n_0\rho_0)(4A_0+\rho_0^2)\rho_0}\left[e^{+\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right]+\cdots, \\
0 &= \alpha_4\left[e^{-\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right]-\frac{3n_0B_0\gamma^2}{2(m_0\sigma_0-n_0\rho_0)(4A_0+\rho_0^2)\rho_0}\left[e^{-\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right]+\cdots, \\
0 &= +(-1)^\nu\frac{\nu\pi A'\gamma\lambda}{2A_0^{\frac{3}{2}}}-\frac{6B_0\alpha_1\gamma}{(4A_0-\sigma_0^2)\sigma_0\sqrt{-1}}\left[(-1)^\nu e^{+\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right] \\
&\quad+[(-1)^\nu-1]c_1^{(2,0)}+\frac{6B_0\alpha_2\gamma}{(4A_0-\sigma_0^2)\sigma_0\sqrt{-1}}\left[(-1)^\nu e^{-\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right] \\
&\quad-\frac{6B_0\alpha_3\gamma}{(4A_0+\rho_0^2)\rho_0}\left[(-1)^\nu e^{\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right]+\frac{6B_0\alpha_4\gamma}{(4A_0+\rho_0^2)\rho_0}\left[(-1)^\nu e^{-\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right] \\
&\quad-(-1)^\nu\frac{\nu\pi}{\sqrt{A_0}}\left\{\frac{3B_0\gamma}{2A_0}\left[2(a_{10}^{(2,0)}+a_{30}^{(2,0)})-(a_{12}^{(2,0)}+a_{32}^{(2,0)})\right]-\frac{9C_0\gamma^3}{16A_0^2}\right\} \\
&\quad-\frac{6B_0a_{11}^{(2,0)}\gamma}{(4A_0-\sigma_0^2)\sigma_0\sqrt{-1}}\left[(-1)^\nu e^{\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right]+\frac{6B_0a_{11}^{(2,0)}\gamma}{(4A_0-\sigma_0^2)\sigma_0\sqrt{-1}} \\
&\quad\times\left[(-1)^\nu e^{-\frac{\nu\pi\sigma_0\sqrt{-1}}{\sqrt{A_0}}}-1\right]-\frac{6B_0a_{31}^{(2,0)}\gamma}{(4A_0+\rho_0^2)\rho_0}\left[(-1)^\nu e^{\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right] \\
&\quad+\frac{6B_0a_{31}^{(2,0)}\gamma}{(4A_0+\rho_0^2)\rho_0}\left[(-1)^\nu e^{-\frac{\nu\pi\rho_0}{\sqrt{A_0}}}-1\right].
\end{aligned}\right\}\tag{24}$$

Upon solving the first four equations of (24) for $\alpha_1, \ldots, \alpha_4$, substituting the result in the last, and reducing by (20), we get

$$0=\frac{\nu\pi A'\gamma\lambda}{2A_0^{\frac{1}{2}}}-\frac{9\nu\pi B_0^2(m_0\rho_0+n_0\sigma_0)\gamma^3}{8(m_0\sigma_0-n_0\rho_0)\sigma_0\rho_0 A_0^{\frac{5}{2}}}+\frac{9\nu\pi B_0^2\gamma^3}{4(4A_0-\sigma_0^2)(4A_0+\rho_0^2)A_0^{\frac{3}{2}}}$$
$$+\frac{9\nu\pi B_0^2(m_0\rho_0+n_0\sigma_0)\sigma_0\rho_0\gamma^3}{16(m_0\sigma_0-n_0\rho_0)(4A_0-\sigma_0^2)(4A_0+\rho_0^2)A_0^{\frac{5}{2}}}-\frac{9\nu\pi C_0\gamma^3}{16A_0^{\frac{3}{2}}}+\cdots.$$

From the characteristic equation and equations (5), we find

$$\left.\begin{aligned} m_0\sigma_0-n_0\rho_0&=-(1+2A_0)\frac{(\sigma_0^2+\rho_0^2)}{2\sigma_0\rho_0}, & m_0\rho_0+n_0\sigma_0&=\frac{\sigma_0^2+\rho_0^2}{2},\\ -\sigma_0^2\rho_0^2&=(1-A_0)(1+2A_0), & -\sigma_0^2+\rho_0^2&=-(2-A_0).\end{aligned}\right\}\tag{25}$$

Hence, after dividing by γ, we have

$$0=A_0A'\lambda+\frac{9}{8}\left[\frac{3B_0^2(1-3A_0+14A_0^2)}{(1+2A_0)(1-7A_0+18A_0^2)}-C_0\right]\gamma^2+\cdots,\tag{26}$$

which can be solved for γ in terms of λ in the form

$$\gamma=\lambda^{\frac{1}{2}}P(\lambda^{\frac{1}{2}}).\tag{27}$$

Upon substituting this result in the series for $\alpha_1, \ldots, \alpha_4$ when they are expressed in terms of γ and λ from (24), we have

$$\alpha_i=\lambda P_i(\lambda^{\frac{1}{2}})\qquad (i=1,\ldots,4).\tag{28}$$

After (27) and (28) are substituted in (10) the coördinates u_i and z become power series in $\lambda^{\frac{1}{2}}$, vanishing with $\lambda^{\frac{1}{2}}$, and they are periodic, since the conditions for periodicity have been satisfied. The series converge for $|\lambda|$ sufficiently small. The radius of convergence depends on μ and μ_0, and it is easy to see from the explicit forms of the equations that it remains finite as μ_0 approaches μ. For $\lambda=\mu-\mu_0$ the orbits belong to the physical problem, and μ_0 can be taken so near μ that the series converge. That is, periodic solutions exist having the form

$$x=\sum_{i=1}^{\infty}x_i\lambda^{\frac{i}{2}},\qquad y=\sum_{i=1}^{\infty}y_i\lambda^{\frac{i}{2}},\qquad z=\sum_{i=1}^{\infty}z_i\lambda^{\frac{i}{2}},\tag{29}$$

where the x_i, y_i, and z_i separately are periodic functions of t having the period $2\pi/\sqrt{A_0}$.

The last two equations of (11) are satisfied by $\gamma=0$. For $T=2\pi/\sqrt{A_0}$ the determinant of the linear terms of the first four equations is distinct from zero; therefore their only solution for $\alpha_1, \ldots, \alpha_4$ as power series in λ, vanishing with λ, is $\alpha_1= \cdots =\alpha_4=0$. But then $u_1, \ldots, u_4$, and z are identically zero. That is, the solutions having the period $2\pi/\sqrt{A_0}$ are in three dimensions and not in two alone. In this respect they agree with the solutions of Class A of Chapter V. It will now be shown that for

$$\lambda=\mu-\mu_0$$

these solutions are those of Class A.

The solutions of Class A were developed as power series in ϵ' of the form

$$x=\sum_{i=1}^{\infty} x_i\epsilon'^i, \qquad y=\sum_{i=1}^{\infty} y_i\epsilon'^i, \qquad z=\sum_{i=1}^{\infty} z_i\epsilon'^i, \tag{30}$$

where x_i, y_i, and z_i are periodic in $\tau=t/(1+\delta)$ with the period $2\pi/\sqrt{A}$. The A_0 appearing in the x_i, y_i, z_i of (29) is different from the A of (30); in the former μ has been replaced by μ_0 in certain places, while in the latter it remains μ. Now let $\mu=\mu_0+\lambda$ in (30) in those places where this transformation was made in the development of (29), and develop the right members as power series in λ. The period of the solutions (30) when expressed in t is $2\pi(1+\delta)/\sqrt{A}$, where δ is a known power series in ϵ'. Make the transformation on μ in this expression so that $A=A_0+p(\lambda)$ and set it equal to the period of (29), viz. $2\pi/\sqrt{A_0}$. Since δ starts with a term of the second degree in ϵ' this equation determines ϵ' as a power series in $\lambda^{\frac{1}{2}}$. Substituting this expression in (30), we have these equations expressed as power series in $\lambda^{\frac{1}{2}}$. For sufficiently small $|\lambda|$ these series converge, the coefficients of each power of $\lambda^{\frac{1}{2}}$ separately are periodic with the period $2\pi/\sqrt{A_0}$, they identically satisfy the correspondingly transformed differential equations, and they are identically equal (in t) to (30) in their original form. Having the same form as (29), it follows from the uniqueness of these solutions that they are identical with them. That is, equations (29) and (30) are two different sets of expressions for the coördinates of the orbits of Class A.

Now return to the consideration of equations (11). Let the last two be satisfied by $\gamma=0$. As we have seen, there is no solution of the first four vanishing with $\lambda=0$ except $\alpha_1= \cdots \alpha_4=0$ if $T=T_1=2\pi/\sqrt{A_0}$. Therefore we take $T=T_2=2\pi/\sigma_0$. The first equation is redundant because of the existence of the integral (8), and will be suppressed. The third and fourth equations can be solved for α_3 and α_4 as power series in α_1, α_2, and λ, vanishing with α_1 and α_2. When these results are substituted in the second equation, we have

$$P(\alpha_1, \alpha_2, \lambda)=0, \tag{31}$$

where P identically vanishes with α_1 and α_2.

We have one equation for the determination of the parameters α_1 and α_2. Consequently we may impose one condition upon them. It will be convenient to take α_1 and α_2 so that

$$x'=0 \text{ at } t=0, \tag{32}$$

a condition which is satisfied in every closed orbit for which the right members of (2) converge. From equations (6) it follows that this condition is

$$\sigma_0\sqrt{-1}\,(\alpha_1-\alpha_2)+\rho_0(\alpha_3-\alpha_4)=0. \tag{33}$$

We shall regard this equation as a relation between α_1 and α_2 which is to be used in connection with (31) for the determination of these parameters.

In order to complete the proof of the existence of the solutions it is necessary to discuss the second and third degree terms of (31). It follows from (7) and (3) that the terms of the second degree are determined by

$$\left.\begin{aligned}
\frac{du_1^{(2)}}{dt}-\sigma_0 i u_1^{(2)}&=-m_0[\ \ ]^{(2)}\,i-\{\ \ \}^{(2)},\\
\frac{du_2^{(2)}}{dt}+\sigma_0 i u_2^{(2)}&=+m_0[\ \ ]^{(2)}\,i-\{\ \ \}^{(2)},\\
\frac{du_3^{(2)}}{dt}-\ \rho_0 u_3^{(2)}&=-\ n_0[\ \ ]^{(2)}\ +\{\ \ \}^{(2)},\\
\frac{du_4^{(2)}}{dt}+\ \rho_0 u_4^{(2)}&=+\ n_0[\ \ ]^{(2)}\ +\{\ \ \}^{(2)},
\end{aligned}\right\} \tag{34}$$

where

$$[\ \ ]^{(2)}=\frac{2A'\lambda x_1+\frac{3}{2}B_0[-2x_1^2+y_1^2]}{2(m_0\sigma_0-n_0\rho_0)},\qquad \{\ \ \}^{(2)}=-\frac{A'\lambda y_1+3B_0x_1y_1}{2(m_0\rho_0+n_0\sigma_0)}. \tag{35}$$

It follows from the forms of the right members of these four equations that their solutions will contain Poisson terms,* whose coefficients involve $\lambda\alpha_1, \ldots, \lambda\alpha_4$ respectively as factors. The coefficients of all the other terms are of the second degree in $\alpha_1, \ldots, \alpha_4$ and linear in B_0, and those which are not periodic involve α_3 or α_4 at least to the first degree. Consequently, when we solve the third and fourth equations of (11) for α_3 and α_4, the results will start with terms of the second degree in α_1, α_2, and λ. When these results are substituted in the second equation of (11), it will contain a term in $\lambda\alpha_2$ and terms of the third degree as the lowest in α_1 and α_2 alone. If we now eliminate α_1 by means of (33), we have an equation whose terms of lowest degree are $\alpha_2\lambda$ and α_2^3. We shall verify first that the coefficient of $\alpha_2\lambda$ is not zero.

*In Celestial Mechanics terms which are of the form of t multiplied by cosine or sine terms are called Poisson terms, from the results in Poisson's theorem on the invariability of the major axes of the planetary orbits.

It follows from (34) and (35) that the Poisson terms in $u_2^{(2)}$ are

$$u_2^{(2)} = -\frac{m_0 A'\lambda a_2 t e^{-\sigma_0\sqrt{-1}t}}{(m_0\sigma_0 - n_0\rho_0)\sqrt{-1}} - \frac{n_0\sqrt{-1}\,A'\lambda a_2\, t e^{-\sigma_0\sqrt{-1}t}}{2(m_0\rho_0 + n_0\sigma_0)}.$$

Hence, so far as these terms are concerned, we find

$$u_2^{(2)}\left(\frac{2\pi}{\sigma_0}\right) - u_2^{(2)}(0) = -\frac{\pi A'\lambda a_2}{\sigma_0\sqrt{-1}}\left\{\frac{2m_0}{m_0\sigma_0 - n_0\rho_0} - \frac{n_0}{m_0\rho_0 + n_0\sigma_0}\right\},$$

which, by equations $(25)_2$ reduces to

$$u_2^{(2)}\left(\frac{2\pi}{\sigma_0}\right) - u_2^{(2)}(0) = \frac{4\pi A'[2m_0\sigma_0\rho_0 + n_0(1+2A_0)]}{(1+2A_0)(\sigma_0^2+\rho_0^2)\sigma_0\sqrt{-1}}\lambda a_2. \tag{36}$$

Therefore the coefficient of λa_2 in (31) is not zero unless $A'=0$. But $A'\neq 0$ except for the center of libration (b) when the finite masses are equal, and in this case a different generalization of the parameter μ can be made to keep it distinct from zero. Consequently, since all the equations are identically satisfied by $a_1 = \cdot\cdot\cdot = a_4 = 0$, the equation obtained by eliminating a_1 between (31) and (33) is divisible by a_2, after which there is a term in λ alone whose coefficient is distinct from zero. Therefore the equation can be solved for a_2 in terms of λ, vanishing with λ, and the periodic solutions exist.

The form of the solution depends upon the degree of the term of the lowest degree in a_2 alone in the final equation after a_1, a_3, and a_4 are eliminated. It is easy to show that the coefficient of a_2^3 in this equation is not identically zero. It has been shown that the terms arising from the solutions of (34) involve B_0 linearly. There are also terms of the third degree in $a_1, \ldots, a_4$ arising from the terms of the third order. The terms of u_2 of the third order are defined by

$$\frac{du_2^{(3)}}{dt} + \sigma_0\sqrt{-1}\,u_2^{(3)} = -\frac{m_0[\quad]^{(3)}\sqrt{-1}}{2(m_0\sigma_0 - n_0\rho_0)} - \frac{\{\quad\}^{(3)}}{2(m_0\rho_0 + n_0\sigma_0)}, \tag{37}$$

where

$$[\quad]^{(3)} = 3B_0[-2x_1x_2 + y_1y_2] + 2C_0[2x_1^3 - 3x_1y_1^2],$$

$$\{\quad\}^{(3)} = 4B_0\{x_1y_2 + y_1x_2\} + \tfrac{3}{2}C_0\{-4x_1^2y_1 + y_1^3\}.$$

The Poisson terms in the solution of (37) involving C_0 as a factor are

$$u_2^{(3)} = -\frac{3C_0m_0(2-n_0^2)\,a_1a_2^2 t e^{-\sigma_0\sqrt{-1}t}}{(m_0\sigma_0 - n_0\rho_0)\sqrt{-1}} + \frac{3C_0n_0(4-3n_0^2)\,a_1a_2^2 t e^{-\sigma_0\sqrt{-1}t}}{4(m_0\rho_0 + n_0\sigma_0)\sqrt{-1}}.$$

Hence we find for these terms, after some reductions,

$$u_2^{(3)}\left(\frac{2\pi}{\sigma_0}\right) - u_2^{(3)}(0) = \frac{3\pi C_0}{(\sigma_0^2+\rho_0^2)\sigma_0\sqrt{-1}}\left\{\frac{4(2-n_0^2)\,m_0\sigma_0\rho_0}{1+2A_0} + (4-3n_0^2)n_0\right\}a_1a_2^2. \tag{38}$$

It follows from equations (5) that $\{\ \}$ of (38) does not identically vanish. Hence the coefficient of $\alpha_1\alpha_2^2$ in the expression for $u_2(2\pi/\sigma_3)-u_2(0)$ consists of terms multiplied by B_0 plus *non-vanishing* terms multiplied by C_0.

Now suppose α_1, α_3, and α_4 are eliminated from the second equation of (11) by means of (33) and the third and fourth equations of (11). It follows from the properties of these equations that in the result the term independent of λ and of lowest degree is of the third degree in α_2. Its coefficient consists of two parts, one of which is (38) and contains C_0 as a factor, while the other contains B_0 as a factor. If the coefficient of B_0 is identically zero, the coefficient of α_2^3 is distinct from zero, because, as we have seen, the part involving C_0 is distinct from zero. Even if the coefficient of B_0 is not zero, it is easy to show that the sum of the two parts of the coefficient of α_2^3 can not be identically zero for each of the three libration points (a), (b), and (c).

Consider the points (a) and (b). The quantities A_0, σ_0, ρ_0, m_0, n_0, and C_0 are the same function of μ_0 for both points, but B_0 is different because of the change of sign in its second term [eq. (3)]. Consequently, the sum of the terms in B_0 and C_0 can not be identically zero in μ_0 for both the points of libration (a) and (b). Hence in this case the second, third, and fourth equations of (11) and (33) are solvable for $\alpha_1, \ldots, \alpha_4$ as power series in $\lambda^{\frac{1}{2}}$, vanishing with λ. Therefore the periodic solutions with the period $2\pi/\sigma_0$ are expansible as power series in $\lambda^{\frac{1}{2}}$.

In a manner similar to that used to prove that (29) are series which represent orbits of Class A, it can be shown that the orbits now under consideration belong to Class B.

101. Direct Construction of the Solutions for Class A.—As in the method of Chapter V, the coördinates in the orbits of Class A are most conveniently obtained from the x, y, and z-equations. Consequently we start from equations (2) and (3). Since the solution is periodic for all $|\lambda^{\frac{1}{2}}|$ sufficiently small, each term of the expansion separately is periodic with the period 2π; and since $z=0$ at $t=0$, each term in the expansion of z separately vanishes at $t=0$.

The coefficients of $\lambda^{\frac{1}{2}}$ are defined by

$$x_1''-2y_1'-(1+2A_0)x_1=0,\quad y_1''+2x_1'-(1-A_0)y_1=0,\quad z_1''+A_0z_1=0. \tag{39}$$

The solutions of these equations satisfying the periodicity and initial conditions are

$$x_1=y_1=0,\qquad z_1=\frac{c_1}{\sqrt{A_0}}\sin\sqrt{A_0}\,t, \tag{40}$$

where c_1 is so far undetermined.

The coefficients of λ are defined by

$$\left.\begin{aligned} x_2''-2\,y_2'-(1+2A_0)\,x_2&=\tfrac{3}{2}\,B_0\,[-2\,x_1^2+y_1^2+z_1^2],\\ y_2''+2\,x_2'-(1-\ \ A_0)\,y_2&=3\,B_0\,x_1\,y_1,\\ z_2''+A_0z_2&=3\,B_0\,x_1\,z_1. \end{aligned}\right\}\quad(41)$$

Upon making use of (40), integrating, and applying the periodicity and initial conditions, we have

$$\left.\begin{aligned} x_2&=-\frac{3B_0\,c_1^2}{4(1+2A_0)\,A_0}+\frac{3B_0(1+3A_0)\,c_1^2}{4(1-7\,A_0+18A_0^2)\,A_0}\cos 2\sqrt{A_0}\,t,\\ y_2&=-\frac{3B_0\,c_1^2}{(1-7A_0+18A_0^2)\,\sqrt{A_0}}\sin 2\sqrt{A_0}\,t,\\ z_2&=+\frac{c_2}{\sqrt{A_0}}\sin\sqrt{A_0}\,t, \end{aligned}\right\}\quad(42)$$

where c_2 is so far arbitrary.

It will be necessary to carry the computation two steps further in order to show how the general term is found. The coefficients of λ^3 are defined by

$$\left.\begin{aligned} x_3''-2y_3'-(1+2\,A_0)x_3&=3B_0z_1z_2=\frac{3B_0c_1c_2}{2A_0}[1-\cos 2\sqrt{A_0}\,t],\\ y_3''+2x_3'-(1-\ \ A_0)y_3&=0,\\ z_3''+A_0\,z_3&=-A'z_1+3B_0x_2z_1+\tfrac{3}{2}\,C_0z_1^3\\ &=-\frac{A'\,c_1}{\sqrt{A_0}}\sin\sqrt{A_0}\,t+\frac{3}{8}\,\frac{C_0c_1^3}{A_0^{\frac{3}{2}}}\Big[3\sin\sqrt{A_0}\,t-\sin 3\sqrt{A_0}\,t\Big]\\ &\quad-\frac{27B_0^2(1-3A_0+14A_0^2)\,c_1^3}{8(1+2A_0)(1-7A_0+18A_0^2)\,A_0^{\frac{3}{2}}}\sin\sqrt{A_0}t\\ &\quad+\frac{9B_0^2(1+3A_0)\,c_1^3}{8(1-7A_0+A_0^2)\,18A_0^{\frac{3}{2}}}\sin 3\sqrt{A_0}t. \end{aligned}\right\}\quad(43)$$

Consider first the solution of the third equation. In order that it shall be periodic the coefficient of $\sin\sqrt{A_0}t$ must be zero, or

$$-\frac{A'}{\sqrt{A_0}}\,c_1+\frac{9}{8}\,\frac{C_0\,c_1^3}{A_0^{\frac{3}{2}}}-\frac{27B_0^2(1-3A_0+14A_0^2)\,c_1^3}{8\,(1+2A_0)\,(1-7\,A_0+18A_0^2)\,A_0^{\frac{3}{2}}}=0.\qquad(44)$$

This equation, which is identical with (26) of the existence proof, has the solutions

$$c_1=0,\qquad c_1=\pm\frac{2\,\sqrt{2A_0\,A'}}{3\sqrt{C_0-\dfrac{3B_0^2(1-3A_0+14A_0^2)}{(1+2A_0)(1-7A_0+18A_0^2)}}}.\qquad(45)$$

The solution $c_1=0$ leads to the trivial case $x\equiv y\equiv z\equiv 0$, as can be shown easily by an induction to the general term. The double sign before the radical plays the same rôle as the double sign before $\lambda^{\frac{1}{2}}$ in the existence. If it is used in one place in the final solution it is superfluous in the other.

With the value of c_1 determined from the second of (45), the solution of (43) satisfying the periodicity and initial conditions is

$$\left.\begin{aligned}
x_3 &= -\frac{3B_0c_1c_2}{2(1+2A_0)A_0}+\frac{3B_0(1+3A_0)c_1c_2}{2(1-7A_0+18A_0^2)A_0}\cos 2\sqrt{A_0}\,t,\\
y_3 &= -\frac{6B_0c_1c_2}{(1-7A_0+18A_0^2)\sqrt{A_0}}\sin 2\sqrt{A_0}\,t,\\
z_3 &= \frac{c_3}{\sqrt{A_0}}\sin\sqrt{A_0}\,t-\frac{9B_0^2(1+3A_0)c_1^3}{64(1-7A_0+18A_0^2)A_0^{\frac{5}{2}}}\sin 3\sqrt{A_0}\,t\\
&\qquad+\frac{3C_0c_1^3}{64A_0^{\frac{5}{2}}}\sin 3\sqrt{A_0}\,t,
\end{aligned}\right\}\tag{46}$$

where c_3 is so far undetermined.

The equation for the determination of z_4 is

$$z_4''+A_0z_4=-A'z_2+3B_0(x_2z_2+x_3z_1)+\tfrac{9}{2}C_0z_1^2z_2.\tag{47}$$

In order that the solution of this equation shall be periodic the coefficient of $\sin\sqrt{A_0}\,t$ in its right member must equal zero. This function of t arises from every term in the right member of (47), and it follows from (40), (42), and (46) that its coefficient carries c_2 linearly and homogeneously. Therefore this condition determines c_2 uniquely by the equation $c_2=0$, whence

$$z_2\equiv x_3\equiv y_3\equiv 0.\tag{48}$$

After the sign of c_1 has been chosen all the other c_i are determined uniquely by the conditions that all the z_i separately shall be periodic. For, suppose that $x_1, \ldots, x_{i-1};\ y_1, \ldots, y_{i-1};\ z_1, \ldots, z_{i-1}$ have been computed and that their coefficients are entirely known except the arbitrary terms $c_{i-2}/\sqrt{A_0}\ \sin\sqrt{A_0}\,t$ in z_{i-2} and $c_{i-1}/\sqrt{A_0}\ \sin\sqrt{A_0}$ in z_{i-1}, and the arbitrary constant c_{i-2}, which enters linearly in x_{i-1} and y_{i-1}. The z_i is defined by

$$z_i''+A_0z_i=-A'z_{i-2}+3B_0(x_2z_{i-2}+x_{i-1}z_1)+\tfrac{9}{2}C_0z_1^2z_{i-2}+\cdots,\tag{49}$$

where the terms not written are completely known. The arbitrary c_{i-2} enters linearly in the coefficient of $\sin\sqrt{A_0}\,t$ in the right member of this equation, which does not involve c_{i-1}, and the constant c_{i-2} is uniquely determined by the condition that this coefficient shall vanish.

It can be shown without difficulty that the solutions have the following properties:

1. The x_{2j+1}, y_{2j+1}, z_{2j} are identically zero $(j=1, 2, \ldots \infty)$.
2. The x_j, y_j, z_j involve c_1 homogeneously to the degree j.
3. The x_{2j} are sums of cosines of even multiples of $\sqrt{A_0}t$, the highest multiple being $2j$.
4. The y_{2j} are sums of sines of even multiples of $\sqrt{A_0}t$, the highest multiple being $2j$.
5. The z_{2j+1} are sums of sines of odd multiples of $\sqrt{A_0}t$, the highest multiple being $2j+1$.
6. Changing the sine of c_1 is equivalent to changing the sign of $\lambda^{\frac{1}{2}}$, which is equivalent to increasing t by $\pi/\sqrt{A_0}$. Therefore the two values of c_1 (or $\lambda^{\frac{1}{2}}$) belong to the same physical orbit, the origin of time being different by half a period in the two cases.
7. The orbits are symmetrical with respect to the x-axis, the xy-plane, and the xz-plane.

It is not necessary to go into the proofs of these properties, which are the same, so far as the comparison can be made, as those found in §87.

102. Direct Construction of the Solutions for Class B.—For these orbits it is advantageous to use the first four equations of (7), the last one being identically zero. We have proved that the solutions are expansible as power series in $\lambda^{\frac{1}{2}}$, that the coefficients of each power of $\lambda^{\frac{1}{2}}$ are periodic with the period $2\pi/\sigma_0$, and that $x'=0$ at $t=0$ for all λ.

The coefficients of $\lambda^{\frac{1}{2}}$ are defined by the differential equations

$$\left.\begin{aligned} &\frac{du_1^{(1)}}{dt}-\sigma_0\sqrt{-1}\,u_1^{(1)}=0, &&\frac{du_3^{(1)}}{dt}-\rho_0 u_3^{(1)}=0,\\ &\frac{du_2^{(1)}}{dt}+\sigma_0\sqrt{-1}\,u_2^{(1)}=0, &&\frac{du_4^{(1)}}{dt}+\rho_0 u_4^{(1)}=0. \end{aligned}\right\} \qquad (50)$$

The periodic solutions of these equations are seen to be

$$u_1^{(1)}=a_1\, e^{\sigma_0\sqrt{-1}\,t}, \qquad u_2^{(1)}=a_2\, e^{-\sigma_0\sqrt{-1}\,t}, \qquad u_3^{(1)}=u_4^{(1)}=0. \qquad (51)$$

From equations (6) and the initial value of x' it is found that

$$a_1=a_2=\frac{1}{2}\,a^{(1)}, \qquad x_1=a^{(1)}\cos\sigma_0 t, \qquad y_1=-n_0\, a^{(1)}\sin\sigma_0 t, \qquad (52)$$

where the coefficient $a^{(1)}$ is so far undetermined.

The coefficients of λ are defined by

$$\left.\begin{aligned}
&\frac{du_1^{(2)}}{dt}-\sigma_0 i u_1^{(2)}=+\frac{m_0[\quad]^{(2)}}{(m_0\sigma_0-n_0\rho_0)\sqrt{-1}}-\frac{\{\quad\}^{(2)}}{(m_0\rho_0+n_0\sigma_0)},\\
&\frac{du_2^{(2)}}{dt}+\sigma_0 i u_2^{(2)}=-\frac{m_0[\quad]^{(2)}}{(m_0\sigma_0-n_0\rho_0)\sqrt{-1}}-\frac{\{\quad\}^{(2)}}{(m_0\rho_0+n_0\sigma_0)},\\
&\frac{du_3^{(2)}}{dt}-\rho_0 u_3^{(2)}=-\frac{n_0[\quad]^{(2)}}{(m_0\sigma_0-n_0\rho_0)}+\frac{\{\quad\}^{(2)}}{(m_0\rho_0+n_0\sigma_0)},\\
&\frac{du_4^{(2)}}{dt}+\rho_0 u_4^{(2)}=+\frac{n_0[\quad]^{(2)}}{(m_0\sigma_0-n_0\rho_0)}+\frac{\{\quad\}}{(m_0\rho_0+n_0\sigma_0)},
\end{aligned}\right\}\quad(53)$$

where

$$\left.\begin{aligned}
&[\quad]^{(2)}=\tfrac{3}{4}B_0[-2x_1^2+y_1^2]=+\tfrac{3}{8}B_0(a^{(1)})^2[-(2-n_0^2)-(2+n_0^2)\cos 2\sigma_0 t],\\
&\{\quad\}^{(2)}=\tfrac{3}{2}B_0 x_1 y_1\qquad\qquad=-\tfrac{3}{4}B_0(a^{(1)})^2 n_0\sin 2\sigma_0 t.
\end{aligned}\right\}\quad(54)$$

The periodic solutions of these equations satisfying $x'=0$ are

$$\left.\begin{aligned}
u_1^{(2)}&=\tfrac{1}{2}a^{(2)}e^{+\sigma_0 it}+a_{10}^{(2)}+a_{12}^{(2)}\cos 2\sigma_0 t-ib_{12}^{(2)}\sin 2\sigma_0 t,\\
u_2^{(2)}&=\tfrac{1}{2}a^{(2)}e^{-\sigma_0 it}+a_{10}^{(2)}+a_{12}^{(2)}\cos 2\sigma_0 t+ib_{12}^{(2)}\sin 2\sigma_0 t,\\
u_3^{(2)}&=\qquad\qquad+a_{30}^{(2)}+a_{32}^{(2)}\cos 2\sigma_0 t+b_{32}^{(2)}\sin 2\sigma_0 t,\\
u_4^{(2)}&=\qquad\qquad+a_{30}^{(2)}+a_{32}^{(2)}\cos 2\sigma_0 t-b_{32}^{(2)}\sin 2\sigma_0 t,
\end{aligned}\right\}\quad(55)$$

where

$$\left.\begin{aligned}
&a^{(2)}\text{ is so far undetermined,}\\
&a_{10}^{(2)}=-\frac{3m_0B_0(2-n_0^2)(a^{(1)})^2}{8(m_0\sigma_0-n_0\rho_0)\sigma_0},\\
&a_{30}^{(2)}=-\frac{3n_0B_0(2-n_0^2)(a^{(1)})^2}{8(m_0\sigma_0-n_0\rho_0)\rho_0},\\
&a_{12}^{(2)}=+\frac{m_0B_0(2+n_0^2)(a^{(1)})^2}{8(m_0\sigma_0-n_0\rho_0)\sigma_0}-\frac{n_0B_0(a^{(1)})^2}{2(m_0\rho_0+n_0\sigma_0)\sigma_0},\\
&b_{12}^{(2)}=-\frac{m_0B_0(2+n_0^2)(a^{(1)})^2}{4(m_0\sigma_0-n_0\rho_0)\sigma_0}+\frac{n_0B_0(a^{(1)})^2}{4(m_0\rho_0+n_0\sigma_0)\sigma_0},\\
&a_{32}^{(2)}=-\frac{3n_0B_0(2+n_0^2)\rho_0(a^{(1)})^2}{8(m_0\sigma_0-n_0\rho_0)(4\sigma_0^2+\rho_0^2)}+\frac{3n_0B_0\sigma_0(a^{(1)})^2}{2(m_0\rho_0+n_0\sigma_0)(4\sigma_0^2+\rho_0^2)},\\
&b_{32}^{(2)}=+\frac{3n_0B_0(2+n_0^2)\sigma_0(a^{(1)})^2}{4(m_0\sigma_0-n_0\rho_0)(4\sigma_0^2+\rho_0^2)}+\frac{3n_0B_0\rho_0(a^{(1)})^2}{4(m_0\rho_0+n_0\sigma_0)(4\sigma_0^2+\rho_0^2)}.
\end{aligned}\right\}\quad(56)$$

The arbitrary $a^{(1)}$ is determined, except as to sign, by the periodicity condition in the next step of the integration; and the double sign is equivalent to the double sign on $\lambda^{\frac{1}{2}}$. After $a^{(1)}$ has been determined, an $a^{(i)}$ is uniquely determined by the periodicity condition at each succeeding step of the integration.

The coefficients of $\lambda^{\frac{3}{2}}$ are defined by

$$\left.\begin{aligned}
\frac{du_1^{(3)}}{dt} - \sigma_0 i u_1^{(3)} &= + \frac{m_0[\quad]^{(3)}}{(m_0\sigma_0 - n_0\rho_0)\sqrt{-1}} - \frac{\{\quad\}^{(3)}}{(m_0\rho_0 + n_0\sigma_0)},\\
\frac{du_2^{(3)}}{dt} + \sigma_0 i u_2^{(3)} &= - \frac{m_0[\quad]^{(3)}}{(m_0\sigma_0 - n_0\rho_0)\sqrt{-1}} - \frac{\{\quad\}^{(3)}}{(m_0\rho_0 + n_0\sigma_0)},\\
\frac{du_3^{(3)}}{dt} - \rho_0 u_3^{(3)} &= - \frac{n_0[\quad]^{(3)}}{(m_0\sigma_0 - n_0\rho_0)} + \frac{\{\quad\}^{(3)}}{(m_0\rho_0 + n_0\sigma_0)},\\
\frac{du_4^{(3)}}{dt} + \rho_0 u_4^{(3)} &= + \frac{n_0[\quad]^{(3)}}{(m_0\sigma_0 - n_0\rho_0)} + \frac{\{\quad\}^{(3)}}{(m_0\rho_0 + n_0\sigma_0)},
\end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned}
[\quad]^{(3)} &= +\ A' x_1 + \tfrac{3}{2} B_0 [-2x_1x_2 + y_1y_2] + \ C_0 [+2x_1^3 - 3x_1y_1^2],\\
\{\quad\}^{(3)} &= -\tfrac{1}{2} A' y_1 + \tfrac{3}{2} B_0 [+x_1y_2 + x_2y_1] + \tfrac{3}{4} C_0 \{-4x_1^2y_1 + y_1^3\}.
\end{aligned}\right\} \tag{58}$$

In order that the solution of the first equation shall be periodic it is necessary that in its right member the coefficient of $e^{\sigma_0\sqrt{-1}t}$ be equal to zero. That part of this coefficient which arises from $A'x_1$ involves $a^{(1)}$ linearly and homogeneously. Those parts which arise from $x_1 y_2$, x_1^3 etc. carry $(a^{(1)})^3$ as a factor and involve it in no other way. Consequently, the condition that the coefficient of $e^{\sigma_0\sqrt{-1}t}$ shall vanish is satisfied by $a^{(1)}=0$, or by an equation of the form

$$P\,(a^{(1)})^2 + Q = 0, \tag{59}$$

where P and Q are known constants. It is easily shown that they are identical with the coefficients of α_2^3 and $\alpha\lambda$ which arise, in §100, in the demonstration of the existence of the solutions. The first determination of a_1 leads to the trivial solution $x \equiv y \equiv 0$; equation (59) gives the double determination for $a^{(1)}$ mentioned on page 210.

In order that the solution of the second equation shall be periodic it is necessary that in its right member the coefficient of $e^{-\sigma_0\sqrt{-1}t}$ be equal to zero. It is easy to see that this condition determines $a^{(1)}$ by an equation which is identical with (59). That is, the same value of $a^{(1)}$ makes the solutions of both the first and the second equations periodic.

The particular integrals of the third and fourth equations are periodic. In solving the third and fourth equations the constants of integration are always to be taken equal to zero.

The right members of the differential equations for the terms of the next order are

$$\left.\begin{aligned}[\ ]^{(4)} &= A'x_2+\tfrac{3}{4}B_0[-2x_2^2+y_2^2-4x_1x_3+2y_1y_3]+C_0[6x_1^2x_2-6x_1y_1y_2-3x_2y_1^2]\,,\\ \{\ \}^{(4)} &= -\tfrac{1}{2}A'y_2+\tfrac{3}{2}B_0\{x_2y_2+x_1y_3+x_3y_1\}+\tfrac{3}{4}C_0\{-8x_1x_2y_1-4x_1^2y_2+3y_1^2y_2\}.\end{aligned}\right\}\quad(60)$$

Before integrating the first equation the coefficient of $e^{\sigma_0\sqrt{-1}t}$ in its right member must be put equal to zero. It is found from an examination of the terms of (60) that $a^{(2)}$ is involved linearly but not homogeneously. Moreover, $a^{(2)}$ is the only unknown quantity in this coefficient. Therefore $a^{(2)}$ is uniquely determined by setting the coefficient of $e^{\sigma_0\sqrt{-1}t}$ equal to zero. The condition that the solution of the second equation shall be periodic is identical with that imposed by the first. The particular integrals of the third and fourth equations are periodic.

At the i^{th} step the right members of the differential equations involve

$$\left.\begin{aligned}[\ ]^{(i)} &= A'x_{i-2}+\tfrac{3}{2}B_0[-2x_1x_{i-1}+y_1y_{i-1}]+C_0[6x_1^2x_{i-2}-6x_1y_1y_{i-2}-3x_{i-2}y_1^2]+\cdots,\\ \{\ \}^{(i)} &= -\tfrac{1}{2}A'y_{i-2}+\tfrac{3}{2}B_0\{x_1y_{i-1}+x_{i-1}y_1\}+\tfrac{3}{4}C_0\{-8x_1y_1x_{i-2}-4x_1^2y_{i-2}+3y_1^2y_{i-2}\}+\cdots,\end{aligned}\right\}(61)$$

where the parts not explicitly written are independent of $a^{(i-2)}$. In order that the solution of the first equation shall be periodic it is necessary that the coefficient of $e^{\sigma_0\sqrt{-1}t}$ in its right member be put equal to zero. This coefficient carries $a^{(i-2)}$ linearly and in general non-homogeneously, and the known factor by which $a^{(i-2)}$ is multiplied is precisely the same as that of $a^{(2)}$ in the equation by which the latter was determined. Therefore the arbitrary $a^{(i-2)}$ is uniquely determined by setting the coefficient of $e^{\sigma_0\sqrt{-1}t}$ equal to zero, for it carries no other unknown. When this condition is satisfied the coefficient of $e^{-\sigma_0\sqrt{-1}t}$ in the second equation is zero, and the entire solution at this step is periodic. Therefore after the sign of $a^{(1)}$ has been chosen the process is unique, and it can be continued indefinitely.

CHAPTER VII.

OSCILLATING SATELLITES WHEN THE FINITE MASSES DESCRIBE ELLIPTICAL ORBITS.

103. The Differential Equations of Motion.—Suppose the finite bodies describe ellipses whose eccentricity is e. Let $1-\mu$ and μ ($\mu \leqq 0.5$) represent their masses, and then determine the linear and time units so that their mean distance apart and the gravitational constant shall be unity. With these units their mean angular motion is unity.

Now refer the system to a set of rectangular axes with the origin at the center of gravity, and let the direction of the axes be so chosen that the $\xi\eta$-plane is the plane of motion of the finite bodies. Suppose the $\xi\eta$-axes rotate with the constant angular rate unity around the ζ-axis in the direction of motion of the finite masses, and suppose $1-\mu$ and μ are on the ξ-axis when they are at the apses of their orbits. Then the differential equations of motion for the infinitesimal body are

$$\left.\begin{aligned}
\frac{d^2\xi}{dt^2}-2\frac{d\eta}{dt}-\xi &= -\frac{(1-\mu)(\xi-\xi_1)}{r_1^3}-\frac{\mu(\xi-\xi_2)}{r_2^3},\\
\frac{d^2\eta}{dt^2}+2\frac{d\xi}{dt}-\eta &= -\frac{(1-\mu)(\eta-\eta_1)}{r_1^3}-\frac{\mu(\eta-\eta_2)}{r_2^3},\\
\frac{d^2\zeta}{dt^2} &= -\frac{(1-\mu)\zeta}{r_1^3}-\frac{\mu\zeta}{r_2^3},
\end{aligned}\right\} \qquad (1)$$

where

$$r_1=\sqrt{(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2}, \qquad r_2=\sqrt{(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2},$$

and where ξ_1, ξ_2, η_1, and η_2 are determined by the fact that the finite bodies move in ellipses.

If we let ρ_1 and v_1 be the polar coördinates of $1-\mu$ referred to fixed axes having their origin at the center of mass of the system, and ρ_2 and v_2 the corresponding coördinates of μ, we have

$$\left.\begin{aligned}
&\xi_1=-\rho_1\cos(v_1-t), \qquad \xi_2=+\rho_2\cos(v_2-t),\\
&\eta_1=-\rho_1\sin(v_1-t), \qquad \eta_2=+\rho_2\sin(v_2-t),\\
&\rho_1=+\mu\left\{1-e\cos t+\frac{e^2}{2}(1-\cos 2t)+\cdots\right\},\\
&\rho_2=(1-\mu)\left\{1-e\cos t+\frac{e^2}{2}(1-\cos 2t)+\cdots\right\},\\
&v_1=v_2=+t+2e\sin t+\frac{5}{4}e^2\sin 2t+\cdots,
\end{aligned}\right\} \qquad (2)$$

where the initial value of t has been so determined that the bodies are at their nearest apses at $t=0$.

104. The Elliptical Solution.—In order to get the Lagrangian elliptical solution for the infinitesimal body we consider first the two-body problem. The equations of motion for the infinitesimal body subject to the attraction of a mass m are, when referred to the rotating axes,

$$\left.\begin{aligned}
&\frac{d^2\xi}{dt^2}-2\frac{d\eta}{dt}-\xi=-k^2m\frac{\xi}{r^3},\\
&\frac{d^2\eta}{dt^2}+2\frac{d\xi}{dt}-\eta=-k^2m\frac{\eta}{r^3},\\
&\frac{d^2\zeta}{dt^2}=-k^2m\frac{\zeta}{r^3},\\
&\quad r=\sqrt{\xi^2+\eta^2+\zeta^2}.
\end{aligned}\right\} \quad (3)$$

We shall consider the solution in the $\xi\eta$-plane. If the eccentricity of the orbit is e and if the mean motion with respect to the fixed axes is unity, then the solution with the same determination of the origin of time and apses as in (2) is

$$\left.\begin{aligned}
&\xi=r\cos(v-t), \qquad \eta=r\sin(v-t), \qquad \zeta=0,\\
&r=k^{\frac{2}{3}}m^{\frac{1}{3}}\left\{1-e\cos t+\frac{e^2}{2}(1-\cos 2t)+\cdots\right\},\\
&v=2e\sin t+\frac{5}{4}e^2\sin 2t+\cdots.
\end{aligned}\right\} \quad (4)$$

It will now be shown that equations (1) will be satisfied if the infinitesimal body moves so that the ratios of its coördinates to the corresponding coördinates of the finite masses have certain constant values. Let the coördinates in these special solutions be represented by ξ_0, η_0, and 0; then

$$\frac{\xi_2-\xi_0}{\xi_0-\xi_1}=\frac{\eta_2-\eta_0}{\eta_0-\eta_1}=M, \qquad \xi_0=\frac{\xi_2+M\xi_1}{1+M}, \qquad \eta_0=\frac{\eta_2+M\eta_1}{1+M}. \quad (5)$$

Upon making use of (2) and (5), it is found that

$$\left.\begin{aligned}
&\xi_0-\xi_1=+\frac{\xi_0}{1-\mu(1-M)}, &&\eta_0-\eta_1=+\frac{\eta_0}{1-\mu(1+M)},\\
&\xi_0-\xi_2=-\frac{M\xi_0}{1-\mu(1+M)}, &&\eta_0-\eta_2=-\frac{M\eta_0}{1-\mu(1+M)},\\
&r_1=+\frac{\sqrt{\xi_0^2+\eta_0^2}}{1-\mu(1+M)}, &&r_2=+\frac{M\sqrt{\xi_0^2+\eta_0^2}}{1-\mu(1+M)}.
\end{aligned}\right\} \quad (6)$$

Then equations (1) become

$$
\left.
\begin{aligned}
\xi_0'' - 2\eta_0' - \xi_0 &= -\frac{[(1-\mu)M^2-\mu]\,[1-\mu(1+M)]^2}{M^2}\,\frac{\xi_0}{r_0^3},\\
\eta_0'' + 2\xi_0' - \eta_0 &= -\frac{[(1-\mu)M^2-\mu]\,[1-\mu(1+M)]^2}{M^2}\,\frac{\eta_0}{r_0^3},\\
\zeta_0'' &= -\frac{[(1-\mu)M^3+\mu]\,[1-\mu(1+M)]^3}{M^3}\,\frac{\zeta_0}{r_0^3},\\
r_0 &= +\sqrt{\xi_0^2+\eta_0^2+\zeta_0^2}\,.
\end{aligned}
\right\}
\tag{7}
$$

The first two of these equations are of the same form as (3), and their solutions corresponding to (4) are

$$
\left.
\begin{aligned}
&\xi_0 = r_0\cos(v-t), \qquad \eta_0 = r_0\sin(v-t), \qquad \zeta_0 = 0,\\
&r_0 = \frac{[(1-\mu)M^2-\mu]^{\frac{1}{3}}\,[1-\mu(1+M)]^{\frac{2}{3}}}{M^{\frac{2}{3}}}\,\{1-e\cos t+\cdots\},\\
&v = t + 2e\sin t + \tfrac{5}{4}e^2\sin 2t + \cdots.
\end{aligned}
\right\}
\tag{8}
$$

From these equations and (2), we find

$$
\frac{\xi_0}{\xi_1} = \frac{\eta_0}{\eta_1} = -\frac{[(1-\mu)M^2-\mu]^{\frac{1}{3}}\,[1-\mu(1+M)]^{\frac{2}{3}}}{\mu M^{\frac{2}{3}}};
$$

and from (6),

$$
\frac{\xi_0}{\xi_1} = \frac{\eta_0}{\eta_1} = -\frac{[1-\mu(1+M)]}{\mu(1+M)}.
$$

On equating these two expressions for the ratio $\xi_0/\xi_1 = \eta_0/\eta_1$, and rationalizing, we have

$$
(1-\mu)M^5 + 3(1-\mu)M^4 + 3(1-\mu)M^3 - 3\mu M^2 - 3\mu M - \mu = 0. \tag{9}
$$

It is easily verified that starting from the expressions for the ratio ξ_0/ξ_2 the same quintic equation is obtained. Therefore, for those values of M satisfying (9), equations (2) and (8) are a particular solution of the three-body problem where one mass is infinitesimal. As is well known, there are three real solutions, one for each ordering of the three masses. As in Chapter V, we shall call them (a), (b), and (c) in the order of decreasing values of their x-coördinates.

Equation (9) is Lagrange's quintic in case one mass is infinitesimal and the units are chosen so that the masses of the finite bodies are $1-\mu$ and μ. For example, if in equation (60), page 216, of *Introduction to Celestial Mechanics*, we put $m_1 = 1-\mu$, $m_2 = 0$, and $m_3 = \mu$, we get equation (9).

105. Equations for the Oscillations.—We shall study the oscillations in the vicinity of the Lagrangian solutions. For this purpose we make the transformation

$$\xi=\xi_0+x, \qquad \eta=\eta_0+y, \qquad \zeta=0+z \tag{10}$$

in equations (1), and expand as power series in x, y, and z. After this transformation and expansion, we let

$$\mu=\mu_0+\lambda \tag{11}$$

in those places where μ appears explicitly. This is not the only way in which the original μ can be divided into the new μ and $\mu_0+\lambda$, and sometimes others are advisable. The coefficients of the various powers of x, y, and z are expansible as power series in e, the terms independent of e being constants, as is seen from (2) and (8). We find from (6) and (8) that

$$\left.\begin{aligned}
\xi_0-\xi_1&=r_1^{(0)}\left[1-e\cos t-\frac{e^2}{2}(1-\cos 2t)+\cdots\right],\\
\xi_0-\xi_2&=r_2^{(0)}\left[1-e\cos t-\frac{e^2}{2}(1-\cos 2t)+\cdots\right],\\
\eta_0-\eta_1&=r_1^{(0)}\left[\qquad +2e\sin t+\frac{e^2}{4}\sin 2t+\cdots\right],\\
\eta_0-\eta_2&=r_2^{(0)}\left[\qquad +2e\sin t+\frac{e^2}{4}\sin 2t+\cdots\right],\\
r_1&=\sqrt{(\xi_0-\xi_1)^2+(\eta_0-\eta_1)^2}=r_1^{(0)}\left[1-e\cos t+\cdots\right],\\
r_2&=\sqrt{(\xi_0-\xi_2)^2+(\eta_0-\eta_2)^2}=r_2^{(0)}\left[1-e\cos t+\cdots\right],\\
r_1^{(0)}&=\frac{1}{M^{\frac{2}{3}}}\left[\frac{(1-\mu)M^2-\mu}{1-\mu(1+M)}\right]^{\frac{1}{3}}, \qquad r_2^{(0)}=M^{\frac{1}{3}}\left[\frac{(1-\mu)M^2-\mu}{1-\mu(1+M)}\right]^{\frac{1}{3}}.
\end{aligned}\right\} \tag{12}$$

Consequently, after making use of these expansions and the transformations (10) and (11), equations (1) become

$$\left.\begin{aligned}
&x''-2y'-\left\{1+2A+6Ae\cos t-3Ae^2(1-5\cos 2t)+\cdots\right\}x\\
&\qquad\qquad-\left\{6Ae\sin t+\tfrac{51}{4}Ae^2\sin 2t+\cdots\right\}y=X,\\
&y''+2x'-\left\{6Ae\sin t+\tfrac{51}{4}Ae^2\sin 2t+\cdots\right\}x\\
&\qquad\qquad-\left\{1-A-3Ae\cos t+\tfrac{3}{2}Ae^2(3-7\cos 2t)+\cdots\right\}y=Y,\\
&z''+\left\{+A+3Ae\cos t+\tfrac{3}{2}Ae^2(1+3\cos 2t)+\cdots\right\}z=Z,
\end{aligned}\right\} \tag{13}$$

where

$$
\left.
\begin{aligned}
A=&+\frac{1-\mu_0}{r_1^{(0)3}}+\frac{\mu_0}{r_2^{(0)3}},\\[2ex]
X=&+\left\{-\frac{1}{r_1^{(0)3}}+\frac{1}{r_2^{(0)3}}-6\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]e\cos t+\cdots\right\}x\lambda\\
&+\left\{-6\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]e\sin t+\cdots\right\}y\lambda\\
&+\left\{-3\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]-12\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\cos t+\cdots\right\}x^2\\
&+\left\{-24\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\sin t+\cdots\right\}xy\\
&+\left\{\frac{3}{2}\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]+6\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\cos t+\cdots\right\}y^2\\
&+\left\{\frac{3}{2}\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]+6\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\cos t+\cdots\right\}z^2\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;\\[2ex]
Y=&+\left\{-6\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]e\sin t+\cdots\right\}x\lambda\\
&+\left\{\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]+3\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]e\cos t+\cdots\right\}y\lambda\\
&+\left\{-12\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\sin t+\cdots\right\}x^2\\
&+\left\{+3\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]+12\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\cos t+\cdots\right\}xy\\
&+\left\{+9\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\sin t+\cdots\right\}y^2\\
&+\left\{+3\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\sin t+\cdots\right\}z^2+\cdots;\\[2ex]
Z=&+\left\{\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]+3\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right]e\cos t+\cdots\right\}z\lambda\\
&+\left\{3\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]+12\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\cos t+\cdots\right\}xz\\
&+\left\{6\left[\substack{+\\ \pm}\frac{1-\mu_0}{r_1^{(0)4}}\substack{\pm\\ -}\frac{\mu_0}{r_2^{(0)4}}\right]e\sin t+\cdots\right\}yz+\cdots,
\end{aligned}
\right\}\quad(14)
$$

the signs in the [ ] being $+\,+$, $+\,-$, $-\,-$, according as the point (a), (b), or (c) is under consideration. All terms up to the second order inclusive in x, y, z, and λ are written.

106. The Symmetry Theorem.—It follows from (2), (8), and (1) that the right members of (12) have the following properties:

(a) The X and Y involve only even powers of z, and Z involves only odd powers of z.

(b) In X the coefficients of all terms involving even powers of y are sums of cosines of integral multiples of t, and the coefficients of all terms involving odd powers of y are sums of sines of integral multiples of t.

(c) In Y the coefficients of all terms involving even powers of y are sums of sines of integral multiples of t, and the coefficients of all terms involving odd powers of y are sums of cosines of integral multiples of t.

(d) In Z the coefficients of all terms involving even powers of y are sums of cosines of integral multiples of t, and the coefficients of all terms involving odd powers of y are sums of sines of integral multiples of t.

Suppose the initial conditions are

$$x=\alpha, \qquad x'=0, \qquad y=0, \qquad y'=\beta, \qquad z=0, \qquad z'=\gamma. \tag{15}$$

Then the solutions of (12) are

$$\left.\begin{array}{ll} x=f\,(\alpha,\ 0,\ 0,\ \beta,\ 0,\ \gamma;\ t), & x'=f'(\alpha,\ 0,\ 0,\ \beta,\ 0,\ \gamma;\ t), \\ y=g\,(\alpha,\ 0,\ 0,\ \beta,\ 0,\ \gamma;\ t), & y'=g'(\alpha,\ 0,\ 0,\ \beta,\ 0,\ \gamma;\ t), \\ z=h(\alpha,\ 0,\ 0,\ \beta,\ 0,\ \gamma;\ t), & z'=h'(\alpha,\ 0,\ 0,\ \beta,\ 0,\ \gamma;\ t). \end{array}\right\} \tag{16}$$

Now make the transformation

$$x=+\bar{x}, \quad x'=-\bar{x}', \quad y=-\bar{y}, \quad y'=+\bar{y}', \quad z=-\bar{z}, \quad z'=+\bar{z}', \quad t=-\bar{t}. \tag{17}$$

It follows from the properties (a) , . . . , (d) that the form of equations (13) is not changed by this transformation. Consequently the solutions with the initial conditions

$$\bar{x}=\alpha, \qquad \bar{x}'=0, \qquad \bar{y}=0, \qquad \bar{y}'=\beta, \qquad \bar{z}=0, \qquad \bar{z}'=\gamma,$$

are identical with (16), and we have, making use of (17),

$$\left.\begin{array}{ll} x\,(t)=\bar{x}\,(\bar{t})=\bar{x}\,(-t)=+x\,(-t), & x'(t)=\bar{x}'(\bar{t})=\bar{x}'(-t)=-x'(-t), \\ y\,(t)=\bar{y}\,(\bar{t})=\bar{y}\,(-t)=-y\,(-t), & y'(t)=\bar{y}'(\bar{t})=\bar{y}'(-t)=+y'(-t), \\ z\,(t)=\bar{z}\,(\bar{t})=\bar{z}\,(-t)=-z\,(-t), & z'(t)=\bar{z}'(\bar{t})=\bar{z}'(-t)=+z'(-t). \end{array}\right\} \tag{18}$$

Therefore, with the initial conditions (15), x, y', and z' are even functions of t, while x', y, and z are odd functions of t. That is, if the infinitesimal body crosses the x-axis perpendicularly when the finite bodies are at an apse, its motion is symmetrical with respect to the x-axis.

107. Integration of Equations (13).—*The Terms of the First Degree.* Suppose the initial conditions are

$$x(0)=a_1,\quad x'(0)=a_2,\quad y(0)=a_3,\quad y'(0)=a_4,\quad z(0)=a_5,\quad z'(0)=a_6. \tag{19}$$

We shall now integrate equations (13) as power series in a_1, a_2, a_3, a_4, a_5, a_6, and λ. Since there are no terms in the differential equations independent of x, y, and z and their derivatives, there will be no terms in the solutions independent of $a_1, \ldots, a_6$.

The terms of the first degree in $a_1, \ldots, a_6$ are defined by the differential equations

$$\left.\begin{aligned} &x_1''-2y_1'-\left[1+2A+6Ae\cos t+\cdots\right]x_1-\left[6Ae\sin t+\cdots\right]y_1=0,\\ &y_1''+2x_1'-\left[6Ae\sin t+\cdots\right]x_1-\left[1-A-3Ae\cos t+\cdots\right]y_1=0,\\ &z_1''+\left[A+3Ae\cos t+\tfrac{3}{2}Ae^2(1+3\cos 2t)+\cdots\right]z_1=0, \end{aligned}\right\} \tag{20}$$

subject to the initial conditions (19). The coefficients are power series in e, periodic with the period 2π, and reduce to constants for $e=0$. The first two equations are independent of the third, and conversely.

For $e=0$ equations (20) become simply

$$x_1''-2y_1'-[1+2A]x_1=0,\qquad y_1''+2x_1'-[1-A]y_1=0,\qquad z_1''+Az_1=0. \tag{21}$$

From the results obtained in §§23–25 it follows that the properties of the solutions of (20) depend upon the character of the roots of the characteristic equation of (21). This equation for the first two of (21) is

$$s^4+(2-A)s^2+(1-A)(1+2A)=0. \tag{22}$$

Two roots of this equation can be equal only if A has one of the values $-1/2$, 0, $8/9$, or 1. The first two are excluded by the fact that A is necessarily positive. When μ_0 is near μ in value, as it will always be taken here, A is greater than unity and, consequently, the last two values are excluded.

Equation (22) has two pairs of roots equal in numerical value but opposite in sign, and for $A>1$ two of them are pure imaginaries and two are real. Let us represent them by $\pm\sigma_0\sqrt{-1}$, $\pm\rho_0$, where σ_0 and ρ_0 are real. We now raise the question whether two of the roots of (22) can differ by an imaginary integer. In order that this may be so we must have

$$s=\sigma_0\sqrt{-1}=\tfrac{1}{2}j\sqrt{-1},$$

where j is an integer. Since this value of s must satisfy (22), we find

$$j^4-4(2-A)j^2+16(1-A)(1+2A)=0;$$

whence

$$16A=j^2+4\pm\sqrt{9j^4-56j^2+144}. \tag{23}$$

The question is whether there are integral values of j giving admissible values for A by (23). To insure the convergence of the final series it will be necessary to take μ_0 nearly equal to μ, and we shall suppose at once that this condition is satisfied. It was shown in §82 that when $\mu_0 = \mu$ the value of A exceeds unity for each of the points (a), (b), and (c) for all values of μ. From the expansions given in equations (42), p. 206, of *Introduction to Celestial Mechanics*, it is seen that for very small values of μ the values of A are approximately 4, 4, and 1 for the points of libration (a), (b), and (c) respectively. · In the numerical example of §90, for which $\mu = 1/11$, it was found that A is 2.25, 6,51. and 1.08 for the points of libration (a), (b), and (c) respectively. In the extreme case of $\mu = 1/2$ we easily find from the formulas of §§76 and 82 that the values of A are 1.56, 8, and 1.56 for the points of libration (a), (b), and (c) respectively. While μ varies from 0 to 1/2 the values of A vary from 4 to 1.56, 4 to 8, and 1 to 1.56 for the points (a), (b), and (c) respectively. From this we see what values of A are possible in the actual problem, and we shall be able to determine whether j takes integral values for any of them.

When the negative sign is taken before the radical in (23), the function $16A$ has its greatest value of zero for $j^2 = 4$. Consequently we get no admissible values of A using this sign before the radical.

When the positive sign is taken before the radical in (23), j and A are found to have the following relations:

If $j =$	1 ,	2 ,	3 ,	4 ,	5 ,	6 ,	. . . ,
then $A =$	0.92,	1.00,	2.01,	3.73,	5.94,	8.70,	. . . ,

and A is larger than 8.7 for all j larger than 6. The values of A for j between 2 and 6 are admissible for either the point (a) or the point (b), but not for the point (c). Consequently since A, and therefore σ, is a continuous function of μ and μ_0, there exist pairs of values of μ and μ_0 for which the difference of the imaginary roots of (22) is an imaginary integer, but they are exceptional.

The characteristic equation for the third equation of (21) is simply

$$s^2 + A = 0.$$

Hence the solution of the third equation of (20) does not take an exceptional form unless $2\sqrt{A}$ is an integer, or

$$A = \frac{j^2}{4} \qquad (j \text{ an integer}).$$

It follows from the discussion above that the admissible values of A satisfying this relation are 2.25, 4.00, and 6.25. These values belong only to the point (a) or the point (b), and in no case can the congruence be satisfied for the point (c).

According to the results obtained in §23, the solutions and characteristic exponents of (20) are always expansible as power series in e except when A has the special values noted above; and according to the results obtained in §24, the same result is true, in general, even if the roots of the characteristic equations differ by imaginary integers. However, in the latter case the construction of the solutions is quite different.

It was proved in §33 that in equations of the type under consideration here the characteristic exponents occur in pairs which are equal numerically but opposite in sign. Therefore the solutions of (20) are of the form

$$\left.\begin{aligned}
x_1 &= a_1 e^{\sigma\sqrt{-1}t}u_1 + a_2 e^{-\sigma\sqrt{-1}t}u_2 + a_3 e^{\rho t}u_3 + a_4 e^{-\rho t}u_4,\\
x_1' &= a_1 e^{\sigma\sqrt{-1}t}[\sigma\sqrt{-1}u_1 + u_1'] + a_2 e^{-\sigma\sqrt{-1}t}[-\sigma\sqrt{-1}u_2 + u_2']\\
&\qquad + a_3 e^{\rho t}[\rho u_3 + u_3'] + a_4 e^{-\rho t}[-\rho u_4 + u_4'],\\
y_1 &= a_1 e^{\sigma\sqrt{-1}t}v_1 + a_2 e^{-\sigma\sqrt{-1}t}v_2 + a_3 e^{\rho t}v_3 + a_4 e^{-\rho t}v_4,\\
y_1' &= a_1 e^{\sigma\sqrt{-1}t}[\sigma\sqrt{-1}v_1 + v_1'] + a_2 e^{-\sigma\sqrt{-1}t}[-\sigma\sqrt{-1}v_2 + v_2']\\
&\qquad + a_3 e^{\rho t}[\rho v_3 + v_3'] + a_4 e^{-\rho t}[-\rho v_4 + v_4'],\\
z_1 &= c_1 e^{\omega\sqrt{-1}t}w_1 + c_2 e^{-\omega\sqrt{-1}t}w_2,\\
z_1' &= c_1 e^{\omega\sqrt{-1}t}[\omega\sqrt{-1}w_1 + w_1'] + c_2 e^{-\omega\sqrt{-1}t}[-\omega\sqrt{-1}w_2 + w_2'],
\end{aligned}\right\} \quad (24)$$

where

$$\left.\begin{aligned}
&a_1, \ldots, a_4, c_1, c_2, \text{ are arbitrary constants of integration},\\
&\sigma = \sigma_0 + \sigma_1 e + \sigma_2 e^2 + \cdots, \qquad u_i = u_i^{(0)} + u_i^{(1)}e + u_i^{(2)}e^2 + \cdots,\\
&\rho = \rho_0 + \rho_1 e + \rho_2 e^2 + \cdots, \qquad v_i = v_i^{(0)} + v_i^{(1)}e + v_i^{(2)}e^2 + \cdots,\\
&\omega = \sqrt{A} + \omega_1 e + \omega_2 e^2 + \cdots, \qquad w_i = w_i^{(0)} + w_i^{(1)}e + w_i^{(2)}e^2 + \cdots,\\
&u_i^{(0)}, v_i^{(0)}, w_i^{(0)} \text{ are constants.}
\end{aligned}\right\} \quad (25)$$

The initial values of the v_i and w_i can be taken equal to unity without loss of generality, and will be so chosen. Moreover, the u_i, v_i, and w_i are periodic with the period 2π, and since this property holds for all e for which the series converge, each $u_i^{(j)}$, $v_i^{(j)}$, and $w_i^{(j)}$ separately is periodic with the period 2π. The coefficients of these series can be found by the methods set forth in §26. The a_i and c_i are uniquely expressible in terms of the initial values of x, x', y, y', z, and z', the α_i of equations (19), because the solutions (24) constitute a fundamental set, by hypothesis, and the determinant of the coefficients of the a_i and c_i is therefore distinct from zero. We may use either the a_i and c_i or the α_i as arbitraries.

The characteristic exponents σ, ρ, and ω are real for e sufficiently small, as we shall now show. The ω arises from the third equation of (21), which is of the same form as that treated in §50, where it was shown that the characteristic exponents are pure imaginaries in this case. The $\pm\sigma\sqrt{-1}$ and $\pm\rho$ are roots of an equation of the form

$$\Delta(\alpha, e) = 0, \tag{26}$$

where Δ is an even function of α and a power series in e [see §22, and in particular equation (98)]. For $e=0$ the solutions of this equation are

$$\alpha=\pm\sigma_0\sqrt{-1}, \qquad \alpha=\pm\rho_0,$$

where σ_0 and ρ_0 are real. Now let

$$\alpha=\sigma_0\sqrt{-1}+\beta\sqrt{-1}$$

and (26) becomes

$$\Delta(\sigma_0\sqrt{-1},\, e)+\Delta'(\sigma_0\sqrt{-1},\, e)\beta\sqrt{-1}-\frac{1}{2}\Delta''(\sigma_0\sqrt{-1},\, e)\beta^2+\cdots=0. \quad (27)$$

All the even derivatives are even functions of $\sigma_0\sqrt{-1}$, and the coefficients of the various powers of β in these terms are therefore real; all the odd derivatives are odd in $\sigma_0\sqrt{-1}$ and are multiplied by odd powers of $\beta\sqrt{-1}$, and the coefficients of the various powers of β in these terms are therefore also real. Consequently, since $\Delta'(\sigma_0\sqrt{-1},\, 0)$ is distinct from zero under the conditions satisfied in this problem, the solution of (27) for β as a power series in e, vanishing with e, gives a unique series whose coefficients are all real. Therefore $\sigma=\alpha$ is real. It can be shown similarly that ρ is a real constant.

Now suppose the initial conditions are given by (15). Then, since

$$y_1(t)=-y_1(-t), \qquad z_1(t)=-z_1(-t),$$

we have

$$\begin{aligned}
&a_1e^{\sigma\sqrt{-1}t}v_1(t)+a_2e^{-\sigma\sqrt{-1}t}v_2(t)+a_3e^{\rho t}v_3(t)+a_4e^{-\rho t}v_4(t)\\
&\quad\equiv-\left[a_1e^{-\sigma\sqrt{-1}t}v_1(-t)+a_2e^{\sigma\sqrt{-1}t}v_2(-t)+a_3e^{-\rho t}v_3(-t)+a_4e^{\rho t}v_4(-t)\right],\\
&c_1e^{\omega\sqrt{-1}t}w_1(t)+c_2e^{-\omega\sqrt{-1}t}w_2(-t)\equiv-\left[c_1e^{-\omega\sqrt{-1}t}w_1(-t)+c_2e^{\omega\sqrt{-1}t}w_2(-t)\right].
\end{aligned}$$

On applying the lemma of §58, with the necessary slight modifications, it follows that

$$a_1=-a_2, \qquad a_3=-a_4, \qquad c_1=-c_2.$$

Then we have from (24)

$$\begin{aligned}
x_1&=+a_1\left[e^{+\sigma\sqrt{-1}t}u_1(+t)-e^{-\sigma\sqrt{-1}t}u_2(+t)\right]+a_3\left[e^{+\rho t}u_3(+t)-e^{-\rho t}u_4(+t)\right]\\
&=-a_1\left[e^{-\sigma\sqrt{-1}t}u_1(-t)-e^{+\sigma\sqrt{-1}t}u_2(-t)\right]+a_3\left[e^{-\rho t}u_3(-t)-e^{+\rho t}u_4(-t)\right],\\
y_1&=+a_1\left[e^{+\sigma\sqrt{-1}t}v_1(+t)-e^{-\sigma\sqrt{-1}t}v_2(+t)\right]+a_3\left[e^{+\rho t}v_3(+t)-e^{-\rho t}v_4(+t)\right]\\
&=-a_1\left[e^{-\sigma\sqrt{-1}t}v_1(-t)-e^{+\sigma\sqrt{-1}t}v_2(-t)\right]-a_3\left[e^{-\rho t}v_3(-t)-e^{+\rho t}v_4(-t)\right],\\
z_1&=+c_1\left[e^{+\omega\sqrt{-1}t}w_1(+t)-e^{-\omega\sqrt{-1}t}w_2(+t)\right]\\
&=-c_1\left[e^{-\omega\sqrt{-1}t}w_1(-t)-e^{+\omega\sqrt{-1}t}w_2(-t)\right].
\end{aligned}$$

Since these relations are identities in t, we have

$$u_1(t) \equiv -u_2(-t),\quad u_3(t) \equiv -u_4(-t),\quad v_1(t) \equiv v_2(-t),\quad v_3(t) \equiv v_4(-t),\quad w_1(t) \equiv w_2(-t).$$

Therefore, if the u_j, v_j, and w_j are arranged as Fourier series, they satisfy the relations

$$\left.\begin{aligned}
u_1 &= +\Sigma\,[A_j \cos jt + B_j \sin jt], & v_1 &= +\Sigma\,[E_j \cos jt + F_j \sin jt],\\
u_2 &= -\Sigma\,[A_j \cos jt - B_j \sin jt], & v_2 &= +\Sigma\,[E_j \cos jt - F_j \sin jt],\\
u_3 &= +\Sigma\,[C_j \cos jt + D_j \sin jt], & v_3 &= +\Sigma\,[G_j \cos jt + H_j \sin jt],\\
u_4 &= -\Sigma\,[C_j \cos jt - D_j \sin jt], & v_4 &= +\Sigma\,[G_j \cos jt - H_j \sin jt],\\
w_1 &= +\Sigma\,[K_j \cos jt + L_j \sin jt], & w_2 &= +\Sigma\,[K_j \cos jt - L_j \sin jt].
\end{aligned}\right\} \quad (28)$$

Since the u_j, v_j, and w_j are defined by linear equations they are independent of the initial conditions. But the equations $a_1 = -a_2$, $a_3 = -a_4$, $c_1 = -c_2$ hold only for the initial conditions (5).

108. The terms of the Second Degree.—The terms of the second degree in the α_i and λ are found from equations (13) and (14) to be defined by

$$\left.\begin{aligned}
&x_2'' - 2y_2' - \left[1 + 2A + 6Ae\cos t + \cdots\right]x_2 - \left[6Ae\sin t + \cdots\right]y_2 = X_2,\\
&y_2'' + 2x_2' - \left[6Ae\sin t + \cdots\right]x_2 - \left[1 - A - 3Ae\cos t + \cdots\right]y_2 = Y_2,\\
&z_2'' + \left[A + 3Ae\cos t + \cdots\right]z_2 = Z_2,
\end{aligned}\right\} \quad (29)$$

where

$$\left.\begin{aligned}
X_2 = &+\left[-2A' - 6A'e\cos t + \cdots\right]x_1\lambda + \left[-6A'e\sin t + \cdots\right]y_1\lambda\\
&+\left[-3B - 12Be\cos t + \cdots\right]x_1^2 + \left[-24Be\sin t + \cdots\right]x_1y_1\\
&+\left[+\tfrac{3}{2}B + 6Be\cos t + \cdots\right]y_1^2 + \left[\tfrac{3}{2}B + 6Be\cos t + \cdots\right]z_1^2 + \cdots,\\
Y_2 = &+\left[-6A'e\sin t + \cdots\right]x_1\lambda + \left[A' + 3A'e\cos t + \cdots\right]y_1\lambda\\
&+\left[-12Be\sin t + \cdots\right]x_1^2 + \left[3B + 12Be\cos t + \cdots\right]x_1y_1\\
&+\left[+9Be\sin t + \cdots\right]y_1^2 + \left[+3Be\sin t + \cdots\right]z_1^2 + \cdots,\\
Z_2 = &+\left[A' + 3A'e\cos t + \cdots\right]z_1\lambda + \left[3B + 12Be\cos t + \cdots\right]x_1z_1\\
&+\left[6Be\sin t + \cdots\right]y_1z_1 + \cdots,\\
&A' = \frac{1}{r_1^{(0)3}} - \frac{1}{r_2^{(0)3}}, \qquad B = \pm\frac{1-\mu_0}{r_1^{(0)4}} \pm \frac{\mu_0}{r_2^{(0)4}}.
\end{aligned}\right\} \quad (30)$$

It is necessary for further work to determine some of the properties of the solutions of equations (29). These properties depend upon the form and properties of their right members, which are given in equations (30). The general character of the coefficients in these equations easily follows from the properties (a), . . . , (d) of §106. On referring to the results which were developed in Chapter I, it is found that:

[1] The solutions of (29) consist in the first place of the complementary functions, and they are identical in form with (24). The arbitraries a_i and c_i which appear are uniquely determined by the conditions $x_2(0)=x_2'(0)=y_2(0)=y_2'(0)=z_2(0)=z_2'(0)=0$, where $x_2, \ldots, z_2'$ are the complete solutions of (29) [§15].

[2] There are terms arising from those parts of the right members which contain λ as a factor. These right members consist of sums of terms which are periodic with the period 2π multiplied by one of the *fundamental exponentials* $e^{+\sigma\sqrt{-1}t}$, $e^{-\sigma\sqrt{-1}t}$, $e^{+\rho t}$, $e^{-\rho t}$, $e^{\omega\sqrt{-1}t}$, and $e^{-\omega\sqrt{-1}t}$. Hence it follows from the results of §30 that the corresponding parts of the solutions consist of sums of periodic terms whose period is 2π multiplied by these same exponentials, plus t times the corresponding part of the complementary function. We shall be particularly interested in those terms which contain t as a factor. They are linear in the a_i and c_i, and for $e=0$ the parts having the period 2π reduce to constants. The expressions for x_2 and y_2 do not depend upon c_1 and c_2, and the expression for z_2 is independent of $a_1, \ldots, a_4$.

[3] Next consider the parts of the right members of (29) which are independent of λ. They consist of sums of periodic terms having the period 2π multiplied by the squares and second-degree products of the fundamental exponentials. Therefore, except in the special case where σ or ω is an integer, the exponents of the exponentials in the right members are not congruent to any of the characteristic exponents mod. $\sqrt{-1}$; hence it follows by §30 that corresponding parts of the solutions consist of sums of periodic terms, period 2π, multiplied by these same exponentials. In particular, there are no terms containing t as a factor. These terms are homogeneous of the second degree in the a_i and the c_i.

109. The Terms of the Third Degree.—The terms of the third degree in the a_i, c_i, and λ are defined by equations whose left members are identical in form with the left members of (29). The right members contain terms which are

(a) linear in λ and of the second degree in x_1, y_1, and z_1;
(b) linear in λ and of the first degree in x_2, y_2, and z_2;
(c) of the third degree in x_1, y_1, and z_1; and
(d) of the first degree in x_1, y_1, z_1 and in x_2, y_2, and z_2.

The solutions have the following properties:

[4] There are the complementary functions identical in form with the expressions (24). The constants which appear in them are determined by the conditions that $x_3(0)=x_3'(0)=y_3(0)=y_3'(0)=z_3(0)=z_3'(0)=0$.

[5] The part of the solution coming from the terms (a) consists of sums of periodic terms, period 2π, multiplied by the squares and second-degree products of the fundamental exponentials.

[6] The terms (b) give rise to terms of two different classes because x_2, y_2, and z_2 consist of terms of two different types. There are terms which contain λ as a factor, and a part of these are sums of periodic terms, period 2π, multiplied by t times the fundamental exponentials; the remaining part lacks the factor t. These terms are homogeneous and linear in the a_i and the c_i. The corresponding parts of the solutions are sums of periodic terms, period 2π, multiplied partly by t^2, partly by t, and partly by t^0. They are all homogeneous and linear in the a_i and the c_i, the x and y-terms being independent of the c_i, and the z-terms of the a_i.

The terms of the solutions coming from the other part of (b) are sums of periodic terms, period 2π, multiplied by squares and second-degree products of the fundamental exponentials. They are homogeneous of the second degree in the a_i and the c_i.

[7] The terms of the type (c) are homogeneous of the third degree in the a_i and the c_i. They consist of terms of two classes, the first of which are sums of periodic terms, period 2π, multiplied by the fundamental exponentials to the first degree; and the second of which are sums of periodic terms, period 2π, multiplied by cubes and non-canceling third-degree products of the fundamental exponentials. The corresponding parts of the solutions consist respectively of t times sums of periodic terms, period 2π, multiplied by the fundamental exponentials to the first degree, and sums of periodic terms, period 2π, multiplied by cubes and non-canceling third-degree products of the fundamental exponentials.

[8] The part of the solution coming from the terms (d) consists of terms of two kinds, the first depending upon those parts of x_2, y_2, and z_2 which contain λ as a factor, and the second depending upon those parts of x_2, y_2, and z_2 which are independent of λ. The parts of the solutions corresponding to the first are homogeneous of the second degree in the a_i and the c_i, and they are sums of periodic terms, period 2π, multiplied by squares and second-degree products of the fundamental exponentials, and some of these products contain t as a factor while others do not. The other parts of the solutions coming from the terms (d) have the properties of those coming from (c).

110. General Properties of the Solutions.—It will be necessary to use the following general properties of the solutions:

[9] Since the right members of the first two equations of (13) involve only even powers of z, it follows that x and y are even functions of c_1 and c_2 taken together.

[10] Since the right member of the third equation of (13) is an odd function of z, it follows that z is an odd function of c_1 and c_2 taken together, and that z identically vanishes for $c_1=c_2=0$.

[11] Since the right members of the first two equations of (13) vanish identically for $x=y=z=0$, but not for $x=y=0$, it follows that x and y vanish identically for $a_1 = \cdots = a_4 = c_1 = c_2 = 0$, but not for $a_1 = \cdots = a_4 = 0$, $c_1 \neq 0$, $c_2 \neq 0$.

[12] Since the equations reduce to those having constant terms for $e=0$, it follows that the sums of the periodic terms, period 2π, reduce to constants for $e=0$.

111. Conditions for the Existence of Symmetrical Periodic Orbits.—The differential equations are periodic in t with the period 2π. Consequently the period of the periodic solutions, if they exist, will be $T=2n\pi$, where n is an integer. When the initial conditions are such that the orbit of the infinitesimal body is symmetrical, as defined in §106, then sufficient conditions for the existence of the periodic solutions are

$$\left.\begin{array}{lll} a_1+a_2=0, & a_3+a_4=0, & c_1+c_2=0, \\ x'\left(\frac{T}{2}\right)-x'(0)=0, & y\left(\frac{T}{2}\right)-y(0)=0, & z\left(\frac{T}{2}\right)-z(0)=0. \end{array}\right\} \quad (31)$$

These equations are power series in $a_1, \ldots, a_4, c_1, c_2$, and λ, and vanish identically with $a_1 = \cdots = a_4 = c_1 = c_2 = 0$. In order that they may have any solution for $a_1, \ldots, a_4, c_1$, and c_2, vanishing with λ, aside from this one, the determinant of the coefficients of the linear terms in $a_1, \ldots, a_4$, c_1, and c_2 must vanish. It follows from (28) that

$$\left.\begin{array}{llll} u_1(0)=-u_2(0), & u_1\left(\frac{T}{2}\right)=-u_2\left(\frac{T}{2}\right), & u_1'(0)=+u_2'(0), & u_1'\left(\frac{T}{2}\right)=+u_2'\left(\frac{T}{2}\right), \\ u_3(0)=-u_4(0), & u_3\left(\frac{T}{2}\right)=-u_4\left(\frac{T}{2}\right), & u_3'(0)=+u_4'(0), & u_3'\left(\frac{T}{2}\right)=+u_4'\left(\frac{T}{2}\right), \\ v_1(0)=+v_2(0), & v_1\left(\frac{T}{2}\right)=+v_2\left(\frac{T}{2}\right), & v_1'(0)=-v_2'(0), & v_1'\left(\frac{T}{2}\right)=-v_2'\left(\frac{T}{2}\right), \\ v_3(0)=+v_4(0), & v_3\left(\frac{T}{2}\right)=+v_4\left(\frac{T}{2}\right), & v_3'(0)=-v_4'(0), & v_3'\left(\frac{T}{2}\right)=-v_4'\left(\frac{T}{2}\right), \\ w_1(0)=+w_2(0), & w_1\left(\frac{T}{2}\right)=+w_2\left(\frac{T}{2}\right), & w_1'(0)=-w_2(0), & w_1'\left(\frac{T}{2}\right)=-w_2'\left(\frac{T}{2}\right), \end{array}\right\} \quad (32)$$

and therefore the determinant of the coefficients of the linear terms of (31) is found from (24) to be

$$\Delta = \Delta_1\,\Delta_2\,, \tag{33}$$

$$\Delta_1 = \begin{vmatrix} 1, & 1, & 0, & 0 \\ 0, & 0, & 1, & 1 \\ A_1 e^{\sigma\sqrt{-1}\frac{T}{2}} - B_1, & A_1 e^{-\sigma\sqrt{-1}\frac{T}{2}} - B_1, & A_3 e^{\rho\frac{T}{2}} - B_3, & A_3 e^{-\rho\frac{T}{2}} - B_3 \\ E_1 e^{\sigma\sqrt{-1}\frac{T}{2}} - F_1, & E_1 e^{-\sigma\sqrt{-1}\frac{T}{2}} - F_1, & E_3 e^{\rho\frac{T}{2}} - F_3, & E_3 e^{-\rho\frac{T}{2}} - F_3 \end{vmatrix}, \tag{34}$$

$$\Delta_2 = \begin{vmatrix} 1, & 1 \\ e^{\omega\sqrt{-1}\frac{T}{2}} w_1\left(\frac{T}{2}\right) - w_1(0), & e^{-\omega\sqrt{-1}\frac{T}{2}} w_1\left(\frac{T}{2}\right) - w_1(0), \end{vmatrix} \tag{35}$$

where

$$A_1 = \sigma\sqrt{-1}\,u_1\left(\frac{T}{2}\right) + u_1'\left(\frac{T}{2}\right), \qquad A_3 = \rho u_3\left(\frac{T}{2}\right) + u_3'\left(\frac{T}{2}\right),$$
$$B_1 = \sigma\sqrt{-1}\,u_3(0) + u_3'(0), \qquad B_3 = \rho u_3(0) + u_3'(0),$$
$$E_1 = v_1\left(\frac{T}{2}\right), \quad F_1 = v_1(0), \quad E_3 = v_3\left(\frac{T}{2}\right), \quad F_3 = v_3(0).$$

On reducing, we find

$$\left.\begin{aligned} \Delta_1 &= \left[e^{\sigma\sqrt{-1}\frac{T}{2}} - e^{-\sigma\sqrt{-1}\frac{T}{2}}\right]\left[e^{\rho\frac{T}{2}} - e^{-\rho\frac{T}{2}}\right]\begin{vmatrix} A_1, & A_3 \\ v_1\left(\frac{T}{2}\right), & v_3\left(\frac{T}{2}\right) \end{vmatrix}, \\ \Delta_2 &= w_1\left(\frac{T}{2}\right)\left[e^{\omega\sqrt{-1}\frac{T}{2}} - e^{-\omega\sqrt{-1}\frac{T}{2}}\right]. \end{aligned}\right\} \tag{36}$$

It will now be shown that $w_1(T/2)$ and

$$\begin{vmatrix} A_1, & A_3 \\ v_1\left(\frac{T}{2}\right), & v_3\left(\frac{T}{2}\right) \end{vmatrix}$$

are distinct from zero. Let D_1 represent the determinant of the fundamental set of solutions of the x and y-equations, and D_2 that of the z-equation. On writing these determinants for the time $t = T/2$ and making use of the relations (32), we find

$$D_1 = \begin{vmatrix} u_1\left(\frac{T}{2}\right), & -u_1\left(\frac{T}{2}\right), & u_3\left(\frac{T}{2}\right), & -u_3\left(\frac{T}{2}\right) \\ A_1, & A_1, & A_3, & A_3 \\ v_1\left(\frac{T}{2}\right), & v_1\left(\frac{T}{2}\right), & v_3\left(\frac{T}{2}\right), & v_3\left(\frac{T}{2}\right) \\ C_1, & -C_1, & C_3, & -C_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} w_1\left(\frac{T}{2}\right), & w_1\left(\frac{T}{2}\right) \\ \omega\sqrt{-1}\,w_1\left(\frac{T}{2}\right) + w_1'\left(\frac{T}{2}\right), & -\omega\sqrt{-1}\,w_1\left(\frac{T}{2}\right) - w_1'\left(\frac{T}{2}\right) \end{vmatrix},$$

where

$$C_1 = \sigma\sqrt{-1}\, v_1\!\left(\frac{T}{2}\right) + v_1'\!\left(\frac{T}{2}\right), \qquad C_3 = \rho v_3\!\left(\frac{T}{2}\right) + v_3'\!\left(\frac{T}{2}\right).$$

The determinants become, after some reductions,

$$\left.\begin{aligned} D_1 &= 4\begin{vmatrix} C_1, & C_3 \\ u_1\!\left(\frac{T}{2}\right), & u_3\!\left(\frac{T}{2}\right)\end{vmatrix} \times \begin{vmatrix} A_1, & A_3 \\ v_1\!\left(\frac{T}{2}\right), & v_3\!\left(\frac{T}{2}\right)\end{vmatrix}, \\ D_2 &= -2\left[\omega\sqrt{-1}\, w_1\!\left(\frac{T}{2}\right) - w_1'\!\left(\frac{T}{2}\right)\right] w_1\!\left(\frac{T}{2}\right). \end{aligned}\right\} \quad (37)$$

Since D_1 and D_2 are determinants of fundamental sets of solutions of linear differential equations for the regular point $t = T/2$, they are distinct from zero. Therefore their second factors are not zero, and equations (36) can be satisfied only by

$$\left[e^{\sigma\sqrt{-1}\frac{T}{2}} - e^{-\sigma\sqrt{-1}\frac{T}{2}}\right] = 0 \text{ or } \left[e^{\omega\sqrt{-1}\frac{T}{2}} - e^{-\omega\sqrt{-1}\frac{T}{2}}\right] = 0. \qquad (38)$$

If either of these equations is satisfied, Δ is zero.

In order that one of equations (38) shall be satisfied it is necessary that either

$$\sigma T = 2N_1\pi, \text{ or } \omega T = 2N_2\pi \qquad (N_1, N_2 \text{ integers}). \qquad (39)$$

Since $T = 2n\pi$, where n is an integer, these conditions become

$$\sigma = \frac{N_1}{n}, \text{ or } \omega = \frac{N_2}{n}. \qquad (40)$$

Hence the conditions for the existence of the symmetrical periodic solutions in question can be satisfied only when σ or ω is rational. These quantities, given in (25), are power series in e and they depend upon μ and μ_0 and the way μ is generalized when the transformation $\mu = \mu_0 + \lambda$ is made. Since σ and ω are continuous functions of μ, μ_0, and e, the rationality of at least one of them at a time can be assured. It should be noted further that σ_0 and ω_0 depend upon μ, μ_0, and the mode of generalization of μ, but that they are independent of e.

In any case $|\lambda|$ can be taken so small that the series will converge, but the periodic solution does not belong to the physical problem except when $\lambda = \mu - \mu_0$. Suppose the series diverge for this value of λ. Theoretically the values of the coördinates for this value of λ can be obtained by analytic continuation with respect to λ as the argument from the periodic solution which exists for a smaller value of λ. There is an exception only if the function has a natural boundary, or if $\lambda = \mu - \mu_0$ is a singular point.

112. The Existence of Three-Dimensional Symmetrical Periodic Orbits.—Suppose ω is the rational number $\omega = N/n$, where N and n are relatively prime integers, and take $T = 2n\pi$. Suppose σT is not an integral multiple of 2π. Then $\Delta_1 \neq 0$ and the first four equations of (31) can be solved for $a_1, \ldots, a_4$ uniquely as power series in c_1, c_2, and λ, vanishing identically for $c_1 = c_2 = 0$, by property [11]. The $a_1, \ldots, a_4$ are even functions of c_1 and c_2 taken together, by property [9]. When these results are substituted in the last two equations of (31), they become power series in c_1, c_2, and λ. These series are of odd degree in c_1 and c_2 taken together, by property [10], and therefore vanish identically for $c_1 = c_2 = 0$. The substitution of the values for $a_1, \ldots, a_4$ does not change the linear terms, for the first four equations were even in c_1 and c_2 alone.

Let c_2 be eliminated by means of the fifth equation of (31). Then c_1 is a factor of the result, which has the form

$$0 = c_1 [\alpha_{01} \lambda + \alpha_{20} c_1^2 + \cdots]. \tag{41}$$

The solution $c_1 = 0$ is trivial and we are interested only in those obtained by setting the other factor equal to zero. The second factor set equal to zero is satisfied by $c_1 = \lambda = 0$. If α_{01} is distinct from zero, solutions for c_1 as power series in fractional powers of λ certainly exist. If $\alpha_{j_1 0}$ is the first α_{j0} which does not vanish, then the solutions are expansible as power series in λ^{1/j_1}. In particular, if α_{20} is distinct from zero the solutions are expansible as power series in $\pm\lambda^{\frac{1}{2}}$. If the number of solutions is odd, only one is real; and if even, only two are real, and these are real only for positive or negative values of λ according as α_{01} and $\alpha_{j_1 0}$ are unlike or like in sign.

It is clear *a priori* that the number of solutions will be even, for there is nothing of a dynamical nature by which to distinguish the two sides of the xy-plane. Consequently, if any initial projection gives rise to a periodic orbit, a symmetrically opposite one with respect to the xy-plane will also produce a periodic orbit.

It remains to show that α_{01} is distinct from zero. To prove this the terms of the second degree in the a_i, c_i, and λ must be considered (§108). It follows from the form of (29) and (30) that α_{01} depends only upon the z-equation, for it is not changed by the substitution of the solutions of the first four equations of (31) for $a_1, \ldots, a_4$ in the last two. Hence α_{01} depends only upon the solution of

$$z_2'' + [A + 3Ae\cos t + \cdots] z_2 = [A' + 3A'e\cos t + \cdots] z_1, \tag{42}$$

where

$$z_1 = c_1 [e^{\omega\sqrt{-1}t} w_1 - e^{-\omega\sqrt{-1}t} w_2].$$

It follows from the properties of w_1, w_2, ω and the general theory of §§29 and 30 that the coefficient α_{01} is a power series in e. For $e = 0$ it was given in equation (26), of Chapter VI, where it was shown to be $(-1)^n A\sigma A'$. This being distinct from zero, e can be taken so small that the series for α_{01} is distinct from zero.

Similarly, a_{20} is made up of a constant part distinct from zero plus a converging series in e, and is therefore distinct from zero for e sufficiently small. Suppose $e=0$ in (41) and let the value of c_1 obtained from solving the resulting equation be $c_1^{(0)}$; then let $c_1=c_1^{(0)}(1+\gamma)$. It is easily found from (41) that $\partial\gamma/\partial e$ is a power series in e and γ which is distinct from zero for $\gamma=e=0$ provided λ has such a value that $c_1^{(0)}\neq 0$. Hence the solution of (41) can be written in the form

$$c_1=\pm\lambda^{\frac{1}{2}}p(\pm\lambda^{\frac{1}{2}}), \tag{43}$$

where $p(\pm\lambda^{\frac{1}{2}})$ is a power series in $\lambda^{\frac{1}{2}}$ whose coefficients are power series in e. On substituting this result in the solutions of the first four equations of (31) for the a_i as power series in c_1 and λ, we have $a_1, \ldots, a_4$ expressed as power series in $\lambda^{\frac{1}{2}}$. But since $a_1, \ldots, a_4$ contain only even powers of c_1, they have λ instead of $\lambda^{\frac{1}{2}}$ as a factor after c_1 is eliminated by (43). The expressions for the coördinates become, since x and y are functions of c_1^2,

$$x=\lambda P_1(\lambda^{\frac{1}{2}};t), \qquad y=\lambda P_2(\lambda^{\frac{1}{2}};t), \qquad z=\lambda^{\frac{1}{2}}P_3(\lambda^{\frac{1}{2}};t), \tag{44}$$

where P_1, P_2, and P_3 are power series in $\lambda^{\frac{1}{2}}$.

Since the problem is dynamically symmetrical with respect to the xy-plane, a solution symmetrically opposite with respect to the xy-plane exists for all $|\lambda|$ sufficiently small. Therefore z must be an odd series in $\lambda^{\frac{1}{2}}$, and x and y even series in $\lambda^{\frac{1}{2}}$. Hence equations (44) become

$$x=\lambda Q_1(\lambda;t), \qquad y=\lambda Q_2(\lambda;t), \qquad z=\lambda^{\frac{1}{2}}Q_3(\lambda;t), \tag{45}$$

where Q_1, Q_2, and Q_3 are power series in λ.

We suppose that $e>0$ and therefore that the finite bodies are at their perifoci* at $t=0$. The infinitesimal body crosses the x-axis perpendicularly at $t=0$ and at $t=T/2$. It follows from the symmetry of the motion that it can cross the x-axis perpendicularly only at the end and middle of the true period. Therefore if n and N are relatively prime, $T=2n\pi=2N\pi/\omega$ is the true period.

The two cases I, n even, and II, n odd, merit a little further discussion. In the first N is odd because n and N are relatively prime; in the second it may be either odd or even.

113. Case I, n even, N odd.—The infinitesimal body crosses the x-axis perpendicularly at $t=0$ and also at $t=T/2=n\pi=2n'\pi$, where n' is an integer. Since at both of these epochs the finite bodies are at their perifoci, the infinitesimal body crosses the x-axis perpendicularly only when the finite bodies are at their perifoci. It follows from this that the infinitesimal body crosses the x-axis perpendicularly at the same point but in the opposite direction with respect to the xy-plane at $t=0$ and $t=T/2$. To

*_Perifoci_ will be used to denote the points at which the finite bodies are nearest each other, and _apofoci_ those at which they are most remote from each other. These points correspond to perihelia and aphelia in planetary motion.

prove it suppose the two points were different. Then we should have four solutions corresponding to $\pm\lambda^{\frac{1}{2}}$ at each of the points, and it is known that there are but two. The values of $z'(0)$ and $z'(T/2)$ are opposite in sign because otherwise the period of the motion would be $T/2$.

It can now be shown that in this case the orbits obtained by taking the two signs before $\lambda^{\frac{1}{2}}$ are geometrically the same one. Consider the orbit defined by the positive sign before $\lambda^{\frac{1}{2}}$. At the time $t=T/2$ the infinitesimal body crosses the x-axis perpendicularly at the point at which it crossed at $t=0$, but in the opposite z-direction. We may consider $T/2$ as an origin of time for defining orbits which cross the x-axis perpendicularly. It has been shown that there is but one with the given z-direction of motion, and at $t=T/2+T/2=T$ the infinitesimal body will again cross the x-axis perpendicularly with the opposite z-direction, viz., with that which it had at $t=0$. Since this orbit was unique, it follows that the two orbits which correspond to the double sign before $\lambda^{\frac{1}{2}}$ are geometrically the same, but that in one the infinitesimal body is half a period ahead of its position in the other one. That is, changing the sign of $\lambda^{\frac{1}{2}}$ in the solution is equivalent to adding $T/2$ to t.

When the solutions are actually constructed and are reduced to the trigonometric form, they involve sines and cosines of $(j\omega+k)t$, where j and k are integers. Since x, y', and z' are even functions of t, they will involve only cosines, and since x', y, and z are odd functions of t, they will involve only sines. Since x and y are even series in $\lambda^{\frac{1}{2}}$, it follows from the foregoing properties that in them

$$\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\Big[(j\omega+k)t\Big]\equiv\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\Big[(j\omega+k)(t+T/2)\Big].$$

These identities are satisfied if, and only if,

$$\cos(j\omega+k)T/2=\cos(j\omega+k)2\pi n'=1 \quad (n' \text{ an integer});$$

therefore

$$(j\omega+k)n'=j\frac{N}{2}+kn'=p \quad (p \text{ an integer}).$$

Since N is odd it follows from this relation that j is necessarily even.

Similarly, in the case of the series for z the relation

$$\sin\Big[(j\omega+k)t\Big]\equiv-\sin\Big[(j\omega+k)(t+T/2)\Big]$$

must be fulfilled. It follows from this identity in t that

$$(j\omega+k)n'=j\frac{N}{2}+kn'=\frac{2p+1}{2} \quad (p \text{ an integer}).$$

This relation can be satisfied only if j is an odd integer.

There are solutions in this case which have not been obtained by the analysis as given. The orbits which have been discussed intersect the x-axis perpendicularly when the finite bodies are at their perifoci, and obliquely, if at all, when they are at their apofoci. Similarly, supposing $t=0$ when the

finite bodies are at their apofoci, it can be proved that there are orbits *with the same period* in which the infinitesimal body crosses the x-axis perpendicularly when the finite bodies are at their apofoci, and obliquely, if at all, when they are at their perifoci.

114. Case II, n odd.—In the present case the infinitesimal body crosses the x-axis perpendicularly when the finite bodies are at their perifoci and also when they are at their apofoci, because $T/2$ is an odd multiple of π. If N is even, the solution for $+\lambda^{\frac{1}{2}}$ is geometrically distinct from that for $-\lambda^{\frac{1}{2}}$. To prove this, we note that $|\lambda|$ can be taken so small that the sign of the series for z' is determined by its first term. If e is small, z' has its sign determined by the constant parts of w_1 and w_2. Now the first parts of these first terms, viz., $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$, have the period $2\pi/\omega$. Since $T=2N\pi/\omega$, and N is an even integer, it follows that the sign of z' is the same at $t=0$ and $t=T/2$. Now suppose λ to increase to the value belonging to the physical problem. If the sign of $z'(T/2)$ changes it must do so by passing through zero. But in this case, since $z(T/2)$ is also zero, z would be identically zero. Therefore $z'(0)$ would also be zero and would change sign for the same value of λ, and z' would still have the same sign at $t=0$ and $t=T/2$. Consequently, the z-component of velocity at $t=T/2$ is not equal to the negative of that at $t=0$. Hence, when n is odd and N is even, the orbits for $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are geometrically distinct, because a single orbit can cross the x-axis perpendicularly but twice.

If n and N are both odd, the infinitesimal body crosses the x-axis perpendicularly, as before, both when the finite bodies are at their perifoci and also when they are at their apofoci. But though in this case the sign of z' at $t=0$ is opposite to that at $t=T/2$, the orbits for $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are distinct; for otherwise the motion of the infinitesimal body would be precisely the same while the finite bodies were moving from perifoci to apofoci as while they were moving from apofoci to perifoci. This is impossible because t enters the differential equations differently in the two cases.

If we set up the problem taking $t=0$ when the finite masses are at their apofoci, we shall find similarly two solutions; but they will be identical with these, for in these the infinitesimal body crosses the x-axis perpendicularly when the finite bodies are at their apofoci.

The numbers n and N have so far been taken relatively prime, and $T=2n\pi$. We shall now inquire whether there are other solutions with the period $T'=\kappa T$, where κ is an integer. The determinant Δ_2 vanishes for this value of t. Solutions having this period certainly exist, for they include as special cases those with the period T. Proceeding as in finding equation (41), the corresponding steps are taken, and it is found that in this case α_{01} and α_{20} differ from their former values only by multiples of 2π. Therefore the number of solutions with the period T' is the same as with the period T. Hence there are no new solutions whose periods are multiples of those considered.

115. Convergence.—The existence of the symmetrical three-dimensional periodic orbits has been proved except for a discussion of the convergence of the series which have been employed, a matter which must now be taken up. The nature of the difficulty will first be pointed out. Equations (13) were integrated as power series in the a_i, c_i, and λ. It was shown in §§ 14–16 that for any preassigned T the moduli of these parameters can be taken so small that the series converge for all $0 \overline{\gtrless} t \leqq T$. The limits on the moduli of the a_i, c_i, and λ are functions of μ, μ_0, and e. But T can not be taken arbitrarily in advance, for it is a discontinuous function of ω, which is in turn a function of μ, μ_0, and e. It is not evident *a priori* that values of μ, μ_0, and e, satisfying the relations $\mu-\mu_0=\lambda$, $\omega=f(\mu, \mu_0, e)$, exist such that all the series which are employed are convergent.

The final parameters of the solutions are μ, μ_0, e, and the mode of generalizing μ is arbitrary. Suppose the ratio of the finite masses is given, that is, that μ is a fixed number. Suppose also that the mode of generalizing μ has been determined. It will be shown that values of μ_0 and e exist such that the series all converge for $\lambda=\mu-\mu_0$.

In equations (25) it was shown that

$$\omega=\sqrt{A}+\omega_1 e+\omega_2 e^2+\cdots,$$

where $\sqrt{A}$ and the ω_i are functions of μ_0. Moreover, a detailed examination of the functional relation shows they are *continuous* functions of μ_0. Suppose for any μ_0 such that $|\mu-\mu_0|\leqq\epsilon_1>0$ the series for ω converges for $|e|\leqq\eta_1$, where η_1 depends on ϵ_1. It is easy to show from the nature of the dependence of the series for ω upon μ and μ_0 that $\eta_1>0$. Take any particular $\mu_0^{(1)}$ such that $|\mu-\mu_0^{(1)}|\leqq\epsilon_1$ and suppose that, while e runs over the range 0 to η_1, the value of $\omega\,(\mu_0, e)$ runs over the range $\omega\,(\mu_0^{(1)}, 0)=\sqrt{A}$ to $\omega\,(\mu_0^{(1)}, \eta_1)$. For brevity, let $\omega^{(0)}$ and $\omega^{(1)}$ represent the smallest and largest values of ω. An examination shows that ω_1, ω_2, . . . are not all identically zero, and it follows from this that $|\omega^{(1)}-\omega^{(0)}|>0$. Let $\mu_0^{(1)}$ take all real values such that $|\mu-\mu_0^{(1)}|\leqq\epsilon_1$ and find the corresponding values of $\omega^{(0)}$ and $\omega^{(1)}$. Let $\omega_g^{(0)}$ be the greatest $\omega^{(0)}$, and $\omega_l^{(1)}$ be the least $\omega^{(1)}$. The value of ϵ_1 can be taken so small that $\omega_l^{(1)}-\omega_g^{(0)}>0$, and hence from the continuity of ω as a function of μ_0 it follows that ω takes all values satisfying the inequalities $\omega_g^{(0)} \overline{\gtrless} \omega \leqq \omega_l^{(1)}$. Take any rational value of ω depending on μ, μ_0, and e which satisfies these inequalities, say

$$\omega_0=\frac{N_0}{n_0}, \tag{46}$$

where N_0 and n_0 are relatively prime integers. Then determine $T=T_0$ by the equation

$$T_0=2n_0\pi=\frac{2N_0\pi}{\omega_0}. \tag{47}$$

Now consider the series (31), in which we put $T=T_0$. Since they vanish identically for $a_i=c_i=0$, the discussion in §§14–16 shows that, for any values of λ and e for which the differential equations are regular, $r_i>0$ and $\rho_i>0$ can be so determined that the series converge for all $0 \gtreqqless T \leqq T_0$ provided $|a_i|<r_i$, $|c_i|<\rho_i$. The r_i and ρ_i are functions of μ_0, λ, and e. We may eliminate λ by the relation $\mu=\mu_0+\lambda$, and we shall take $|\mu-\mu_0| \leqq \epsilon_1$ and $0 \gtreqqless e \leqq \eta_1$, where ϵ_1 and η_1 are both distinct from zero and have such values that the differential equations are regular for $|\mu-\mu_0| \leqq \epsilon_1$, $e \gtreqqless \eta_1$. Let $r_i^{(0)}$ and $\rho_i^{(0)}$ be the least values of r_i and ρ_i as μ_0 and e take all values satisfying the inequalities $|\mu-\mu_0| \leqq \epsilon_1$, $0 \gtreqqless e \leqq \eta_1$. It has been shown that when the solutions of (31) exist they have the form

$$a_i=\lambda p_i(\lambda^{\frac{1}{2}}, e) \qquad\qquad c_1=-c_2=\lambda^{\frac{1}{2}} p_5(\lambda^{\frac{1}{2}}, e), \tag{48}$$

where the p_i are power series in $\lambda^{\frac{1}{2}}$ and e. These series will converge and give $|a_i|<r_i^{(0)}$, $|c_i|<\rho_i^{(0)}$ provided $0<|\lambda| \leqq \epsilon_2 \leqq \epsilon_1$, for all $e<\eta_1$. That is, since the series vanish identically for $\lambda=0$, the limits *on the a_i and c_i can be controlled by λ alone.* Choose any λ satisfying the inequality $|\lambda| \leqq \epsilon_2$ and determine μ_0 by the relation $\mu-\mu_0=\lambda$. It was shown above that for this μ_0 there exists an $e<\eta_1$ such that $\omega(\mu_0, e)=\omega_0=N_0/n_0$. Hence for this μ_0 and e all the series employed are convergent; that is, the existence of certain solutions of the type in question is proved.

The question might be asked whether the solutions exist if both μ and e are given in advance. It is not easy to make the answer in general, but it is clear that the mode of generalization of μ into μ and $\mu_0+\lambda$ opens a wide range of possibilities. This is so unless, indeed, the realm of validity of the results is independent of this process. Suppose μ, e, and the initial conditions are given such that the motion is periodic. The coördinates may be represented by

$$x=F_1(\mu, t), \qquad y=F_2(\mu, t), \qquad z=F_3(\mu, t).$$

Consider the expansions of the functions F_i as power series in λ, where $\mu=\mu_0+\lambda$ in at least part of the places in which μ occurs. It is clear that the realm of convergence of the series depends upon the manner of this transformation. For example, if $F_1=F_2=F_3=\sin t/(1+\lambda)(2+\mu)$, and if we put $1+\mu=1+\mu_0+\lambda$, $2+\mu=2+\mu$, then the functions are expansible as power series in λ which converge if $|\lambda|<1+\mu_0$, where λ and μ_0 are subject to the condition $\mu_0+\lambda=\mu$. But if we put $1+\mu=1+\mu$, $2+\mu=2+\mu_0+\lambda$, then the series in λ are convergent if $|\lambda|<2+\mu_0$ where λ and μ_0 are subject to the condition $\mu_0+\lambda=\mu$. For example, if $\mu=1/3$ in the first case the series converge if $|\lambda|<2/3$, and in the second case if $|\mu|<7/6$. Now it is clear from the variety of ways in which λ can be introduced that convergence of the series can be secured in many, if not all, cases when μ and e are given in advance.

116. The Existence of Two-Dimensional Symmetrical Periodic Orbits.— The last two equations of (31) are satisfied identically by $c_1 = c_2 = 0$, in which case the orbits become plane curves. We have to consider the solution of the first four equations for $a_1, \ldots, a_4$ in terms of λ.

The determinant Δ, equation (33), now becomes simply $\Delta = \Delta_1$. It follows from (34) that the condition $\Delta_1 = 0$ can be satisfied only by $\sigma = N/n$, where N and n are integers which we shall suppose are relatively prime.

We shall solve the first three equations of (31) for a_2, a_3, and a_4 as power series in a_1 and λ. This solution exists and is unique provided the determinant of the coefficients of the linear terms in a_2, a_3, and a_4 is distinct from zero. It follows from (34) that this determinant is

$$D = -A_3\left[e^{\rho\frac{T}{2}} - e^{-\rho\frac{T}{2}}\right],$$

which is distinct from zero unless A_3 is zero.

If A_3 were zero we should use the first, second, and fourth equations of (31). The determinant of the coefficients of the linear terms of a_2, a_3, and a_4 in these equations is found from (34) to be

$$D = -v_3\left(\frac{T}{2}\right)\left[e^{\rho\frac{T}{2}} - e^{-\rho\frac{T}{2}}\right],$$

which is distinct from zero unless $v_3(T/2)$ is zero. But A_3 and $v_3(T/2)$ can not both vanish, for then D_1 of (37) would be zero, whereas it is distinct from zero. Therefore a_2, a_3, and a_4 can always be eliminated by means of three of equations (31), leaving a single equation of the form

$$0 = a_1[\beta_{01}\lambda + \beta_{10}a_1 + \beta_{11}a_1\lambda + \beta_{20}a_1^2 + \cdots]. \tag{49}$$

This equation carries a_1 as a factor because equations (31) are identically satisfied by $a_1 = \cdots = a_4 = 0$.

Now consider equation (49). The trivial solution $a_1 = 0$ will be rejected, and we shall attempt to solve for a_1 as power series in $\lambda^{1/j}$, where j is an integer. If β_{01} is distinct from zero such a solution certainly exists, and j is determined by the first β_{j0} which is distinct from zero. The coefficient β_{01} is a power series in e whose term independent of e was found in Chapter VI, equation (26), to be distinct from zero. Consequently for e sufficiently small β_{01} is distinct from zero, and the solutions exist.

Now consider β_{10}. It was shown in Chapter VI, §100, that the part of this coefficient independent of e is zero. It will now be shown that it is identically zero. The solutions of the first three equations of (31) for a_2, a_3, and a_4 contain no terms of the first degree in a_1 alone, for, if we eliminate a_2 and a_3 by means of the first two equations, we have

$$A_3\left[e^{\rho\frac{T}{2}} - e^{-\rho\frac{T}{2}}\right]a_4 = 0 + \text{terms of second and higher degrees.} \tag{50}$$

It follows from property [3] of §108 that the terms in the solutions of the second degree in the a_i do not contain t as a factor. Hence, the only terms of the second degree in the a_i which are not periodic with the period T are those

which contain a_3^2 or a_4^2 as a factor. Hence the solution (50) for a_4 has a term of at least the third degree in a_1 as the term of lowest degree in a_1 alone. Consequently, when the first three equations of (31) are solved for a_2, a_3, and a_4 and substituted in the last one, the result contains no term of the second degree in a_1. That is, β_{10} is identically zero.

It is not necessary to consider β_{11} if β_{20} is distinct from zero. In Chapter VI, § 100, it was shown that the part of β_{20} which is independent of e is distinct from zero. It is also distinct from zero for e sufficiently small. Therefore the solutions are expansible as power series in $\pm\lambda^{\frac{1}{2}}$ of the form

$$x = \pm\lambda^{\frac{1}{2}} P(\pm\lambda^{\frac{1}{2}}; t), \qquad y = \pm\lambda^{\frac{1}{2}} Q(\pm\lambda^{\frac{1}{2}}; t).$$

One value of the double sign belongs to the orbit when the infinitesimal body crosses the axis in one direction, and the other when it crosses it in the other direction.

There are two cases to be considered, according as n is even or odd in the expression $\sigma = N/n$.

117. Case I, n even.—The infinitesimal body crosses the x-axis perpendicularly at $t=0$ and at $t=T/2=n\pi=2n'\pi$, where n' is an integer. Since $2n'\pi$ is an integral multiple of the period of revolution of the finite bodies, and since the infinitesimal body crosses the x-axis perpendicularly only at the beginning and middle of the period, it follows that it crosses the x-axis perpendicularly only when the finite bodies are at their perifoci. It follows, as in the case of the three dimensional orbits, that the orbit belonging to $-\lambda^{\frac{1}{2}}$ is not geometrically distinct from that belonging to $+\lambda^{\frac{1}{2}}$; in particular, that one orbit can be obtained from the other by increasing t by $T/2$.

Consider the terms which are even in $\lambda^{\frac{1}{2}}$. They are not altered by changing the sign of $\lambda^{\frac{1}{2}}$. Consequently in these terms

$$\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left[(j\sigma+k)t\right] \equiv + \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left[(j\sigma+k)(t+T/2)\right],$$

from which it follows that j is an even integer. In the terms of odd degree in $\lambda^{\frac{1}{2}}$ we have

$$\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left[(j\sigma+k)t\right] \equiv - \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left[(j\sigma+k)(t+T/2)\right],$$

from which it follows that j is an odd integer.

If we should set up the problem starting the infinitesimal body perpendicularly to the x-axis when the finite bodies are at their apofoci, we should find similarly two geometrically identical orbits in which the infinitesimal body crosses the x-axis obliquely when the finite bodies are at their perifoci. That is, when n is even there are two classes of geometrically distinct orbits of given period which intersect the x-axis perpendicularly; in one, the periodic orbits intersect it thus only when the finite bodies are at their perifoci, and in the other only when they are at their apofoci.

118. Case II, n odd.—If in the case where n is odd the infinitesimal body crosses the x-axis perpendicularly at $t=0$ and the finite bodies are at their perifoci, then it also crosses the x-axis perpendicularly at $t=T/2$ when the finite bodies are at their apofoci. If N is even, the infinitesimal body crosses the x-axis in the same direction at $t=0$ and $t=T/2$. Hence the orbits for $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are in this case geometrically distinct.

If N is odd, the orbits for $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are also distinct, because otherwise the motion of the infinitesimal body would be the same while the finite bodies are moving from perifoci to apofoci as while they are moving from apofoci to perifoci. This is impossible because t enters the differential equations differently in the two cases.

Similarly, starting when the finite bodies are at their apofoci, two geometrically distinct orbits are obtained, in both of which the infinitesimal body crosses the x-axis perpendicularly when the finite bodies are at their apofoci, and also when they are at their perifoci. These orbits, therefore, are identical with those obtained starting when the finite bodies were at their perifoci.

CONSTRUCTION OF THREE-DIMENSIONAL PERIODIC SOLUTIONS.

119. Defining Properties of the Solutions.—It has been shown that the periodic solutions have the form

$$x=\sum_{j=1}^{\infty} x_{2j}\lambda^{j}, \qquad y=\sum_{j=1}^{\infty} y_{2j}\lambda^{j}, \qquad z=\sum_{j=1}^{\infty} z_{2j-1}\lambda^{\frac{2j-1}{2}}, \tag{51}$$

where the x_j, y_j, and z_j are all periodic with the period $2\pi/\omega$. It has been shown that symmetrical orbits exist, two for each value of λ, and their coefficients are uniquely determined by the periodicity conditions and $z(0)=0$. It follows from this last relation and from the fact that the series for x, y, and z converge for all $|\lambda|$ sufficiently small, that each $z_j(0)$ separately is zero.

120. Coefficient of $\lambda^{\frac{1}{2}}$.—It follows from (13) that this term is defined by

$$z_1''+\left[A+3Ae\cos t+\tfrac{3}{2}Ae^2(1+3\cos 2t)+\cdots\right]z_1=0. \tag{52}$$

This equation is of the type of that treated in §53, and its general solution is of the form

$$z_1=c_1^{(1)}e^{\omega\sqrt{-1}t}w_1(e;t)+c_2^{(1)}e^{-\omega\sqrt{-1}t}w_2(e;t), \tag{53}$$

where

$$\omega=\sqrt{A}+\omega_1 e+\omega_2 e^2+\cdots, \qquad w_1=1+w_1^{(1)}e+w_1^{(2)}e^2+\cdots,$$

while w_2 differs from w_1 only in the sign of $\sqrt{-1}$, and where each $w_1^{(j)}$ is separately periodic with the period 2π. Since (53) is unchanged by changing the signs of both t and $\sqrt{-1}$, it follows that in the expressions for w_1 and w_2

the coefficients of the cosine terms are real, and those of the sine terms are purely imaginary. Since $w_1(e;\,0)=w_2(e;\,0)=1$ for all values of e, it follows from the condition $z_1(0)=0$ that $c_1^{(1)}+c_2^{(1)}=0$.

It will now be shown that ω is a series in even powers of e. The coefficient of z_1 in (52) is derived from the expansion of r_1^{-3} and r_2^{-3}. Now the expressions for r_1 and r_2 are unchanged if in them e is replaced by $-e$ and t by $t+\pi$. Consequently, if $e^{\omega(e)\sqrt{-1}t}w_1(e;\,t)$ is a solution of (52), then also $e^{\omega(-e)\sqrt{-1}(t+\pi)}w_1(-e;\,t+\pi)$ is a solution. Since any solution can be expressed linearly in terms of the two solutions in (53), and since the solutions now under consideration hold for all e sufficiently small, and, for $e=0$, differ only by the factor $e^{\omega_0\sqrt{-1}\,2\pi}$, it follows that for any e they differ only by a constant factor. Therefore

$$e^{\omega(e)\sqrt{-1}t}w_1\,(e;\,t)-Ce^{\omega(-e)\sqrt{-1}(t+\pi)}w_1\,(-e;\,t+\pi)=0.$$

This relation is satisfied identically in t and e. Since w_1 is periodic with the period 2π, it follows also that

$$e^{\omega(e)\sqrt{-1}\,2\pi}e^{\omega(e)\sqrt{1-}t}w_1\,(e;\,t)-e^{\omega(-e)\sqrt{-1}\,2\pi}Ce^{\omega(-e)\sqrt{-1}(t+\pi)}w_1\,(-e;\,t+\pi)=0.$$

Hence we have two homogeneous linear equations in $e^{\omega(e)\sqrt{-1}t}w_1\,(e;\,t)$ and $Ce^{\omega(-e)\sqrt{-1}(t+\pi)}w_1(-e;\,t+\pi)$, and as these quantities are not identically zero, the determinant

$$\begin{vmatrix} 1, & 1 \\ e^{\omega(e)\sqrt{-1}\,2\pi}, & e^{\omega(-e)\sqrt{-1}\,2\pi} \end{vmatrix} = e^{\omega(-e)\sqrt{-1}\,2\pi}-e^{\omega(e)\sqrt{-1}\,2\pi}$$

must vanish. Since by definition ω must reduce to $\sqrt{A}$ for $e=0$, we have $\omega(-e)\equiv\omega(e)$; that is, ω is a function of e^2.

Upon carrying out the computation by the method of §53, we find

$$\left.\begin{aligned}
\omega &= \sqrt{A}+0.e-\frac{3\sqrt{A}\,(1-A)}{4(1-4A)}\,e^2+0.e^3+\cdots,\\
w_1 &= 1-\left\{\frac{3A}{1-4A}\,(1-\cos t)+\frac{6A^{\frac{3}{2}}\sqrt{-1}}{1-4A}\sin t\right\}e\\
&\quad+\left\{-\frac{9A(1-15A+18A^2+8A^3)}{8(1-A)(1-4A)^2}-\frac{9A^2}{(1-4A)^2}\cos t+\frac{18A^{\frac{5}{2}}\sqrt{-1}}{(1-4A)^2}\sin t\right.\\
&\quad\left.+\frac{9A(1-3A-2A^2)}{8(1-A)(1-4A)}\cos 2t-\frac{9A^{\frac{3}{2}}(1-5A)\sqrt{-1}}{8(1-A)(1-4A)}\sin 2t\right\}e^2+\cdots
\end{aligned}\right\}\quad(54)$$

It will be noticed that the coefficients of $\cos jt$ and $\sin jt$ have e^j as a factor so far as they are written. This is a general property of the differential equations, and an examination of the process of integration, as explained in §53, shows it is also a general property of the solutions.

If we had solved the differential equations for x_1 and y_1, we should have found that these quantities are identically zero because of the periodicity conditions to which the solutions are subject.

121. Coefficients of λ.—It follows from (13) and (14) that these terms are defined by

$$\left.\begin{aligned} x_2''-2y_2'-\left[1+2A+6Ae\cos t+\cdots\right]x_2-\left[6Ae\sin t+\cdots\right]y_2&=Y_2,\\ y_2''+2x_2'-\left[6Ae\sin t+\cdots\right]x_2-\left[1-A-3Ae\cos t+\cdots\right]y_2&=X_2,\end{aligned}\right\} \quad (55)$$

where

$$X_2=\left[\tfrac{3}{2}B+6Be\cos t+\cdots\right]z_1^2,$$

$$Y_2=\left[3Be\sin t+\cdots\right]z_1^2,$$

$$B=\begin{smallmatrix}+\\+\\-\end{smallmatrix}\frac{1-\mu_0}{r_1^{(0)4}}\begin{smallmatrix}+\\-\\-\end{smallmatrix}\frac{\mu_0}{r_2^{(0)4}},$$

the signs in B being the first, second, or third pair according as the point (a), (b), or (c) is in question. By means of (53) and (54), we find

$$\left.\begin{aligned} z_1^2 =\; & +(c_1^{(1)})^2 e^{2\omega\sqrt{-1}t}\left\{1-\left[\frac{6A}{1-4A}(1-\cos t)+\frac{12A^{3/2}}{1-4A}\sqrt{-1}\sin t\right]e+\cdots\right\}\\ & +2c_1^{(1)}c_2^{(1)}\left\{1-\left[\frac{6A}{1-4A}(1-\cos t)\right]e+\cdots\right\}\\ & +(c_2^{(1)})^2 e^{-2\omega\sqrt{-1}t}\left\{1-\left[\frac{6A}{1-4A}(1-\cos t)-\frac{12A^{3/2}}{1-4A}\sqrt{-1}\sin t\right]e+\cdots\right\},\\ X_2 =\; & 3B(c_1^{(1)})^2 e^{2\omega\sqrt{-1}t}\left\{\frac{1}{2}+\left[\frac{-3A}{1-4A}+\frac{2-5A}{1-4A}\cos t+\frac{6A^{3/2}}{1-4A}\sqrt{-1}\sin t\right]e+\cdots\right\}\\ & +6Bc_1^{(1)}c_2^{(1)}\left\{\frac{1}{2}+\left[\frac{-3A}{1-4A}+\frac{2-5A}{1-4A}\cos t\right]e+\cdots\right\}\\ & +3B(c_2^{(1)})^2 e^{-2\omega\sqrt{-1}t}\left\{\frac{1}{2}+\left[\frac{-3A}{1-4A}+\frac{2-5A}{1-4A}\cos t-\frac{6A^{3/2}}{1-4A}\sqrt{-1}\sin t\right]e+\cdots\right\},\\ Y_2 =\; & 3B(c_1^{(1)})^2 e^{2\omega\sqrt{-1}t}\left\{[\sin t]e+\cdots\right\}+6Bc_1^{(1)}c_2^{(1)}\left\{[\sin t]e+\cdots\right\}\\ & +3Bc_2^{(1)2}e^{-2\omega\sqrt{-1}t}\left\{[\sin t]e+\cdots\right\}.\end{aligned}\right\} \quad (56)$$

The character of the solutions of equations of the type to which (55) belong was discussed in §29. It was there shown that they consist of the complementary function plus terms of the same character as X_2 and Y_2. It follows from the hypothesis that the imaginary characteristic exponent $\sigma\sqrt{-1}$, arising in the solution of (55), and $\omega\sqrt{-1}$ are incommensurable, that the constants of integration must all be put equal to zero. The particular integrals can be found most conveniently by assuming their form with undetermined coefficients, and then defining them by the conditions that the equations shall be identically satisfied.

We shall need the following properties of the solutions of equations (55). They are homogeneous of the second degree in $c_1^{(1)} e^{\omega\sqrt{-1}t}$ and $c_2^{(1)} e^{-\omega\sqrt{-1}t}$. The terms in the solutions multiplied by $(c_2^{(1)})^2$ differ from those multiplied by $(c_1^{(1)})^2$ only in the sign of $\sqrt{-1}$, because this is a property of the right members of the differential equations. If throughout equations (55) we change y_2 into $-y_2$, $\sqrt{-1}$ into $-\sqrt{-1}$, and t into $-t$, the equations are unchanged. Therefore, in the expression for x_2 the coefficients of the cosine terms are real and the coefficients of the sine terms are purely imaginary. The opposite is true for y_2. The terms in x_2 having $c_1^{(1)} c_2^{(1)}$ as a factor involve only cosines and are independent of the exponentials $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$, while those in y_2 having $c_1^{(1)} c_2^{(1)}$ as a factor involve only sines and are also independent of the exponentials $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$. It follows from these properties and $c_1^{(1)} = -c_2^{(1)}$, that $x_2'(0) = y_2(0) = 0$.

Certain divisors are introduced in the integration of (55). When the right members of these equations are omitted, the solutions are of the form (24). On using the method of the variation of parameters, we find

$$a_i' = \frac{F_i(t)}{\Delta} \qquad (i=1, \ldots, 4),$$

where $F_i(t)$ has the form of X_2 and Y_2, and where Δ is the determinant of the fundamental set of solutions (24). It follows from the principles of §18 applied to this case that Δ is constant. The expressions $F_i(t)$ contain terms of the types given in (56). Consequently the a_i contain terms of the types

$$\begin{aligned}\Delta a_i = (c_1^{(1)})^2 \int e^{2\omega\sqrt{-1}t}\left[A_j \cos jt + \sqrt{-1} B_j \sin jt\right] dt \\ + (c_2^{(1)})^2 \int e^{-\omega\sqrt{-1}t}\left[A_j \cos jt - \sqrt{-1} B_j \sin jt\right] dt + c_1^{(1)} c_2^{(1)} \int C_j \cos jt\, dt;\end{aligned}$$

or, performing the indicated integrations,

$$\begin{aligned}\Delta a_i = &+ \frac{(c_1^{(1)})^2 A_j}{j^2 - 4\omega^2} e^{+2\omega\sqrt{-1}t}\left[2\omega\sqrt{-1}\cos jt + j \sin jt\right] \\ &- \frac{(c_1^{(1)})^2 B_j e^{+2\omega\sqrt{-1}t}}{j^2 - 4\omega^2}\left[j\sqrt{-1}\cos jt + 2\omega \sin jt\right] \\ &+ \frac{(c_2^{(1)})^2 A_j}{j^2 - 4\omega^2} e^{-2\omega\sqrt{-1}t}\left[-2\omega\sqrt{-1}\cos jt + j \sin jt\right] \\ &+ \frac{(c_2^{(1)})^2 B_j e^{-2\omega\sqrt{-1}t}}{j^2 - 4\omega^2}\left[j\sqrt{-1}\cos jt + 2\omega \sin jt\right] + \frac{c_1^{(1)} c_2^{(1)}}{j} C_j \sin jt.\end{aligned}$$

Therefore, the divisor $j^2 - 4\omega^2$ appears in terms involving t in the form $e^{\pm 2\omega\sqrt{-1}t} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jt$, and the divisor j in those involving t in $\sin jt$.

It was seen that in the expression for z_1 the coefficients of $\cos jt$ and $\sin jt$ carry e^j as a factor. Consequently it is true also for z_1^2, and therefore for the A_j, B_j, and C_j.

122. Coefficient of $\lambda'^{3/2}$.—It is found from (13), (14), and (51) that

$$
\left.
\begin{aligned}
&z_3''+\left[A+3Ae\cos t+\cdots\right]z_3=Z_3,\\
&Z_3=+\left\{\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}+\cdots\right\}z_1+\left\{3B+\cdots\right\}x_2z_1\\
&\quad+\left\{\frac{3}{2}\left[\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right]+\cdots\right\}z_1^3+\left\{6Be\sin t+\cdots\right\}y_2z_1.
\end{aligned}
\right\}\tag{57}
$$

The coefficients of z_1, x_2z_1, and z_1^3 are series involving only cosines, and the coefficient of y_2z_1 is a series involving only sines.

Now consider the solutions of (57). The general solution of the left member set equal to zero is

$$
z_3=C_1e^{\omega\sqrt{-1}t}w_1+C_2e^{-\omega\sqrt{-1}t}w_2,
$$

where C_1 and C_2 are the constants of integration. By the method of the variation of parameters, the conditions on C_1 and C_2 that (57) shall be satisfied when its right member is included are

$$
\left.
\begin{aligned}
&e^{\omega\sqrt{-1}t}w_1C_1'+e^{-\omega\sqrt{-1}t}w_2C_2'=0,\\
&e^{\omega\sqrt{-1}t}[\omega\sqrt{-1}\,w_1+w_1']C_1'+e^{-\omega\sqrt{-1}t}[-\omega\sqrt{-1}\,w_2+w_2']C_2'=Z_3.
\end{aligned}
\right\}\tag{58}
$$

Upon solving (58) for C_1' and C_2', we get

$$
\Delta C_1'=-w_2Z_3e^{-\omega\sqrt{-1}t},\qquad \Delta C_2'=+w_1Z_3e^{\omega\sqrt{-1}t},\tag{59}
$$

where

$$
\Delta=\begin{vmatrix} w_1\quad, & w_2\\ \omega\sqrt{-1}\,w_1+w_1', & -\omega\sqrt{-1}\,w_2+w_2'\end{vmatrix}.
$$

It follows from the results of §18 that in this case Δ is constant, and since w_1 and w_2 are power series in e the determinant Δ is also a power series in e.

In order that the solutions of (57) shall be periodic with the period T it is necessary and sufficient that the right members of (59) contain no constant terms. We must therefore pick out the terms in $-w_2Z_3$ and w_1Z_3 which are constants multiplied by $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ respectively, and set their coefficients equal to zero.

Consider the first term of Z_3. Upon substituting the value of z_1 from (53), we see that the constant part of the coefficient of $e^{\omega\sqrt{-1}t}$ coming from $-w_2Z_3$ is the constant part of the product

$$
-c_1^{(1)}w_1w_2\left\{\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}+3Be\cos t+\cdots\right\}.\tag{60}
$$

The corresponding part of the coefficient of $e^{-\omega\sqrt{-1}t}$ coming from $+w_1 Z_3$ is the constant part of

$$+c_2^{(1)} w_1 w_2 \left\{ \frac{1}{r_1^{(0)3}} - \frac{1}{r_2^{(0)3}} + 3Be\cos t + \cdots \right\}. \tag{61}$$

It follows from the properties of w_1 and w_2 that their product involves only cosines and that the coefficients of their product are all real. Hence the constant parts of (60) and (61) are power series in e which, aside from the coefficients $c_1^{(1)}$ and $c_2^{(1)}$, differ only in sign, and the parts independent of e are respectively

$$-c_1^{(1)}\left(\frac{1}{r_1^{(0)3}} - \frac{1}{r_2^{(0)3}}\right), \qquad +c_2^{(1)}\left(\frac{1}{r_1^{(0)3}} - \frac{1}{r_2^{(0)3}}\right).$$

Consider now the terms coming from that part of Z_3 which contains $x_2 z_1$ as a factor. The constant parts of the coefficients of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ in the products

$$-w_2 x_2 z_1 \left[3B + 12Be\cos t \cdots\right]$$

and

$$+w_1 x_2 z_1 \left[3B + 12Be\cos t \cdots\right]$$

respectively are required. It follows from the properties of w_1 and w_2 that

$$w_1 \left[3B + 12Be\cos t + \cdots\right]$$

and

$$w_2 \left[3B + 12Be\cos t + \cdots\right]$$

differ only in the sign of $\sqrt{-1}$, which is a factor of all the sine terms, while the coefficients of the cosine terms are all real. These products do not involve the exponentials $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$. In the product $x_2 z_1$ the coefficients of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ are multiplied by $(c_1^{(1)})^2 c_2^{(1)}$ and $c_1^{(1)}(c_2^{(1)})^2$ respectively. It follows from the properties of x_2 and z_1 that in this product the coefficients of $(c_1^{(1)})^2 c_2^{(1)} e^{\omega\sqrt{-1}t}$ and $c_1^{(1)}(c_2^{(1)})^2 e^{-\omega\sqrt{-1}t}$ differ only in the sign of $\sqrt{-1}$, which is a factor of all the sine terms, while the coefficients of the cosine terms are all real.

The typical terms of the products $-w_2[3B + 12Be\cos t \cdots]$ and that part of the product $x_2 z_1$ which contain $e^{\omega\sqrt{-1}t}$ as a factor are respectively

$$-A_j\cos jt - \sqrt{-1}B_j\sin jt, \qquad (c_1^{(1)})^2 c_2^{(1)}[a_j\cos jt + \sqrt{-1}b_j\sin jt].$$

The corresponding terms from $+w_1[3B + 12Be\cos t + \cdots]$ and the part of $x_2 z_1$ containing $e^{-\omega\sqrt{-1}t}$ as a factor are respectively

$$+A_j\cos jt - \sqrt{-1}B_j\sin jt, \qquad c_1^{(1)}(c_2^{(1)})^2[a_j\cos jt - \sqrt{-1}b_j\sin jt].$$

The constant parts of the products of these terms are respectively

$$-\frac{1}{2}(c_1^{(1)})^2 c_2^{(1)}[A_j a_j - B_j b_j], \qquad +\frac{1}{2}c_1^{(1)}(c_2^{(1)})^2[A_j a_j - B_j b_j].$$

Since these properties hold each term individually, it follows that the constant parts of the coefficients of $e^{\omega\sqrt{-1}t}$ and $e^{-\sqrt{\omega-1}t}$ in $-w_2 x_2 z_1[3B+\cdots]$ and $+w_1 x_2 z_1[3B+\cdots]$ are respectively of the form

$$-(c_1^{(1)})^2 c_2^{(1)}[C_0+C_1 e+\cdots], \qquad +c_1^{(1)}(c_2^{(1)})^2[C_0+C_1 e+\cdots]. \quad (62)$$

The corresponding discussion shows that the same result is true for the third and fourth terms of Z_3. Therefore, in order that the solutions of (57) shall be periodic, we must impose the conditions

$$0=Kc_1^{(1)}+L(c_1^{(1)})^2 c_2^{(1)}, \qquad 0=Kc_2^{(1)}+Lc_1^{(1)}(c_2^{(1)})^2, \quad (63)$$

where K and L are constants and power series in e. It follows from the corresponding work in Chapter VI that both K and L have terms independent of e which are distinct from zero. The solutions of (63) are

$$c_1^{(1)}=c_2^{(1)}=0, \qquad K+Lc_1^{(1)}c_2^{(1)}=0. \quad (64)$$

The former leads to the trivial solution $x\equiv y\equiv z\equiv 0$ and need not be considered. Since $c_2^{(1)}=-c_1^{(1)}$ the latter gives

$$c_1^{(1)}=\pm\sqrt{\frac{K}{L}}=\pm[K_0+K_1 e+\cdots], \quad (65)$$

and $c_1^{(1)}$ and $c_2^{(1)}$ are determined except as to sign. The orbits which correspond to the two values of $c_1^{(1)}$ are geometrically identical or distinct according as T is an odd or even multiple of 2π (see §§113–114).

After the conditions (63) are satisfied, the solutions of (57) are

$$z_3=c_1^{(3)}e^{\omega\sqrt{-1}t}w_1(e;t)+c_2^{(3)}e^{-\omega\sqrt{-1}t}w_2(e;t)+P_1^{(3)}(t)+P_3^{(3)}(t), \quad (66)$$

where $P_1^{(3)}$ is linear and homogeneous in $c_1^{(1)}e^{\omega\sqrt{-1}t}$ and $c_2^{(1)}e^{-\omega\sqrt{-1}t}$, and where $P_3^{(3)}$ is homogeneous of the third degree in $c_1^{(1)}e^{\omega\sqrt{-1}t}$ and $c_2^{(1)}e^{-\omega\sqrt{-1}t}$. In the right member of (57) the coefficients of all cosine terms are real, and the coefficients of all sine terms are purely imaginary; and moreover those which are multiplied by $c_1^{(1)}$, $(c_1^{(1)})^3$, $(c_1^{(1)})^2 c_2^{(1)}$ differ from those respectively which are multiplied by $c_2^{(1)}$, $(c_2^{(1)})^3$, $c_1^{(1)}(c_2^{(1)})^2$ only in the sign of $\sqrt{-1}$. It follows that $P_1^{(3)}$ and $P_3^{(3)}$ have these properties also. The w_1 and w_2 are given in (53) and (54).

The $P_1^{(3)}$ and $P_3^{(3)}$ are entirely known functions, while $c_1^{(3)}$ and $c_2^{(3)}$ are subject to the relation

$$z_3(0)=c_1^{(3)}+c_2^{(3)}+P_1^{(3)}(0)+P_3^{(3)}(0)=0.$$

It follows from the properties of the expressions $P_1^{(3)}(t)$ and $P_3^{(3)}(t)$ and $c_1^{(1)} = -c_2^{(1)}$ that $P_1^{(3)}(0) = P_3^{(3)}(0) = 0$. Hence $c_1^{(3)}$ and $c_2^{(3)}$ are subject to the condition

$$z_3(0) = c_1^{(3)} + c_2^{(3)} = 0. \tag{67}$$

Therefore but one undetermined constant remains in (66).

The divisors introduced at this step can be found as they were in the preceding step. They are the determinant of the fundamental set of solutions of the z-equation, $j^2 - \omega^2$, and $j^2 - 9\omega^2$, the $j^2 - \omega^2$ appearing with terms which involve $e^{\pm\omega\sqrt{-1}t} \frac{\cos}{\sin} jt$, and the $j^2 - 9\omega^2$ with those which involve $e^{\pm 3\omega\sqrt{-1}t} \frac{\cos}{\sin} jt$. The coefficients of $\frac{\cos}{\sin} jt$ contain e^j as a factor.

123. Coefficients of λ^2.—The differential equations at this step are

$$\left.\begin{aligned} x_4'' - 2y_4' - \left[1 + 2A + 6Ae\cos t + \cdots\right]x_4 - \left[6Ae\sin t + \cdots\right]y_4 &= X_4, \\ y_4'' + 2x_4' - \left[6Ae\sin t + \cdots\right]x_4 - \left[1 - A - 3Ae\cos t + \cdots\right]y_4 &= Y_4, \end{aligned}\right\} \tag{68}$$

where X_4 and Y_4 are of even degree in $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ considered together. It follows, as in the case of x_2 and y_2, that in the right member of the first equation the coefficients of all cosine terms are real, and those of all sine terms are purely imaginary; the opposite is true in the right member of the second equation. The coefficients of $c_1^{(1)} c_1^{(3)}$ and $(c_1^{(1)})^4$ differ from those of $c_2^{(1)} c_2^{(3)}$ and $(c_2^{(1)})^4$ respectively only in the sign of $\sqrt{-1}$. The coefficients of $(c_1^{(1)})^2 (c_2^{(1)})^2$ are real, independent of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$, and involve only cosines. The solutions of (68) have the same properties. It follows from these properties and $c_1^{(1)} = -c_2^{(1)}$, that $x_4'(0) = y_4(0) = 0$.

The divisors introduced in the solution are the determinant of the fundamental set of solutions of the x and y-equations, $j^2 - 16\omega^2$, $j^2 - 4\omega^2$, and j in connection with the terms involving $e^{\pm 4\omega\sqrt{-1}t} \frac{\cos}{\sin} jt$, $e^{\pm\omega\sqrt{-1}t} \frac{\cos}{\sin} jt$, and $\frac{\cos}{\sin} jt$ respectively. The coefficients of these terms contain e^j as a factor.

124. Coefficient of $\lambda^{5/2}$.—It is necessary to consider this step in order to be able to make the general discussion. The differential equation defining z_5 is

$$z_5'' + \left[A + 3Ae\cos t + \cdots\right]z_5 = Z_5, \tag{69}$$

where it is found from (14) and properties (a) and (d) of § 106 that Z_5 is made up of real cosine series multiplied into z_3, $x_2 z_3$, $x_4 z_1$, $x_2^2 z_1$, $y_2^2 z_1$, $z_1^2 z_3$, z_1^5, and of real sine series multiplied into $y_2 z_3$ and $y_4 z_1$. It follows from the properties of z_1, z_3, x_2, y_2, x_4, and y_4 that Z_5 is of odd degree in $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ taken together; that the coefficients of the cosine terms

are real, and those of the sine terms purely imaginary; that $c_1^{(3)}$ and $c_2^{(3)}$ enter linearly; that the coefficients of $c_1^{(3)}$, $(c_1^{(1)})^2c_2^{(3)}$, $c_1^{(1)}c_2^{(1)}c_1^{(3)}$, $(c_2^{(1)})^3(c_2^{(1)})^2$ differ from the coefficients of $c_2^{(3)}$, $(c_2^{(1)})^2c_1^{(3)}$, $c_1^{(1)}c_2^{(1)}c_2^{(3)}$ and $(c_1^{(1)})^2(c_2^{(1)})^3$ respectively only in the sign of $\sqrt{-1}$; and that the terms which involve $\cos jt$ and $\sin jt$ have e^j as a factor in their coefficients.

Now consider the solution of (69). The discussion proceeds as in the case of (57), and it is found that, in order that the solution shall be periodic, the constant terms in $-w_2 Z_5 e^{-\omega\sqrt{-1}t}$ and $+w_1 Z_5 e^{\omega\sqrt{-1}t}$ must be zero. It follows from the properties of Z_5 enumerated above that these conditions are of the form

$$\left.\begin{aligned}
&[A_1+A_2c_1^{(1)}c_2^{(1)}]\,c_1^{(3)}+A_3\,(c_1^{(1)})^2\,c_2^{(3)}+A_4(c_1^{(1)})^3(c_1^{(1)})^2=0,\\
&[A_1+A_2c_1^{(1)}c_2^{(1)}]\,c_2^{(3)}+A_3\,(c_2^{(1)})^2\,c_1^{(3)}+A_4(c_1^{(1)})^2(c_2^{(1)})^3=0,
\end{aligned}\right\}\quad(70)$$

where $A_1, \ldots, A_4$ are power series in e. Upon reducing by means of the last equation of (64), we get

$$\left.\begin{aligned}
&\left[A_1-\frac{A_2K}{L}\right]c_1^{(3)}+\frac{A_3K^2}{L^2}\,c_2^{(3)}+\frac{A_4K^2}{L^2}\,c_1^{(1)}=0,\\
&\left[A_1-\frac{A_2K}{L}\right]c_2^{(3)}+\frac{A_3K^2}{L^2}\,c_1^{(3)}+\frac{A_4K^2}{L^2}\,c_2^{(1)}=0.
\end{aligned}\right\}\quad(71)$$

Since $c_2^{(1)}=-c_1^{(1)}$ and $c_2^{(3)}=-c_1^{(3)}$, these equations are equivalent and define $c_1^{(3)}$ provided

$$A_1-\frac{A_2K}{L}-\frac{A_3K^2}{L^2}$$

is distinct from zero. In the case $e=0$ of Chapter VI the corresponding coefficient was distinct from zero. Since the coefficient in the present case is a power series in e and reduces to that of the former case for $e=0$, it follows that it is distinct from zero for e sufficiently small.

When equations (71) are satisfied, the solution of (69) has the form

$$z_5=c_1^{(5)}\,e^{\omega\sqrt{-1}t}\,w_1+c_2^{(5)}\,e^{-\omega\sqrt{-1}t}\,w_2+P_1^{(5)}(t)+P_3^{(5)}(t)+P_5^{(5)}(t),\qquad(72)$$

where $P_3^{(5)}(t)$ is linear and homogeneous in $c_1^{(3)}e^{\omega\sqrt{-1}t}$ and $c_2^{(3)}e^{-\omega\sqrt{-1}t}$ taken together, and $P_3^{(5)}(t)$ and $P_5^{(5)}(t)$ are homogeneous of the third and fifth degrees respectively in $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ taken together.

It follows from the facts noted above that Z_5 is unchanged if $c_1^{(3)}$, $c_1^{(1)}$, $+\sqrt{-1}$ are interchanged with $c_2^{(3)}$, $c_2^{(1)}$, $-\sqrt{-1}$, and that the coefficients of the cosine terms are real; hence the solution (72) has the same properties. Consequently, since $c_2^{(1)}=-c_1^{(1)}$, $c_1^{(3)}=-c_1^{(3)}$, it follows that $P_1^{(5)}(0)=P_3^{(5)}(0)=P_5^{(5)}(0)=0$; and since $z_5(0)=0$, that

$$c_1^{(5)}+c_2^{(5)}=0.\qquad(73)$$

125. The General Step for the x and y-Equations.—Suppose z_1, x_2, y_2, . . . , $z_{2\nu-1}$, $x_{2\nu-2}$, $y_{2\nu-2}$ have been computed and that they have the following properties:

(A) The x_{2j} and y_{2j} are even functions of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ taken together, and the z_{2j+1} are odd functions of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ taken together.

(B) In the x_{2j} and z_{2j+1} the constant parts of the coefficients of all cosine terms are real, and those of all sine terms are purely imaginary.

(C) In the y_{2j} the constant parts of the coefficients of all cosine terms are purely imaginary, and those of all sine terms are real.

(D) In the x_{2j} and y_{2j} the highest powers of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ are $2j$, and in z_{2j+1} they are $2j+1$.

(E) The coefficients of $(c_1^{(1)})^{j_1}(c_1^{(3)})^{j_3} \cdots (c_1^{(l)})^{j_l}(c_2^{(1)})^{k_1}(c_2^{(3)})^{k_3} \cdots (c_2^{(m)})^{k_m}$ in the expressions for x_{2j}, y_{2j}, and z_{2j+1} differ from the coefficients of $(c_2^{(1)})^{j_1}(c_2^{(3)})^{j_3} \cdots (c_2^{(l)})^{j_l}(c_1^{(1)})^{k_1}(c_1^{(3)})^{k_3} \cdots (c_1^{(m)})^{k_m}$ only in the sign of $\sqrt{-1}$.

(F) The constants of integration arising at the step $2j+1$, viz., $c_1^{(2j+1)}$ and c_2^{2j+1} $(j=1, \ldots, \nu-2)$, must satisfy the relations

$$c_1^{(2j+1)} = M^{(2j+1)}\, c_1^{(1)}, \qquad c_2^{(2j+1)} = M^{(2j+1)}\, c_2^{(1)} = -M^{(2j+1)}\, c_1^{(1)},$$

where $M^{(2j+1)}$ is a power series in e, in order that the solution at the step $2j+3$ shall be periodic. The constants $c_1^{(2\nu-1)}$ and $c_2^{(2\nu-1)}$ remain arbitrary at the step 2ν.

(G) The divisors introduced at the step $2j$ are the determinant of the fundamental set of solutions of the x and y-equations and k, $k^2-4\omega^2$, $k^2-16\omega^2$, . . . , $k^2-4j^2\omega^2$ in the coefficients of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} kt$; and at the step $2j+1$ they are the determinant of the fundamental set of solutions of the z-equation and k, $k^2-\omega^2$ $k^2-9\omega^2$, . . . , $k^2-(2j+1)^2\omega^2$.

The differential equations which define $x_{2\nu}$ and $y_{2\nu}$ are

$$\left.\begin{aligned} &x''_{2\nu}-2y'_{2\nu}-[1+2A+6Ae\cos t\cdots]x_{2\nu}-[6Ae\sin t\cdots]y_{2\nu}=X_{2\nu}\\ &y''_{2\nu}+2x'_{2\nu}-\quad[6Ae\sin t\cdots]x_{2\nu}-[1-A-3Ae\cos t\cdots]y_{2\nu}=Y_{2\nu}\end{aligned}\right\}\quad(74)$$

It follows from the properties (a), (b), (c), and (d) of §106 that $X_{2\nu}$ and $Y_{2\nu}$ are even functions of z_1, . . . , $z_{2\nu-1}$ taken together; that in $X_{2\nu}$ the coefficients of all terms which are of even degree in y_2, y_4, . . . , $y_{2\nu-2}$ taken together are sums of cosines of integral multiples of t; that in $X_{2\nu}$ the coefficients of all terms which are of odd degree in y_2, y_4, . . . , $y_{2\nu-2}$ taken together are sums of sines of integral multiples of t; that the last two properties are reversed in the case of $Y_{2\nu}$; and that if in $X_{2\nu}$ or $Y_{2\nu}$ the general term has as a factor

$$x_2^{\lambda_2}\cdots x_{2j}^{\lambda_{2j}}\cdot y_2^{\mu_2}\cdots y_{2\kappa}^{\mu_{2\kappa}}\cdot z_1^{\nu_1}\cdots z_{2l-1}^{\nu_{2l-1}},$$

then

$$2\lambda_2+\cdots+2j\lambda_{2j}+2\mu_2+\cdots+2\kappa\mu_{2\kappa}+\nu_1+\cdots(2l-1)\nu_{2l-1} \gtreqless 2\nu.$$

It follows from these properties and (A), . . . , (G) that $X_{2\nu}$ and $Y_{2\nu}$ are even functions of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ taken together; that in $X_{2\nu}$ the constant parts of the coefficients of all cosine terms are real, and those of all sine terms are purely imaginary; that in $Y_{2\nu}$ the last property is reversed; that in $X_{2\nu}$ and $Y_{2\nu}$ the highest powers of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ are 2ν; and that the coefficients of $(c_1^{(1)})^{j_1}(c_1^{(3)})^{j_3} \cdots (c_1^{(l)})^{j_l}(c_2^{(1)})^{k_1}(c_2^{(3)})^{k_3} \cdots (c_2^{(m)})^{k_m}$ differ from the coefficients of $(c_2^{(1)})^{j_1}(c_2^{(3)})^{j_3} \cdots (c_2^{(l)})^{j_l}(c_1^{(1)})^{k_1}(c_1^{(3)})^{k_3} \cdots (c_1^{(m)})^{k_m}$ only in the sign of $\sqrt{-1}$.

It follows from the form of (74) and these properties that the periodic solutions of equations (74) (i. e., the particular integrals) have the properties (A), . . . , (G) so far as they pertain to x_{2j} and y_{2j}; and then from $c_1^{(1)} = -c_2^{(1)}$, that $x'_{2\nu}(0) = y_{2\nu}(0) = 0$.

126. The General Step for the z-Equation.—The differential equation defining $z_{2\nu+1}$ is

$$z''_{2\nu+1} + [A + 3Ae\cos t + \cdots]z_{2\nu+1} = Z_{2\nu+1}. \tag{75}$$

It follows from the properties (a), . . . , (g) of §106 that $Z_{2\nu+1}$ is of odd degree in $z_1, z_3, \ldots, z_{2\nu-1}$ taken together, and that it contains $z_{2\nu-1}$ linearly. In $Z_{2\nu+1}$ the coefficients of terms which are of even degree in $y_2, \ldots, y_{2\nu}$ taken together are cosines of integral multiples of t, and the coefficients of those terms which are odd in the same quantities are sines of integral multiples of t.

It follows from these properties and (A), . . . , (G) of §125 that $Z_{2\nu+1}$ is an odd function of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ taken together; that the highest power of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ is $2\nu+1$; that $c_1^{(2\nu-1)}$ and $c_2^{(2\nu-1)}$ enter linearly; that the constant parts of the coefficients of all cosine terms are real, and that those of all sine terms are purely imaginary; and that the coefficients of $(c_1^{(1)})^{j_1} \cdots (c_1^{(l)})^{j_l} (c_2^{(1)})^{k_1} \cdots (c_2^{(m)})^{k_m}$ differ from the coefficients of $(c_2^{(1)})^{j_1} \cdots c_2^{(l)})^{j_l} (c_1^{(1)})^{k_1} \cdots (c_1^{(m)})^{k_m}$ only in the sign of $\sqrt{-1}$.

Now consider the solution of (75). The conditions that it shall be periodic with the period T are that in $-w_2 Z_{2\nu+1}$ and $+w_1 Z_{2\nu+1}$ the constant parts of the coefficients of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ respectively shall be equal to zero. The $z_{2\nu-1}$ enters $Z_{2\nu+1}$ linearly and has the same coefficient that z_3 has in Z_5. Hence, from the relations of the preceding paragraph and equations (70), we have

$$\left.\begin{aligned} [A_1 + A_2 c_1^{(1)} c_2^{(1)}]\, c_1^{(2\nu-1)} + A_3\, (c_1^{(1)})^2\, c_2^{(2\nu-1)} + P_{2\nu+1}(c_1^{(1)},\, c_2^{(1)}) = 0, \\ [A_1 + A_2 c_1^{(1)} c_2^{(1)}]\, c_2^{(2\nu-1)} + A_3\, (c_2^{(1)})^2\, c_1^{(2\nu-1)} + P_{2\nu+1}(c_2^{(1)},\, c_1^{(1)}) = 0, \end{aligned}\right\} \tag{76}$$

where $P_{2\nu+1}$ is a polynomial of odd degree in $c_1^{(1)}$ and $c_2^{(1)}$ taken together. It is supposed that $c_1^{(2j+1)}$ and $c_2^{(2j+1)}$ $(j=1, \ldots, \nu-2)$ have been eliminated at the successive steps by the equations corresponding to (71). If the general term in $P_{2\nu+1}$ is $(c_1^{(1)})^j(c_2^{(1)})^k$, then j and k satisfy the relation

$$j = k + 1. \tag{77}$$

On reducing (76) by means of (64), making use of (77) and (F) of §125, and the fact that $c_2^{(1)}=-c_1^{(1)}$, it is seen that equations (76) are equivalent, and that $c_1^{(2\nu-1)}=c_2^{(2\nu-1)}$ is uniquely determined as a power series in e. Since in $Z_{2\nu+1}$ the coefficient of $(c_1^{(1)})^{j_1}\cdots(c_1^{(l)})^{j_l}(c_2^{(1)})^{k_1}\cdots(c_2^{(m)})^{k_m}$ differs from that of $(c_2^{(1)})^{j_1}\cdots(c_2^{(l)})^{j_l}(c_1^{(1)})^{k_1}\cdots(c_1^{(m)})^{k_m}$ only in the sign of $\sqrt{-1}$, it follows that the solution has the same property, and from this that $z_{2\nu+1}(0)=0$.

For small values of ν there will be no other terms than those considered above in $-w_2Z_{2\nu+1}$ and $+w_1Z_{2\nu+1}$, which are constants multiplied by the exponentials $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ respectively. When there are no other terms and when equations (76) are satisfied, the solution of (75) is periodic and $z_{2\nu+1}$ has all the properties of z_{2j-1} specified in §125.

But since $\omega=N/n$, where n and N integers, there is a value of ν for which other terms of the type in question can arise. It follows from the properties of $Z_{2\nu+1}$ that $-w_2Z_{2\nu+1}$ and $+w_1Z_{2\nu+1}$ contain respectively the terms

$$\left.\begin{aligned}
&-C^{(\nu)}(c_1^{(1)})^{2\nu+1}e^{+(2\nu+1)\omega\sqrt{-1}t}[a_0+a_1\cos t+\cdots+a_k\cos kt+\cdots\\
&\qquad\qquad-\sqrt{-1}\,b_1\sin t-\cdots-\sqrt{-1}\,b_k\sin kt-\cdots],\\
&+C^{(\nu)}(c_2^{(1)})^{2\nu+1}e^{-(2\nu+1)\omega\sqrt{-1}t}[a_0+a_1\cos t+\cdots+a_k\cos kt+\cdots\\
&\qquad\qquad+\sqrt{-1}\,b_1\sin t+\cdots+\sqrt{-1}\,b_k\sin kt+\cdots].
\end{aligned}\right\}\quad(78)$$

Now $e^{(2\nu+1)\omega\sqrt{-1}t}=e^{\omega\sqrt{-1}t}[\cos 2\nu\omega t+\sqrt{-1}\sin 2\nu\omega t]$. Consequently terms of the type in question will arise if $2\nu\omega=k$, k being an integer. Upon substituting the value of ω, this relation becomes $2\nu n=kN$, which can be satisfied when 2ν becomes a multiple of N. Suppose the integer n is odd. In this case when $2\nu=N$, the smallest k satisfying the relation, viz. $k=n$, is obtained. The term in which this occurs is multiplied by $\lambda^{2\nu+1/2}e^n=\lambda^{N+1/2}e^n$. But if n is even and N odd, the relation is satisfied first for increasing values of ν when $2\nu=2N$, and then $k=2n$. The term in which this relation occurs is multiplied by $\lambda^{2\nu+1/2}e^{2n}=\lambda^{2N+1/2}e^{2n}$. After terms of this type once appear they in general occur similarly at all subsequent steps of the integration.

The coefficients of $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$ obtained from (78) are respectively $-C_{2\nu+1}(c_1^{(1)})^{2\nu+1}$ and $+C_{2\nu+1}(c_2^{(1)})^{2\nu+1}$, where $C_{2\nu+1}$ is a constant multiplied by e^n or e^{2n} according as N is even or odd. Therefore, when these terms arise we have in place of equations (76)

$$\left.\begin{aligned}
&[A_1+A_2c_1^{(1)}c_2^{(1)}]\,c_1^{(2\nu-1)}+A_3(c_1^{(1)})^2c_2^{(2\nu-1)}+P_{2\nu+1}(c_1^{(1)},\ c_2^{(1)})-C_{2\nu+1}(c_1^{(1)})^{2\nu+1}=0,\\
&[A_1+A_2c_1^{(1)}c_2^{(1)}]\,c_2^{(2\nu-1)}+A_3(c_2^{(1)})^2c_1^{(2\nu-1)}+P_{2\nu+1}(c_2^{(1)},\ c_1^{(1)})-C_{2\nu+1}(c_2^{(1)})^{2\nu+1}=0.
\end{aligned}\right\}\quad(79)$$

Consequently $c_1^{(2\nu-1)}$ and $c_2^{(2\nu-1)}$ are determined in this case as well as in that in which the terms multiplied by $C_{2\nu+1}$ do not arise. This completes the proof of the possibility of constructing the solutions.

APPLICATION OF THE INTEGRAL.

127. Form of the Integral.—Equations (1) can be written in the form

$$\left.\begin{array}{ll}\dfrac{d^2\xi}{dt^2}-2\dfrac{d\eta}{dt}=\dfrac{\partial U}{\partial\xi}, & \dfrac{d^2\zeta}{dt^2}=\dfrac{\partial U}{\partial\zeta},\\[2ex] \dfrac{d^2\eta}{dt^2}+2\dfrac{d\xi}{dt}=\dfrac{\partial U}{\partial\eta}, & U=\frac{1}{2}(\xi^2+\eta^2)+\dfrac{1-\mu}{r_1}+\dfrac{\mu}{r_2}.\end{array}\right\} \tag{80}$$

After the transformations (10) and (11), these equations have the form

$$x''-2y'=\frac{\partial U}{\partial x},\qquad y''+2x'=\frac{\partial U}{\partial y},\qquad z''=\frac{\partial U}{\partial z}, \tag{81}$$

where now U is a power series in x, y, and z^2 and contains no terms lower than the second degree in x, y, and z. It contains terms independent of λ and others, for the particular transformation (11), multiplied by λ to the first degree only. The coefficients in the series for U are power series in e whose coefficients, in turn, are periodic in t with the period 2π, and which reduce to constants for $t=0$.

The first terms of U are seen from equations (10) and (11) to be

$$\left.\begin{aligned}U=&+\frac{1}{2}\Big[1+2A+6Ae\cos t+\cdots\Big]x^2+\Big[6Ae\sin t+\cdots\Big]xy\\
&+\frac{1}{2}\Big[1-A-3Ae\cos t+\cdots\Big]y^2-\frac{1}{2}\Big[A+3Ae\cos t+\cdots\Big]z^2\\
&+\left[\frac{-1}{r_1^{(0)3}}+\frac{1}{r_2^{(0)3}}-3\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\cos t+\cdots\right]x^2\lambda\\
&-\left[6\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\sin t+\cdots\right]xy\lambda\\
&+\frac{1}{2}\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}+3\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\cos t+\cdots\right]y^2\lambda\\
&+\frac{1}{2}\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}+3\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\cos t+\cdots\right]z^2\lambda\\
&+\left[\frac{3}{8}\left(\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right)+\frac{3}{2}\left(\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right)e\cos t+\cdots\right]z^4+\cdots.\end{aligned}\right\} \tag{82}$$

The integral of equations (81), analogous to the Jacobi integral in the case where U does not involve t explicitly, is

$$x'^2+y'^2+z'^2=2U-2\int\left(\frac{\partial U}{\partial t}\right)dt+C, \tag{83}$$

where $(\partial U/\partial t)$ is the partial derivative of U with respect to t so far as t occurs explicitly, and not as it enters through x, y, and z. This partial derivative is zero for e equal to zero, and therefore it contains e as a factor.

128. The Integral in Case of the Periodic Solutions.—In the periodic solutions which have been considered, x, x', y, and y' are expansible as power series in λ, while z and z' are expansible in odd powers of $\lambda^{\frac{1}{2}}$. It follows from property (a) of §106 that U is a power series in z^2. Therefore when the expansions of all these variables are substituted in (83), the result is a power series in integral powers of λ of the form

$$F = F_1\lambda + F_2\lambda^2 + \cdots + F_\nu\lambda^\nu + \cdots = C, \tag{84}$$

where, of course, F and the F_j involve the integral sign which arises from the right members of (83), and where C is the constant of integration.

Since (84) converges and is satisfied for all $|\lambda|$ sufficiently small, it follows that

$$F_\nu = C_\nu \qquad (\nu = 1, \ldots \infty), \tag{85}$$

the C_ν being constants. Since the series for x, y, and z have the forms

$$\begin{aligned} x &= \lambda\,[x_2 + x_4\lambda + \cdots + x_{2j}\,\lambda^{j-1} + \cdots], \\ y &= \lambda\,[y_2 + y_4\lambda + \cdots + y_{2j}\,\lambda^{j-1} + \cdots], \\ z &= \lambda^{\frac{1}{2}}[z_1 + z_3\,\lambda + \cdots + z_{2j+1}\lambda^{j} + \cdots], \end{aligned}$$

it follows that F_ν involves x_2, x_2', y_2, y_2', $\ldots$, $x_{2\nu-2}$, $x_{2\nu-2}'$, $y_{2\nu-2}$, $y_{2\nu-2}'$, z_1, z_1', $\ldots$, $z_{2\nu-1}$, $z_{2\nu-1}'$. Equation (85) therefore has the form

$$P_\nu(x_{2j}, x_{2j}', y_{2j}, y_{2j}', z_{2j+1}, z_{2j+1}', t) + \int \frac{\partial Q_\nu}{\partial t}(x_{2j}, y_{2j}, z_{2j+1}, t)\,dt = C_\nu, \tag{86}$$

where P_ν and Q_ν are polynomials in the indicated arguments with t entering the coefficients in sines and cosines. The subscript j runs from 0 to $\nu-1$. The derivatives enter (86) only in the form $x_{2j-2}'x_{2\nu-2j}'$, $y_{2j-2}'y_{2\nu-2j}'$, and $z_{2j-1}'z_{2\nu-2j}'$. Suppose the general term of P_ν or Q_ν is

$$x_{2\mu_1}^{\lambda_1} \cdots x_{2\mu_\kappa}^{\lambda_\kappa} \cdot y_{2\mu_1'}^{\lambda_1'} \cdots y_{2\mu_{\kappa'}'}^{\lambda_{\kappa'}'} \cdot z_{2\mu_1''+1}^{\lambda_1''} \cdots z_{2\mu_{\kappa''}''+1}^{\lambda_{\kappa''}''}.$$

The exponents and subscripts satisfy the relation

$$\begin{aligned} 2[\lambda_1\mu_1 + \cdots + \lambda_\kappa\mu_\kappa + \lambda_1'\mu_1' + \cdots \lambda_{\kappa'}'\mu_{\kappa'}' + \lambda_1''\mu_1'' + \cdots + \lambda_{\kappa''}''\mu_{\kappa''}''] \\ + [\lambda_1'' + \cdots + \lambda_{\kappa''}''] = 2\nu. \end{aligned}$$

The partial derivative of Q_n with respect to t is taken only so far as t enters explicitly in the coefficients of x_{2j}, $\ldots$, z_{2j+1}, but the integral must be computed for t entering both explicitly and also implicitly through the x_{2j}, $\ldots$, z_{2j+1}.

It was shown in §125 that the x_{2j}, the y_{2j}, and the z_{2j+1} have the form

$$x_{2j} = \sum_{k=-j}^{+j} x_{2j}^{(2k)} e^{2k\omega\sqrt{-1}t}, \quad y_{2j} = \sum_{k=-j}^{+j} y_{2j}^{(2k)} e^{2k\omega\sqrt{-1}t}, \quad z_{2j+1} = \sum_{k=-j-1}^{+j} z_{2j+1}^{(2k+1)} e^{(2k+1)\omega\sqrt{-1}t}, \tag{87}$$

where the $x_{2j}^{(2k)}$, the $y_{2j}^{(2k)}$, and the $z_{2j+1}^{(2k+1)}$ are power series in e whose coefficients are periodic with the period 2π. In $x_{2j}^{(2k)}$ and $z_{2j+1}^{(2k+1)}$ the coefficients of the cosine terms are real, and those of the sine terms are purely imaginary; the opposite is true in $y_{2j}^{(2k)}$. It follows from these properties and those of F_ν enumerated on page 254 that P_ν and $\partial Q_\nu/\partial t$ can be written in the form

$$P_\nu = \sum_{k=-\nu}^{+\nu} P_\nu^{(k)} e^{2k\omega\sqrt{-1}t}, \qquad \frac{\partial Q_\nu}{\partial t} = \sum_{k=-\nu}^{+\nu} R_\nu^{(k)} e^{2k\omega\sqrt{-1}t}, \tag{88}$$

where $P_\nu^{(k)}$ and $R_\nu^{(k)}$ are periodic with the period 2π. Since in U the coefficients of odd powers of y are multiplied by sine series, it follows from the properties of the $x_{2j}^{(2k)}$, the $y_{2j}^{(2k)}$, and the $z_{2j+1}^{(2k+1)}$ that in $P_\nu^{(k)}$ the coefficients of the cosine terms are real, and those of the sine terms are purely imaginary; the opposite is true in $R_\nu^{(k)}$.

The integrals coming from $\int \frac{\partial Q_\nu}{\partial t} dt$ are of the types

$$\left.\begin{aligned}\sqrt{-1}\int e^{2k\omega\sqrt{-1}t}\cos jt\,dt &= -\frac{2k\omega e^{2k\omega\sqrt{-1}t}}{j^2-4k^2\omega^2}\cos jt + \frac{j\sqrt{-1}e^{2k\omega\sqrt{-1}t}}{j^2-4k^2\omega^2}\sin jt,\\ \int e^{2k\omega\sqrt{-1}t}\sin jt\,dt &= -\frac{je^{2k\omega\sqrt{-1}t}}{j^2-4k^2\omega^2}\cos jt + \frac{2k\omega\sqrt{-1}e^{2k\omega\sqrt{-1}t}}{j^2-4k^2\omega^2}\sin jt.\end{aligned}\right\} \tag{89}$$

Therefore we have

$$\int \frac{\partial Q_\nu}{dt} dt = \sum_{k=-\nu}^{+\nu} S_\nu^{(k)} e^{2k\omega\sqrt{-1}t}, \tag{90}$$

where the $S_\nu^{(k)}$ are periodic with the period 2π. Moreover, the coefficients of the cosine terms are real, and those of the sine terms are purely imaginary. It follows from these properties that (85) can be written in the form

$$F_\nu = \sum_{k=-\nu}^{+\nu} F_\nu^{(k)} e^{2k\omega\sqrt{-1}t} = C_\nu, \tag{91}$$

where the $F_\nu^{(k)}$ are periodic with the period 2π. The coefficients of the cosine terms are real, and those of the sine terms are purely imaginary. If we let

$$e^{4k\omega\sqrt{-1}\pi} = \sigma_k \tag{92}$$

and make use of the fact that the $F_\nu^{(k)}$ are periodic with the period 2π, we get from (91)

$$\sum_{k=-\nu}^{+\nu} F_\nu^{(k)} e^{2k\omega\sqrt{-1}t}\sigma_k = C_\nu, \quad \sum_{k=-\nu}^{+\nu} F_\nu^{(k)} e^{2k\omega\sqrt{-1}t}\sigma_k^2 = C_\nu, \cdots, \quad \sum_{k=-\nu}^{+\nu} F_\nu^{(k)} e^{2k\omega\sqrt{-1}t}\sigma_k^{2\nu} = C_\nu. \tag{93}$$

These equations and (91) can be satisfied only if either

$$F_\nu^{(0)} = C, \qquad F_\nu^{(k)} = 0 \qquad (k = -\nu, \ldots, -1, +1, \ldots, +\nu), \tag{94}$$

or

$$\Delta = \begin{vmatrix} 1, & \ldots, & 1, & 1, & 1, & \ldots, & 1 \\ \sigma_{-\nu}, & \ldots, & \sigma_{-1}, & 1, & \sigma_1, & \ldots, & \sigma_\nu \\ \sigma_{-\nu}^2, & \ldots, & \sigma_{-1}^2, & 1, & \sigma_1^2, & \ldots, & \sigma_\nu^2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \sigma_{-\nu}^{2\nu}, & \ldots, & \sigma_{-1}^{2\nu}, & 1, & \sigma_1^{2\nu}, & \ldots, & \sigma_\nu^{2\nu} \end{vmatrix} = 0.$$

This determinant is the well-known product of the differences of the $\sigma_{\pm j}$ taken in all possible pairs, and is distinct from zero unless a relation of the form $\sigma_i=\sigma_j$ is satisfied. Since $\sigma_i=\sigma_1^i$, $\sigma_j=\sigma_1^j$, this relation can be satisfied only if $\sigma_1^{i-j}=1$; or, because of (92), only if

$$e^{4(i-j)\omega\sqrt{-1}\pi}=1.$$

Since $\omega=N/n$, this equation can be fulfilled only if $4(i-j)/n$ is an integer. But then two or more of the exponentials of (91) are equal in value, and the number of terms under the summation sign is reduced by combining similar ones. With this understanding as to the reduction of (91), Δ can not vanish and equations (94) must be fulfilled. It is clear that when these equations are satisfied all relations of the form of (93) are satisfied.

The $F_\nu^{(k)}$ are explicit power series in e, and they also involve e implicitly in ω, which is a power series in this same parameter. Now ω enters in two ways. It is introduced as a factor of certain terms either to the first or second degree by the derivatives which occur in (83), and to the first degree by the integral, as is shown by (89). The integral also introduces it in the denominators in the form $j^2-4k^2\omega^2$. We shall substitute for ω its series in e wherever it enters in the first way. This will not change the character of the convergence. But where ω enters in the second way we shall regard e as an independent parameter and leave it implicitly in ω. Then equations (94) can be written in the form

$$F_\nu^{(0)}=\sum_{j=0}^{\infty}F_{\nu,j}^{(0)}\,e^j=C_\nu=\sum_{j=0}^{\infty}C_{\nu,j}\,e^j,$$

$$F_\nu^{(k)}=\sum_{j=0}^{\infty}F_{\nu,j}^{(k)}\,e^j=0 \qquad (k=-\nu,\ \ldots,\ -1,\ +1,\ \ldots,\ +\nu).$$

Since these equations are identically satisfied in e, we have

$$F_{\nu,j}^{(0)}=C_{\nu,j}, \qquad F_{\nu,j}^{(k)}=0. \tag{95}$$

The $F_{\nu,j}^{(k)}$ are sines and cosines of integral multiples of t. On making use of the properties established in § 126, we have

$$F_{\nu,j}^{(0)}=\sum_{p=0}^{j}[a_{\nu,j,p}^{(0)}\cos pt+\sqrt{-1}\,b_{\nu,j,p}^{(0)}\sin pt]=C_{\nu,j},$$

$$F_{\nu,j}^{(k)}=\sum_{p=0}^{j}[a_{\nu,j,p}^{(k)}\cos pt+\sqrt{-1}\,b_{\nu,j,p}^{(k)}\sin pt]=0.$$

Since these equations are identities in t, we have finally

$$\left.\begin{array}{lll} a_{\nu,j,0}^{(0)}=C_{\nu,j}, & a_{\nu,j,p}^{(0)}=b_{\nu,j,p}^{(0)}=0 & (p=1,\ \ldots,\ j), \\ a_{\nu,j,p}^{(k)}=b_{\nu,j,p}^{(k)}=0 & & (k\neq 0;\ p=0,\ \ldots,\ j). \end{array}\right\} \tag{96}$$

129. Determination of the Coefficients of z_{2j+1} when $e=0$.—Equations (96) are relations among the coefficients of the solutions and may be used for checking the computations. The control is very effective because at each step all the preceding coefficients are in general involved. But equations (96) can also be used, step by step, for the determination of the coefficients of the expansion for z when the coefficients of the expansions for x and y are determined, alternately with those for z, from the first two equations of (13). Before taking up the general problem we shall treat the case of $e=0$. When this condition is satisfied the integral becomes

$$\left.\begin{aligned} x'^2+y'^2+z'^2 = \Big[& x^2+y^2+A\,(2x^2-y^2-z^2)+3B\,z^2 \\ & +\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)(-x^2+y^2+z^2)\,\lambda+\tfrac{3}{4}C_0\,z^4+\cdots\Big]+C, \\ C_0 = \frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}. & \end{aligned}\right\} \quad (97)$$

In this case $\omega=\sqrt{A}$ and equations (87) become

$$x_{2j}=\sum_{k=-j}^{+j}\alpha_{2j}^{(2k)}e^{2k\sqrt{A}\sqrt{-1}t},\quad y_{2j}=\sum_{k=-j}^{+j}\beta_{2j}^{(2k)}e^{2k\sqrt{A}\sqrt{-1}t},\quad z_{2j+1}=\sum_{k=-j-1}^{+j}\gamma_{2j+1}^{(2k+1)}\,e^{(2k+1)\sqrt{A}\sqrt{-1}t}, \quad (98)$$

where the $\alpha_{2j}^{(2k)}$, $\beta_{2j}^{(2k)}$, and $\gamma_{2j}^{(2k+1)}$ are constants. We shall show how to compute the $\gamma_{2j+1}^{(2k+1)}$ for successive values of j.

Terms for $j=0$. In this case, since $x_0\equiv y_0=0$, equation (97) becomes, as a consequence of the last of (98),

$$\begin{aligned} & -A[(\gamma_1^{(-1)})^2\,e^{-2\sqrt{A}\sqrt{-1}t}-2\gamma_1^{(-1)}\,\gamma_1^{(1)}+(\gamma_1^{(1)})^2\,e^{2\sqrt{A}\sqrt{-1}t}]= \\ & -A[(\gamma_1^{(-1)})^2\,e^{-2\sqrt{A}\sqrt{-1}t}+2\gamma_1^{(-1)}\,\gamma_1^{(1)}+(\gamma_2^{(1)})^2\,e^{2\sqrt{A}\sqrt{-1}t}]+C_1. \end{aligned}$$

Since this equation is an identity in t, we get $4A\gamma_1^{(-1)}\gamma_1^{(1)}=C_1$. Since C_1 is unknown, this equation imposes no relation on $\gamma_1^{(-1)}$ and $\gamma_1^{(1)}$. But since $z_1(0)=0$, it follows that $\gamma_1^{(-1)}=-\gamma_1^{(1)}$, and there remains the single undetermined constant $\gamma_1^{(1)}$.

Terms in x and y for $j=1$. It follows from (13) and (14) that x_2 and y_2 are defined by the equations

$$\left.\begin{aligned} & x_2''-2y_2'-[1+2A]\,x_2=\tfrac{3}{2}B\,z_1^2=\tfrac{3}{2}B(\gamma_1^{(1)})^2[e^{-2\sqrt{A}\sqrt{-1}t}+e^{2\sqrt{A}\sqrt{-1}t}-2], \\ & y_2''+2x_2'-[1-\ \ A]\,y_2=0. \end{aligned}\right\} \quad (99)$$

The solutions of these equations, which have the period $2\pi/\sqrt{A}$, are

$$\left.\begin{aligned} x_2 &= -\frac{3B(2+3A)\,(\gamma_1^{(1)})^2}{2(1-7A+18A^2)}\,e^{-2\sqrt{A}\sqrt{-1}t}+\frac{3B(\gamma_1^{(1)})^2}{1+2A}-\frac{3B(1+3A)\,(\gamma_1^{(1)})^2}{2(1-7A+18A^2)}\,e^{2\sqrt{A}\sqrt{-1}t}, \\ y_2 &= +\frac{6B\sqrt{A}\sqrt{-1}(\gamma_1^{(1)})^2}{(1-7A+18A^2)}\,[e^{-2\sqrt{A}\sqrt{-1}t}-e^{2\sqrt{A}\sqrt{-1}t}]. \end{aligned}\right\} \quad (100)$$

Terms in z for $j=1$. It follows from (97) that the terms for $j=1$ are

$$\left.\begin{aligned}2z_1'z_3'+2Az_1z_3=&-[x_2'^2+y_2'^2]+(1+2A)x_2^2+(1-A)y_2^2\\&+3Bx_2z_1^2+\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)z_1^2+\tfrac{3}{4}C_0z_1^4+C_2.\end{aligned}\right\}\quad(101)$$

Since z_3 has the form

$$z_3=\gamma_3^{(-3)}e^{-3\sqrt{A}\sqrt{-1}t}+\gamma_3^{(-1)}e^{-\sqrt{A}\sqrt{-1}t}+\gamma_3^{(1)}e^{\sqrt{A}\sqrt{-1}t}+\gamma_3^{(3)}e^{3\sqrt{A}\sqrt{-1}t},\qquad(102)$$

equation (101) gives rise to the relations

$$\left.\begin{aligned}4A\gamma_1^{(1)}\gamma_3^{(-3)}&=-\frac{9(1+3A)B^2(\gamma_1^{(1)})^4}{4(1-7A+18A^2)}+\tfrac{3}{4}C_0(\gamma_1^{(1)})^4=-4A\gamma_1^{(1)}\gamma_3^{(3)},\\8A\gamma_1^{(1)}\gamma_3^{(-3)}&=+\frac{9B^2(\gamma_1^{(1)})^4}{1+2A}+\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)(\gamma_1^{(1)})^2-3C_0(\gamma_1^{(1)})^4=-8A\gamma_1^{(1)}\gamma_3^{(3)}.\end{aligned}\right\}\quad(103)$$

There is another equation which is useless for present purposes because it involves the unknown constant C_2. Since $z_3(0)=0$, we have also

$$\gamma_3^{(-3)}+\gamma_3^{(-1)}+\gamma_3^{(1)}+\gamma_3^{(3)}=0.\qquad(104)$$

It follows from equations (103) and (104) that

$$\gamma_3^{(-3)}=-\gamma_3^{(3)},\qquad\gamma_3^{(-1)}=-\gamma_3^{(1)}.\qquad(105)$$

In order that $\gamma_3^{(-3)}$ shall be the same as determined by both equations of (103), we must impose the condition

$$\frac{27(1-3A+14A^2)B^2(\gamma_1^{(1)})^4}{2(1+2A)(1-7A+18A^2)}-\tfrac{9}{2}C_0(\gamma_1^{(1)})^4+\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)(\gamma_1^{(1)})^2=0,\qquad(106)$$

which determines $\gamma_1^{(1)}$ except as to sign. Then $\gamma_3^{(-3)}$ and $\gamma_3^{(3)}$ are uniquely determined in terms of $\gamma_1^{(1)}$, but $\gamma_3^{(1)}$ remains so far arbitrary.

The problem for $e=0$ was treated in Chapter VI. If we compare equations (100) of the present work with equations (42) of page 211, we find that $\gamma_1^{(1)}$ and c_1 are related by the equation

$$c_1^2=4A(\gamma_1^{(1)})^2.$$

Upon making use of this relation, it is seen that equations (106) of this chapter and (44) of Chapter VI are identical.

Terms in x and y for $j=2$. It follows from (13) and (14) that we have in this case

$$x_4''-2y_4'-[1+2A]x_4=3Bz_1z_3+P_4,\qquad y_4''+2x_4'-[1-A]y_4=Q_4,\qquad(107)$$

where P_4 and Q_4 are entirely known periodic functions of t having the period $2\pi/\sqrt{A}$. We wish the details of the solutions of these equations only so

far as they depend upon the undetermined constant $\gamma_3^{(1)} = -\gamma_3^{(-1)}$. So far as these terms are involved, the right member of the first equation is $3B\gamma_1^{(1)}\gamma_3^{(1)}[e^{-2\sqrt{A}\sqrt{-1}t}+e^{2\sqrt{A}\sqrt{-1}t}]$. Hence the solutions of (107) are

$$\left.\begin{aligned} x_4 &= -\frac{3B(1+3A)\gamma_1^{(1)}\gamma_3^{(1)}}{1-7A+18A^2}[e^{-2\sqrt{A}\sqrt{-1}t}+e^{2\sqrt{A}\sqrt{-1}t}]+\frac{6B\gamma_1^{(1)}\gamma_3^{(1)}}{1+2A}+\overline{P}_4, \\ y_4 &= +\frac{12B\sqrt{A}\sqrt{-1}\gamma_1^{(1)}\gamma_3^{(1)}}{1-7A+18A^2}[e^{-2\sqrt{A}\sqrt{-1}t}-e^{2\sqrt{A}\sqrt{-1}t}]+\overline{Q}_4, \end{aligned}\right\} \quad (108)$$

where $\overline{P}_4$ and $\overline{Q}_4$ are known periodic functions of t.

Terms in z for $j=2$. It follows from (97) that these terms are defined by

$$\left.\begin{aligned} 2z_1'z_5'+2Az_1z_5 = &-2x_2'x_4'-2y_2'y_4'+2(1+2A)x_2x_4+2(1-A)y_2y_4 \\ &+6Bx_2z_1z_3+2\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)z_1z_3+3C_0z_1^3z_3+R_5, \end{aligned}\right\} \quad (109)$$

where R_5 is a known periodic function of t. The expression for z_5 has the form

$$\left.\begin{aligned} z_5 = \gamma_5^{(-5)}e^{-5\sqrt{A}\sqrt{-1}t}&+\gamma_5^{(-3)}e^{-3\sqrt{A}\sqrt{-1}t}+\gamma_5^{(-1)}e^{-\sqrt{A}\sqrt{-1}t} \\ &+\gamma_5^{(1)}e^{\sqrt{A}\sqrt{-1}t}+\gamma_5^{(3)}e^{3\sqrt{A}\sqrt{-1}t}+\gamma_5^{(5)}e^{5\sqrt{A}\sqrt{-1}t}. \end{aligned}\right\} \quad (110)$$

On substituting this expression in (109) and equating the coefficients of the several powers of $e^{\sqrt{A}\sqrt{-1}t}$, beginning with $e^{-5\sqrt{A}\sqrt{-1}t}$, we get

$$\left.\begin{aligned} &8A\gamma_1^{(1)}\gamma_5^{(-5)} = \text{known function of } \gamma_1^{(1)} = -8A\gamma_1^{(1)}\gamma_5^{(5)}, \\ &4A\gamma_1^{(1)}[\gamma_5^{(-3)}+3\gamma_5^{(-5)}] = 3C_0(\gamma_1^{(1)})^3\gamma_3^{(1)}+\text{function of } \gamma_1^{(1)} = -4A\gamma_1^{(1)}[\gamma_5^{(3)}+3\gamma_5^{(5)}], \\ &8A\gamma_1^{(1)}\gamma_5^{(-3)} = \left[\frac{-18B^2(1+3A)}{1-7A+18A^2}-12C_0\right](\gamma_1^{(1)})^3\gamma_3^{(1)}+\text{function of } \gamma_1^{(1)} = -8A\gamma_1^{(1)}\gamma_5^{(3)}. \end{aligned}\right\} \quad (111)$$

There is also an equation, coming from the terms independent of $e^{\sqrt{A}\sqrt{-1}t}$, which involves the unknown C_5 and need not be written. The first equation uniquely defines $\gamma_5^{(-5)}$, which equals $-\gamma_5^{(5)}$. In order that the second and third equations shall be consistent, we must impose the relation

$$\left[\frac{3B^2(1+3A)}{1-7A+18A^2}+C_0\right](\gamma_1^{(1)})^3\gamma_3^{(1)} = \text{known function of } \gamma_1^{(1)}. \quad (112)$$

Therefore $\gamma_3^{(1)}$ is uniquely determined, since its coefficient in this equation is positive; and then the second or third of (111) defines $\gamma_5^{(-3)}$, which is the negative of $\gamma_5^{(3)}$.

Now, on imposing the condition that $z_5(0)=0$, we get $\gamma_5^{(-1)} = -\gamma_5^{(1)}$, and $\gamma_5^{(1)}$ remains undetermined.

All succeeding steps are precisely similar to the one which has just been explained. The parts of the equations which contain undetermined coefficients differ from those of (111) and (112) only in the subscripts.

130. Case when $e \neq 0$. General Equations for z_1.—It will now be shown that when the coefficients of the series for x and y are determined from equations (13) and (14), the coefficients of the expansion for z can be determined from the integral (83). We shall need the partial derivative of U with respect to t so far as this variable occurs explicitly. We find from (82) that

$$\left.\begin{aligned}\left(\frac{\partial U}{\partial t}\right) &= -\left[3Ae\sin t+\cdots\right]x^2+\left[6Ae\cos t+\cdots\right]xy+\left[\tfrac{3}{2}Ae\sin t+\cdots\right]y^2\\ &+\left[\tfrac{3}{2}Ae\sin t+\cdots\right]z^2+\left[3\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\sin t+\cdots\right]x^2\lambda\\ &-\left[6\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\cos t+\cdots\right]xy\lambda-\left[\frac{3}{2}\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\sin t+\cdots\right]y^2\lambda\\ &-\left[\frac{3}{2}\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\sin t+\cdots\right]z^2\lambda-\left[\frac{3}{2}\left(\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right)e\sin t+\cdots\right]z^4+\cdots\end{aligned}\right\}\quad(113)$$

Since $x_0=y_0=0$, we find from (82), (83), and (113) that the integral is

$$z_1'^2=-[A+3Ae\cos t+\cdots]z_1^2-\int[3Ae\sin t+\cdots]z_1^2\,dt+C_1. \quad (114)$$

In the notation of equations (87), the expression for z_1 has the form

$$z_1=z_1^{(-1)}e^{-\omega\sqrt{-1}t}+z_1^{(1)}e^{\omega\sqrt{-1}t},$$

where

$$\left.\begin{aligned}z_1^{(-1)}&=\sum_{k=0}^{\infty}z_{1,k}^{(-1)}e^k, \qquad z_1^{(1)}=\sum_{k=0}^{\infty}z_{1,k}^{(1)}e^k,\\ z_{1,k}^{(-1)}&=\sum_{p=0}^{k}[a_{1,k,p}^{(-1)}\cos pt+\sqrt{-1}\beta_{1,k,p}^{(-1)}\sin pt],\\ z_{1,k}^{(1)}&=\sum_{p=0}^{k}[a_{1,k,p}^{(1)}\cos pt+\sqrt{-1}\beta_{1,k,p}^{(1)}\sin pt].\end{aligned}\right\}\quad(115)$$

On substituting these expressions in equation (114), it is found that

$$\left.\begin{aligned}&+\left[-\omega^2(z_1^{(-1)})^2+(z_1'^{(-1)})^2-2\omega\sqrt{-1}z_1^{(-1)}z_1'^{(-1)}\right]e^{-2\omega\sqrt{-1}t}\\ &+\left[-\omega^2(z_1^{(1)})^2+(z_1'^{(1)})^2+2\omega\sqrt{-1}z_1^{(1)}z_1'^{(1)}\right]e^{+2\omega\sqrt{-1}t}\\ &+\left[2\omega^2z_1^{(-1)}z_1^{(1)}+2z_1'^{(-1)}z_1'^{(1)}-2\omega\sqrt{-1}(z_1^{(-1)}z_1'^{(1)}-z_1^{(1)}z_1'^{(-1)})\right]=\\ &-\left[A+3Ae\cos t+\cdots\right]\left[(z_1^{(-1)})^2e^{-2\omega\sqrt{-1}t}+(z_1^{(1)})^2e^{2\omega\sqrt{-1}t}+2z_1^{(-1)}z_1^{(1)}\right]\\ &-\int\left[3Ae\sin t+\cdots\right]\left[(z_1^{(-1)})^2e^{-2\omega\sqrt{-1}t}+(z_1^{(1)})^2e^{2\omega\sqrt{-1}t}+2z_1^{(-1)}z_1^{(1)}\right]dt+C_1.\end{aligned}\right\}\quad(116)$$

Before the integration the series for $z_1^{(-1)}$ and $z_1^{(1)}$ must be substituted from (115). Consider the coefficient of $e^{-2\omega\sqrt{-1}t}$ under the integral sign. It is a sum of cosines and sines of integral multiples of t. Suppose $\sqrt{-1}A_j$ and B_j are the coefficients of $\cos jt$ and $\sin jt$ respectively. It follows from (89) that in the integral we have in place of these terms

$$\frac{2A_j\omega}{j^2-4\omega^2}\cos jt+\frac{j\sqrt{-1}A_j}{j^2-4\omega^2}\sin jt, \qquad \frac{-jB_j}{j^2-4\omega^2}\cos jt-\frac{2\omega\sqrt{-1}B_j}{j^2-4\omega^2}\sin jt \tag{117}$$

respectively. The corresponding formulas for the coefficient of $e^{2\omega\sqrt{-1}t}$ are obtained from these simply by changing the sign of ω. The terms independent of the exponentials which involve the cosine and sine of jt are divided by $+j$ and $-j$ respectively. Consequently, in all cases it is easy to write down the explicit equations for the identities.

In the notation of (91) the coefficients of $e^{-2\omega\sqrt{-1}t}$, $e^{+2\omega\sqrt{-1}t}$, and e^0 in (116) are respectively $F_1^{(-1)}$, $F_1^{(1)}$, and $F_1^{(0)}$, and we have

$$F_1^{(-1)}=F_1^{(1)}=0, \qquad F_1^{(0)}=C_1.$$

Since these functions are power series in e, we have in accordance with the notation of (95)

$$F_{1,j}^{(-1)}=F_{1,j}^{(1)}=0, \qquad F_{1,j}^{(0)}=C_{1,j}. \tag{118}$$

And these functions in turn have the form

$$\begin{aligned}
F_{1,j}^{(-1)} &= \sum_{p=0}^{j}[a_{1,j,p}^{(-1)}\cos pt+\sqrt{-1}\,b_{1,j,p}^{(-1)}\sin pt]\equiv 0,\\
F_{1,j}^{(1)} &= \sum_{p=0}^{j}[a_{1,j,p}^{(1)}\cos pt+\sqrt{-1}\,b_{1,j,p}^{(1)}\sin pt]\equiv 0,\\
F_{1,j}^{(0)} &= \sum_{p=0}^{j}[a_{1,j,p}^{(0)}\cos pt+\sqrt{-1}\,b_{1,j,p}^{(0)}\sin pt]\equiv C_{1,j}.
\end{aligned}$$

Since these equations are all identities in t, we have

$$\left.\begin{aligned}
&a_{1,j,0}^{(0)}=C_{1,j}, \qquad a_{1,j,p}^{(0)}=0 \qquad (j=0,\ldots\infty;\ p=0,\ldots,j),\\
&a_{1,j,p}^{(-1)}=b_{1,j,p}^{(-1)}=a_{1,j,p}^{(1)}=b_{1,j,p}^{(1)}=0 \qquad (j=0,\ldots\infty;\ p=0,\ldots,j).
\end{aligned}\right\} \tag{119}$$

In order to get the explicit values of these constants in terms of the $\alpha_{1,k,p}^{(-1)}$, $\beta_{1,k,p}^{(-1)}$, $\alpha_{1,k,p}^{(1)}$, and $\beta_{1,k,p}^{(1)}$, we must refer to equation (116). We find from (115) that

$$\begin{aligned}
z_1^{(-1)} &= z_{1,0}^{(-1)}+z_{1,1}^{(-1)}e+z_{1,2}^{(-1)}e^2+\cdots,\\
z_1^{(1)} &= z_{1,0}^{(1)}+z_{1,1}^{(1)}e+z_{1,2}^{(1)}e^2+\cdots,\\
z_1'^{(-1)} &= 0+z_{1,1}'^{(-1)}e+z_{1,2}'^{(-1)}e^2+\cdots,\\
z_1'^{(1)} &= 0+z_{1,1}'^{(1)}e+z_{1,2}'^{(1)}e^2+\cdots,\\
\omega &= \omega_0+\omega_1 e+\omega_2 e^2+\cdots;
\end{aligned}$$

whence

$$\left.\begin{aligned}
(z_1^{(-1)})^2 &= (z_{1,0}^{(-1)})^2 + 2z_{1,0}^{(-1)}z_{1,1}^{(-1)}e + \left[2z_{1,0}^{(-1)}z_{1,2}^{(-1)} + (z_{1,1}^{(-1)})^2\right]e^2 + \cdots,\\
z_1^{(-1)}z_1'^{(-1)} &= 0 + z_{1,0}^{(-1)}z_{1,1}'^{(-1)}e + \left[z_{1,0}^{(-1)}z_{1,2}'^{(-1)} + z_{1,1}^{(-1)}z_{1,1}'^{(-1)}\right]e^2 + \cdots,\\
(z_1'^{(-1)})^2 &= 0 + 0 \qquad \left[+ (z_{1,1}'^{(-1)})^2\right]e^2 + \cdots,\\
(z_1^{(1)})^2 &= (z_{1,0}^{(1)})^2 + 2z_{1,0}^{(1)}z_{1,1}^{(1)}e + \left[2z_{1,0}^{(1)}z_{1,2}^{(1)} + (z_{1,1}^{(1)})^2\right]e^2 + \cdots,\\
z_1^{(1)}z_1'^{(1)} &= 0 + z_{1,0}^{(1)}z_{1,1}'^{(1)}e + \left[z_{1,0}^{(1)}z_{1,2}'^{(1)} + z_{1,1}^{(1)}z_{1,1}'^{(1)}\right]e^2 + \cdots,\\
(z_1'^{(1)})^2 &= 0 + 0 \qquad \left[+ (z_{1,1}'^{(1)})^2\right]e^2 + \cdots,\\
z_1^{(-1)}z_1^{(1)} &= z_{1,0}^{(-1)}z_{1,0}^{(1)} + (z_{1,0}^{(-1)}z_{1,1}^{(1)} + z_{1,0}^{(1)}z_{1,1}^{(-1)})e\\
&\qquad + \left[z_{1,0}^{(-1)}z_{1,2}^{(1)} + z_{1,1}^{(-1)}z_{1,1}^{(1)} + z_{1,2}^{(-1)}z_{1,0}^{(1)}\right]e^2 + \cdots,\\
z_1'^{(-1)}z_1'^{(1)} &= 0 + 0 \qquad \left[+ z_{1,1}'^{(-1)}z_{1,1}'^{(1)}\right]e^2 + \cdots,\\
z_1^{(-1)}z_1'^{(1)} &= 0 + z_{1,0}^{(-1)}z_{1,1}'^{(1)}e + \left[z_{1,0}^{(-1)}z_{1,2}'^{(1)} + z_{1,1}^{(-1)}z_{1,1}'^{(1)}\right]e^2 + \cdots,\\
z_1^{(1)}z_1'^{(-1)} &= 0 + z_{1,0}^{(1)}z_{1,1}'^{(-1)}e + \left[z_{1,0}^{(1)}z_{1,2}'^{(-1)} + z_{1,1}^{(1)}z_{1,1}'^{(-1)}\right]e^2 + \cdots,\\
\omega^2 &= \omega_0^2 + 2\omega_0\omega_1 e + \left[\omega_1^2 + 2\omega_0\omega_2\right]e^2 + \cdots.
\end{aligned}\right\} \quad (120)$$

Equating to zero the coefficients of the various exponentials of (116), we get

$$\left.\begin{aligned}
&(A-\omega^2)(z_1^{(-1)})^2 + (z_1'^{(-1)})^2 - 2\omega\sqrt{-1}\,z_1^{(-1)}z_1'^{(-1)} = -(z_1^{(-1)})^2\left[3Ae\cos t + \cdots\right]\\
&\qquad\qquad - \bar{\int}\left[3Ae\sin t + \cdots\right](z_1^{(-1)})^2\,dt\\
&(A-\omega^2)(z_1^{(1)})^2 + (z_1'^{(1)})^2 + 2\omega\sqrt{-1}\,z_1^{(1)}z_1'^{(1)} = -(z_1^{(1)})^2\left[3Ae\cos t + \cdots\right]\\
&\qquad\qquad - \bar{\int}\left[3Ae\sin t + \cdots\right](z_1^{(1)})^2\,dt\\
&2(A-\omega^2)z_1^{(-1)}z_1^{(1)} + 2z_1'^{(-1)}z_1'^{(1)} - 2\omega\sqrt{-1}\left[z_1^{(-1)}z_1'^{(1)} - z_1^{(1)}z_1'^{(-1)}\right] =\\
&\qquad -2z_1^{(-1)}z_1^{(1)}\left[3Ae\cos t + \cdots\right] - 2\bar{\int} z_1^{(-1)}z_1^{(1)}\left[3Ae\sin t + \cdots\right] + C_1,
\end{aligned}\right\} \quad (121)$$

where the coefficients under the integral sign $\bar{\int}$ must be transformed by equations (117) instead of forming the ordinary integrals, and where C_1 is an undetermined constant. These equations are power series in e, and setting their coefficients equal to zero we have equations (119).

131. Coefficients of e^0.—On referring to (120), we find for these terms

$$-\omega_0^2(z_{1,0}^{(-1)})^2 = -A(z_{1,0}^{(-1)})^2, \quad -\omega_0^2(z_{1,0}^{(1)})^2 = -A(z_{1,0}^{(1)})^2, \quad 2\omega_0^2 z_{1,0}^{(-1)}z_{1,0}^{(1)} = -2Az_{1,0}^{(-1)}z_{1,0}^{(1)} + C_{1,0}.$$

Since $z_1(0) = 0$, we must add to these equations $z_{1,0}^{(-1)} + z_{1,0}^{(1)} = 0$. It follows from these equations that

$$\omega_0^2 = A, \qquad z_{1,0}^{(-1)} = -z_{1,0}^{(1)}, \qquad 4A(z_{1,0}^{(1)})^2 = C_{1,0}, \qquad (122)$$

and $z_{1,0}^{(1)} = a_{1,0,0}^{(1)}$ remains as yet undetermined since $C_{1,0}$ is an unknown constant.

132. Coefficients of ***e.***—On referring to (120), (117), and (115), we get at this step

$$
\left.\begin{aligned}
&-2\omega_0^2 z_{1,0}^{(-1)}\left[\alpha_{1,1,0}^{(-1)}+\alpha_{1,1,1}^{(-1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(-1)}\sin t\right]-2\omega_0\,\omega_1\,(z_{1,0}^{(-1)})^2\\
&-2\omega_0\sqrt{-1}\,z_{1,0}^{(-1)}\left[-\alpha_{1,1,1}^{(-1)}\sin t+\sqrt{-1}\,\beta_{1,1,1}^{(-1)}\cos t\right]=-3\,(z_{1,0}^{(-1)})^2A\cos t\\
&-2z_{1,0}^{(-1)}A\left[\alpha_{1,1,0}^{(-1)}+\alpha_{1,1,1}^{(-1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(-1)}\sin t\right]+\frac{3(z_{1,0}^{(-1)})^2A}{1-4\omega^2}\left[\cos t+2\omega_0\sqrt{-1}\sin t\right],\\[1ex]
&-2\omega_0^2 z_{1,0}^{(1)}\left[\alpha_{1,1,0}^{(1)}+\alpha_{1,1,1}^{(1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(1)}\sin t\right]-2\omega_0\,\omega_1\,(z_{1,0}^{(1)})^2\\
&+2\omega_0\sqrt{-1}\,z_{1,0}^{(1)}\left[-\alpha_{1,1,1}^{(1)}\sin t+\sqrt{-1}\,\beta_{1,1,1}^{(1)}\cos t\right]=-3\,(z_{1,0}^{(1)})^2A\cos t\\
&-2z_{1,0}^{(1)}A\left[\alpha_{1,1,0}^{(1)}+\alpha_{1,1,1}^{(1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(1)}\sin t\right]+\frac{3(z_{1,0}^{(1)})^2A}{1-4\omega^2}\left[\cos t-2\omega_0\sqrt{-1}\sin t\right],\\[1ex]
&+4\omega_0\omega_1 z_{1,0}^{(-1)}z_{1,0}^{(1)}+2\omega_0^2\Big\{z_{1,0}^{(-1)}\left[\alpha_{1,1,0}^{(1)}+\alpha_{1,1,1}^{(1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(1)}\sin t\right]\\
&+z_{1,0}^{(1)}\left[\alpha_{1,1,0}^{(-1)}+\alpha_{1,1,1}^{(-1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(-1)}\sin t\right]\Big\}-2\omega_0\sqrt{-1}\Big\{z_{1,0}^{(-1)}\Big[-\alpha_{1,1,1}^{(1)}\sin t\\
&+\sqrt{-1}\,\beta_{1,1,1}^{(1)}\cos t\Big]-z_{1,0}^{(1)}\left[-\alpha_{1,1,1}^{(-1)}\sin t+\sqrt{-1}\,\beta_{1,1,1}^{(-1)}\cos t\right]\Big\}=\\
&-6z_{1,0}^{(-1)}z_{1,0}^{(1)}A\cos t-2A\Big\{z_{1,0}^{(-1)}\left[\alpha_{1,1,0}^{(1)}+\alpha_{1,1,1}^{(1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(1)}\sin t\right]\\
&+z_{1,0}^{(1)}\left[\alpha_{1,1,0}^{(-1)}+\alpha_{1,1,1}^{(-1)}\cos t+\sqrt{-1}\,\beta_{1,1,1}^{(-1)}\sin t\right]\Big\}+6z_{1,0}^{(-1)}z_{1,0}^{(1)}A\cos t+C_{1,1}.
\end{aligned}\right\}\quad(123)
$$

Since these equations are identities in t, we find, after making use of equations (122), that

$$
\begin{aligned}
&-2A\,z_{1,0}^{(-1)}\alpha_{1,1,0}^{(-1)}-2\sqrt{A}\,\omega_1(z_{1,0}^{(-1)})^2=-2A\,z_{1,0}^{(-1)}\alpha_{1,1,0}^{(-1)},\\
&-2A\,z_{1,0}^{(-1)}\alpha_{1,1,1}^{(-1)}+2\sqrt{A}\,z_{1,0}^{(-1)}\beta_{1,1,1}^{(-1)}=-3A(z_{1,0}^{(-1)})^2-2A\,z_{1,0}^{(-1)}\alpha_{1,1,1}^{(-1)}+\frac{3A(z_{1,0}^{(-1)})^2}{1-4\omega^2},\\
&-2A\,z_{1,0}^{(-1)}\beta_{1,1,1}^{(-1)}+2\sqrt{A}\,z_{1,0}^{(-1)}\alpha_{1,1,1}^{(-1)}=-2A\,z_{1,0}^{(-1)}\beta_{1,1,1}^{(-1)}+\frac{6A^{3/2}(z_{1,0}^{(-1)})^2}{1-4\omega^2};
\end{aligned}
$$

$$
\begin{aligned}
&-2A\,z_{1,0}^{(1)}\alpha_{1,1,0}^{(1)}-2\sqrt{A}\,\omega_1(z_{1,0}^{(1)})^2=-2A\,z_{1,0}^{(1)}\alpha_{1,1,0}^{(1)},\\
&-2A\,z_{1,0}^{(1)}\alpha_{1,1,1}^{(1)}-2\sqrt{A}\,z_{1,0}^{(1)}\beta_{1,1,1}^{(1)}=-3A(z_{1,0}^{(1)})^2-2A\,z_{1,0}^{(1)}\alpha_{1,1,1}^{(1)}+\frac{3A(z_{1,0}^{(1)})^2}{1-4\omega^2},\\
&-2A\,z_{1,0}^{(1)}\beta_{1,1,1}^{(1)}-2\sqrt{A}\,z_{1,0}^{(1)}\alpha_{1,1,1}^{(1)}=-2A\,z_{1,0}^{(1)}\beta_{1,1,1}^{(1)}-\frac{6A^{3/2}(z_{1,0}^{(1)})^2}{1-4\omega^2};
\end{aligned}
$$

$$
\begin{aligned}
&-4\sqrt{A}\,\omega_1(z_{1,0}^{(1)})^2+2A\,z_{1,0}^{(-1)}\alpha_{1,1,0}^{(1)}+2A\,z_{1,0}^{(1)}\alpha_{1,1,0}^{(-1)}=-2A\,z_{1,0}^{(-1)}\alpha_{1,1,0}^{(1)}-2A\,z_{1,0}^{(1)}\alpha_{1,1,0}^{(-1)}+C_{1,1},\\
&+2A\,z_{1,0}^{(-1)}\alpha_{1,1,1}^{(1)}+2A\,z_{1,0}^{(1)}\alpha_{1,1,1}^{(-1)}+2\sqrt{A}\,z_{1,0}^{(-1)}\beta_{1,1,1}^{(1)}-2\sqrt{A}\,z_{1,0}^{(1)}\beta_{1,1,1}^{(-1)}=+6A\,(z_{1,0}^{(1)})^2\\
&\qquad\qquad-2A\,z_{1,0}^{(-1)}\alpha_{1,1,1}^{(1)}-2A\,z_{1,0}^{(1)}\alpha_{1,1,1}^{(-1)}-6A\,(z_{1,0}^{(1)})^2,\\[1ex]
&+2A\,z_{1,0}^{(-1)}\beta_{1,1,1}^{(1)}+2A\,z_{1,0}^{(1)}\beta_{1,1,1}^{(-1)}+2\sqrt{A}\,z_{1,0}^{(-1)}\alpha_{1,1,1}^{(1)}-2\sqrt{A}\,z_{1,0}^{(1)}\alpha_{1,1,1}^{(-1)}\\
&\qquad\qquad=-2A\,z_{1,0}^{(-1)}\beta_{1,1,1}^{(1)}-2A\,z_{1,0}^{(1)}\beta_{1,1,1}^{(-1)}.
\end{aligned}
$$

From the first two sets of these equations we get

$$\omega_1 = 0, \quad \alpha_{1,1,1}^{(1)} = -\alpha_{1,1,1}^{(-1)} = \frac{3A\, z_{1,0}^{(1)}}{1-4\omega^2}, \quad \beta_{1,1,1}^{(1)} = +\beta_{1,1,1}^{(-1)} = -\frac{6A^{3/2}}{1-4\omega^2} z_{1,0}^{(1)} = -2\sqrt{A}\,\alpha_{1,1,1}^{(1)}. \tag{124}$$

The first of the last three equations imposes no condition upon the unknown coefficients since it involves the undetermined $C_{1,1}$, and the second and third equations of the last set become identities. The coefficients $\alpha_{1,1,0}^{(-1)}$ and $\alpha_{1,1,0}^{(1)}$, which are still undetermined, are not involved in these equations.

The fact that ω_1 is zero was known in advance, for it was proved in §120 that ω is a series in even powers of e. It has also been shown that $z_1^{(-1)}$ and $z_1^{(1)}$, aside from constant factors, differ only in the sign of $\sqrt{-1}$. Since $z(0)=0$ these constant factors differ only in sign, from which it follows that $\alpha_{1,j,p}^{(-1)}$ differs from $\alpha_{1,j,p}^{(1)}$ only in sign, while $\beta_{1,j,p}^{(-1)}$ and $\beta_{1,j,p}^{(1)}$ are equal for all j and p. Applying the condition $z(0)=0$ to the terms under consideration at present, we have $z_{1,1}^{(-1)}(0) + z_{1,1}^{(1)}(0) = 0$. On making use of (124), this equation leads to the result

$$\alpha_{1,1,0}^{(-1)} = -\alpha_{1,1,0}^{(1)}, \tag{125}$$

and $\alpha_{1,1,0}^{(1)}$ alone remains undetermined at this step. Of course, it should be noted that $\alpha_{1,1,1}^{(1)}$ and $\beta_{1,1,1}^{(1)}$ are expressed linearly in terms of the undetermined constant $z_{1,0}^{(1)} = \alpha_{1,0,0}^{(1)}$, whose value will be fixed when we treat the coefficient of λ^2 in the integral.

133. Coefficients of e^2 and e^k.—From the consideration of this step we can infer the character of the process in general. Because of the relations between $\alpha_{1,j,p}^{(-1)}$, $\beta_{1,j,p}^{(-1)}$ and $\alpha_{1,j,p}^{(1)}$, $\beta_{1,j,p}^{(1)}$ it is sufficient to equate to zero the coefficient of $e^{2\omega\sqrt{-1}t}$ in (116). Since we are interested only in the possibility of determining the unknown coefficients, it will be sufficient to write out the equations explicitly only so far as they involve these unknowns. Upon equating to zero the terms independent of t and the coefficients of $\cos t$, $\cos 2t$, $\sin t$, and $\sin 2t$ in order in the coefficient of e^2, we find

$$\left.\begin{aligned} -2\sqrt{A}(z_{1,0}^{(1)})^2\omega_2 = f^{(1)}(z_{1,0}^{(1)})^2, \quad -2\sqrt{A}\,z_{1,0}^{(1)}\beta_{1,2,1}^{(1)} = g_{1,2,1}^{(0)}, \quad -4\sqrt{A}\,z_{1,0}^{(1)}\beta_{1,2,2}^{(1)} = g_{1,2,2}^{(0)}, \\ -2\sqrt{A}\,z_{1,0}^{(1)}\alpha_{1,2,1}^{(1)} = f_{1,2,1}^{(0)}, \quad -4\sqrt{A}\,z_{1,0}^{(1)}\alpha_{1,2,2}^{(1)} = f_{1,2,2}^{(0)}, \end{aligned}\right\} \tag{126}$$

where $f^{(1)}$ is known and where $g_{1,2,1}^{(0)}, \ldots, f_{1,2,2}^{(0)}$ are homogeneous functions of the second degree in $z_{1,0}^{(1)} = \alpha_{1,0,0}^{(1)}$ and $\alpha_{1,1,0}^{(1)}$, and are linear in $\alpha_{1,1,0}^{(1)}$ alone. The first equation uniquely determines ω_2; the remainder determine $\beta_{1,2,1}^{(1)}, \ldots, \alpha_{1,2,2}^{(1)}$ uniquely when $z_{1,0}^{(1)}$ and $\alpha_{1,1,0}^{(1)}$ become known, as they do when the coefficient of λ^2 is considered. The coefficient $\alpha_{1,2,0}^{(1)}$ is so far entirely arbitrary.

The coefficients of e^k lead to similar equations. The first has ω_k in place of ω_2, and the left members of the remainder involve $\beta_{1,k,1}^{(1)}, \ldots, \beta_{1,k,k}^{(1)}$, $\alpha_{1,k,1}^{(1)}, \ldots, \alpha_{1,k,k}^{(1)}$, the numerical coefficient of $\beta_{1,k,p}^{(1)}$ and $\alpha_{1,k,p}^{(1)}$ being $-2p\sqrt{A}\,z_{1,0}^{(1)}$. The right members are homogeneous second-degree functions of $\alpha_{1,0,0}^{(0)}, \ldots, \alpha_{1,k-1,0}^{(1)}$. The $\alpha_{1,k,0}^{(1)}$ remains arbitrary.

134. General Equations for $\nu=1$.—The coefficients of z_3 are determined from $F_2=0$. From (82), (83), and (113) we find explicitly that

$$\left.\begin{aligned}
&2z_1'z_3'+2\left[A+3Ae\cos t+\cdots\right]z_1z_3=-(x_2')^2-(y_2')^2+\left[1+2A+6Ae\cos t+\cdots\right]x_2^2\\
&\quad+2\left[6Ae\sin t+\cdots\right]x_2y_2+\left[6Ae\sin t+\cdots\right]y_2^2\\
&\quad+\left[\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}+3\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\cos t+\cdots\right]z_1^2+\left[\frac{3}{8}\left(\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right)\right.\\
&\quad\left.+\frac{3}{2}\left(\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right)e\cos t+\cdots\right]z_1^4+2\int\left[3Ae\sin t+\cdots\right]x_2^2\,dt\\
&\quad-2\int\left[6Ae\cos t+\cdots\right]x_2y_2\,dt-\int\left[3Ae\sin t+\cdots\right]y_2^2\,dt\\
&\quad-2\int\left[3Ae\sin t+\cdots\right]z_1z_3\,dt+\int\left[3\left(\frac{1}{r_1^{(0)3}}-\frac{1}{r_2^{(0)3}}\right)e\sin t+\cdots\right]z_1^2\,dt\\
&\quad+3\int\left[\left(\frac{1-\mu_0}{r_1^{(0)5}}+\frac{\mu_0}{r_2^{(0)5}}\right)e\sin t+\cdots\right]z_1^4\,dt+\cdots+C_2.
\end{aligned}\right\}\quad(127)$$

The x_2 and y_2 are determined from equations (55), and it is seen from these equations that they are homogeneous of the second degree in the coefficients of z_1. Other properties of the solutions are given in §121, among which is that they are of even degree in $e^{\omega\sqrt{-1}t}$ and $e^{-\omega\sqrt{-1}t}$.

The expression for z_3 has the form

$$z_3=z_3^{(-3)}e^{-3\omega\sqrt{-1}t}+z_3^{(-1)}e^{-\omega\sqrt{-1}t}+z_3^{(1)}e^{\omega\sqrt{-1}t}+z_3^{(3)}e^{3\omega\sqrt{-1}t}. \quad (128)$$

The coefficients of $z_3^{(-3)}$ and $z_3^{(-1)}$ differ from those of $z_3^{(3)}$ and $z_3^{(1)}$ respectively, aside from constant factors, only in the sign of $\sqrt{-1}$, and these constant factors differ only in sign. Hence it is sufficient to determine the coefficients of $z_3^{(1)}$ and $z_3^{(3)}$, which have the form

$$\left.\begin{aligned}
z_3^{(1)}&=\sum_{k=0}^{\infty}z_{3,k}^{(1)}e^k, & z_{3,k}^{(1)}&=\sum_{p=0}^{k}\left[\alpha_{3,k,p}^{(1)}\cos pt+\sqrt{-1}\,\beta_{3,k,p}^{(1)}\sin pt\right],\\
z_3^{(3)}&=\sum_{k=0}^{\infty}z_{3,k}^{(3)}e^k, & z_{3,k}^{(3)}&=\sum_{p=0}^{k}\left[\alpha_{3,k,p}^{(3)}\cos pt+\sqrt{-1}\,\beta_{3,k,p}^{(3)}\sin pt\right].
\end{aligned}\right\}\quad(129)$$

135. Terms Independent of e.—Equations (128) and (129) are to be substituted in (127) and the coefficients of $e^{2\omega\sqrt{-1}t}$ and $e^{4\omega\sqrt{-1}t}$ set equal to zero. These terms are power series in e whose coefficients separately must be set equal to zero. We are now interested in the terms which are independent of e. These results were worked out in §129, where the parts of x_2 and y_2 independent of e were derived. The explicit results were given in equations (103), the relations in the present notations being

$$\alpha_{1,0,0}^{(1)}=\gamma_1^{(1)},\qquad \alpha_{3,0,0}^{(1)}=\gamma_3^{(1)},\qquad \alpha_{3,0,0}^{(3)}=\gamma_3^{(3)}.$$

The condition for the consistency of the two expressions for $\gamma_3^{(3)}$ is equation (106), which determines $\gamma_1^{(1)} = \alpha_{1,0,0}^{(1)}$ except as to sign. Therefore, by §132, $\alpha_{1,1,1}^{(1)}$ and $\beta_{1,1,1}^{(1)}$ are also determined except as to sign, and $\alpha_{3,0,0}^{(3)}$ is defined. And by §133 it is seen that the $\alpha_{1,k,p}^{(1)}$, $\beta_{1,k,p}^{(1)}$ $(p \neq 0)$ are all determined except as to sign, while the $\alpha_{1,k,0}^{(1)}$ remain as yet undetermined. The equations corresponding to (103) determine $\gamma_3^{(3)}$ uniquely, but $\gamma_3^{(1)}$ remains so far arbitrary.

136. Coefficients of *e*.—We shall write explicitly only the terms which involve those coefficients $\alpha_{3,k,p}^{(1)}$, $\beta_{3,k,p}^{(1)}$, $\alpha_{3,k,p}^{(3)}$, and $\beta_{3,k,p}^{(3)}$ which, at the successive steps, are unknown. The quantity z_3 is involved in the right member of (127) under the integral sign, but since this term is multiplied by e, it introduces at this step only $\alpha_{3,0,0}^{(1)}$ as an unknown coefficient, and this undetermined constant enters linearly. It is determined from the terms which are independent of e when $\nu = 2$.

Consider first the parts of the coefficients of $e^{2\omega\sqrt{-1}t}$ and $e^{4\omega\sqrt{-1}t}$ which are independent of $\sin t$ and $\cos t$. We find from (127) that

$$4A\,\alpha_{1,0,0}^{(-1)}\alpha_{3,1,0}^{(3)} = -4A\,\alpha_{1,1,0}^{(-1)}\alpha_{3,0,0}^{(3)} + f_{3,1,0}^{(-1)}, \quad -2A\,\alpha_{1,0,0}^{(1)}\alpha_{3,1,0}^{(3)} = 2A\,\alpha_{1,1,0}^{(1)}\alpha_{3,0,0}^{(3)} + f_{3,1,0}^{(1)}, \tag{130}$$

where $f_{3,1,0}^{(-1)}$ and $f_{3,1,0}^{(1)}$ are linear functions of $\alpha_{1,1,0}^{(1)}$, which is the only unknown that they involve. Since $\alpha_{1,0,0}^{(-1)} = -\alpha_{1,0,0}^{(1)}$, the condition that equations (130) shall be consistent is a condition on their right members which a detailed discussion shows uniquely determines the coefficient $\alpha_{1,1,0}^{(1)}$. Then equations (130) uniquely define $\alpha_{3,1,0}^{(3)}$.

Now we set equal to zero the coefficients of $e^{2\omega\sqrt{-1}t}\sin t$, $e^{4\omega\sqrt{-1}t}\sin t$, $e^{2\omega\sqrt{-1}t}\cos t$, and $e^{4\omega\sqrt{-1}t}\cos t$. The explicit expressions are found from equations (127) to be, respectively,

$$\left.\begin{aligned}
\sqrt{-1}\sqrt{A}\,[+\alpha_{1,0,0}^{(-1)}\,\alpha_{3,1,1}^{(3)} + 4\sqrt{A}\,\alpha_{1,0,0}^{(-1)}\,\beta_{3,1,1}^{(3)} - \alpha_{1,0,0}^{(1)}\,\alpha_{3,1,1}^{(1)}] &= \varphi_1, \\
\sqrt{-1}\sqrt{A}\,[-\alpha_{1,0,0}^{(1)}\,\alpha_{3,1,1}^{(3)} - 2\sqrt{A}\,\alpha_{1,0,0}^{(1)}\,\beta_{3,1,1}^{(3)}] &= \varphi_2, \\
\sqrt{A}\,[+4\sqrt{A}\,\alpha_{1,0,0}^{(-1)}\,\alpha_{3,1,1}^{(3)} + \alpha_{1,0,0}^{(-1)}\,\beta_{3,1,1}^{(3)} - \alpha_{1,0,0}^{(1)}\,\beta_{3,1,1}^{(1)}] &= \varphi_3, \\
\sqrt{A}\,[-2\sqrt{A}\,\alpha_{1,0,0}^{(1)}\,\alpha_{3,1,1}^{(3)} - \alpha_{1,0,0}^{(1)}\,\beta_{3,1,1}^{(3)}] &= \varphi_4,
\end{aligned}\right\} \tag{131}$$

where $\varphi_1, \ldots, \varphi_4$ are functions of known quantities and the arbitrary $\alpha_{3,0,0}^{(1)}$, which enters linearly. This constant remains undetermined until the equations are derived for $\nu = 2$. The unknowns in the left members of (131) are $\alpha_{3,1,1}^{(3)}$, $\beta_{3,1,1}^{(3)}$, $\alpha_{3,1,1}^{(1)}$, and $\beta_{3,1,1}^{(1)}$, which enter linearly. On making use of the fact that $\alpha_{1,0,0}^{(-1)} = -\alpha_{1,0,0}^{(1)}$, the determinant of their coefficients becomes

$$\Delta = -A^2(\alpha_{1,0,0}^{(1)})^4[4A - 1], \tag{132}$$

which is distinct from zero. Therefore these quantities are uniquely determined as linear functions of the arbitrary $\alpha_{3,0,0}^{(1)}$.

This illustrates sufficiently the method of determining the coefficients from the integral. The complexity of the details makes it unprofitable to carry the explicit results further.

DIRECT CONSTRUCTION OF THE TWO-DIMENSIONAL SYMMETRICAL PERIODIC SOLUTIONS.

137. Terms in $\lambda^{1/2}$.—It was shown in § 116 that the periodic solutions exist and are expansible as power series in $\lambda^{1/2}$. It is found from equations (13) that the terms of the first degree in $\lambda^{1/2}$ are defined by

$$\left.\begin{aligned} &x_1''-2y_1'-[1+2A+6Ae\cos t+\cdots]x_1-[6Ae\sin t+\cdots]y_1=0,\\ &y_1''+2x_1'-[6Ae\sin t+\cdots]x_1-[1-A-3Ae\cos t+\cdots]y_1=0. \end{aligned}\right\} \quad (133)$$

The general solutions of equations (133) are known from the general theory to have the form

$$\left.\begin{aligned} &x_1=a_1^{(1)}e^{\sigma\sqrt{-1}t}u_1+a_2^{(1)}e^{-\sigma\sqrt{-1}t}u_2+a_3^{(1)}e^{\rho t}u_3+a_4^{(1)}e^{-\rho t}u_4,\\ &y_1=a_1^{(1)}e^{\sigma\sqrt{-1}t}v_1+a_2^{(1)}e^{-\sigma\sqrt{-1}t}v_2+a_3^{(1)}e^{\rho t}v_3+a_4^{(1)}e^{-\rho t}v_4,\\ &\sigma=\sigma_0+\sigma_2e^2+\cdots,\\ &u_i=u_i^{(0)}+u_i^{(1)}e+\cdots \qquad (i=1,\ldots,4),\\ &v_i=v_i^{(0)}+v_i^{(1)}e+\cdots, \end{aligned}\right\} \quad (134)$$

where $a_1^{(1)}, \ldots, a_4^{(1)}$ are arbitrary constants, and where the u_i and the v_i are periodic functions of t with the period 2π.

In order that the solutions shall be periodic we must first impose the conditions

$$a_3^{(1)}=a_4^{(1)}=0. \quad (135)$$

The constant σ is a continuous function of μ, μ_0, and e. It will be supposed that these parameters have such values that σ is a rational number. Then, since $u_i^{(j)}$ and $v_i^{(j)}$ are periodic with the period 2π, the solution at this step is periodic with the period T, where T is a multiple of 2π and $2\pi/\sigma$.

In the symmetrical solutions, $x'(0)=y(0)=0$. Since these relations are identities in $\lambda^{\frac{1}{2}}$, we have $x_1'(0)=y_1(0)=0$. Since in the symmetrical orbits y_1 changes sign with a change of sign of t while its numerical value remains unaltered, we have

$$\begin{aligned} a_1^{(1)}e^{\sigma\sqrt{-1}t}v_1(t)+a_2^{(1)}e^{-\sigma\sqrt{-1}t}v_2(t)+a_3^{(1)}e^{\rho t}v_3(t)+a_4^{(1)}e^{-\rho t}v_4(t)&\equiv-a_1^{(1)}e^{-\sigma\sqrt{-1}t}v_1(-t)\\ -a_2^{(1)}e^{\sigma\sqrt{-1}t}v_2(-t)-a_3^{(1)}e^{-\rho t}v_3(-t)-a_4^{(1)}e^{\rho t}v_4(-t)&. \end{aligned}$$

It follows from this identity that

$$a_1^{(1)}v_1(t)=-a_2^{(1)}v_2(-t), \qquad a_3^{(1)}v_3(t)=-a_4^{(1)}v_4(-t).$$

Without restricting the generality of the results, we may suppose that $v_1(0)=v_2(0)=v_3(0)=v_4(0)=1$. Therefore we have

$$\left.\begin{aligned}
&a_2^{(1)}=-a_1^{(1)},\\
&a_4^{(1)}=-a_3^{(1)}(=0 \text{ in case of periodic orbits}),\\
&x_1=+a_1^{(1)}\left[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2\right]+a_3^{(1)}\left[e^{\rho t}u_3-e^{-\rho t}u_4\right],\\
&y_1=+a_1^{(1)}\left[e^{\sigma\sqrt{-1}t}v_1-e^{-\sigma\sqrt{-1}t}v_2\right]+a_3^{(1)}\left[e^{\rho t}v_3-e^{-\rho t}v_4\right].
\end{aligned}\right\} \quad (136)$$

We shall suppose that the initial conditions are real as well as such as to give the symmetrical orbits. Then, since changing the sign of $\sqrt{-1}$ in (133) does not alter these equations, with the same initial conditions the solutions will be identical with (136). Therefore we have

$$\begin{aligned}
&u_1(-\sqrt{-1})=-u_2(\sqrt{-1}), \quad u_3(-\sqrt{-1})=u_3(\sqrt{-1}), \quad u_4(-\sqrt{-1})=u_4(\sqrt{-1}),\\
&v_1(-\sqrt{-1})=-v_2(\sqrt{-1}), \quad v_3(-\sqrt{-1})=v_3(\sqrt{-1}), \quad v_4(-\sqrt{-1})=v_4(\sqrt{-1}).
\end{aligned}$$

If we change the sign of both t and y_1, equations (133) are unaltered. With the same initial conditions as before, which this transformation does not affect, since $y_1(0)=0$, we have an identical solution except that y_1 is changed in sign. Therefore

$$\begin{aligned}
&u_1(-t)=-u_2(t), \qquad u_3(-t)=-u_4(t),\\
&v_1(-t)=+v_2(t), \qquad v_3(-t)=+v_4(t).
\end{aligned}$$

Now if $\sqrt{-1}$, t, and y_1 are changed in sign the differential equations are unchanged, and hence it follows that

$$\begin{aligned}
&u_1(-\sqrt{-1},-t)=+u_1(\sqrt{-1},t), \qquad v_1(-\sqrt{-1},-t)=-v_1(\sqrt{-1},t),\\
&u_2(-\sqrt{-1},-t)=+u_2(\sqrt{-1},t), \qquad v_2(-\sqrt{-1},-t)=-v_2(\sqrt{-1},t),\\
&u_3(-\sqrt{-1},-t)=-u_4(\sqrt{-1},t), \qquad v_3(-\sqrt{-1},-t)=+v_4(\sqrt{-1},t),\\
&u_4(-\sqrt{-1},-t)=-u_3(\sqrt{-1},t); \qquad v_4(-\sqrt{-1},-t)=+v_3(\sqrt{-1},t).
\end{aligned}$$

It follows from the last three sets of relations that $u_4, \ldots, u_4$, $v_1, \ldots, v_4$, when expressed as Fourier series, have the form

$$\left.\begin{aligned}
&u_1=\Sigma[+a_j\cos jt+\sqrt{-1}\,b_j\sin jt], \qquad u_3=\Sigma[+c_j\cos jt+d_j\sin jt],\\
&u_2=\Sigma[-a_j\cos jt+\sqrt{-1}\,b_j\sin jt], \qquad u_4=\Sigma[-c_j\cos jt+d_j\sin jt],\\
&v_1=\Sigma[+\sqrt{-1}\,\alpha_j\cos jt+\beta_j\sin jt], \qquad v_3=\Sigma[+\gamma_j\cos jt+\delta_j\sin jt],\\
&v_2=\Sigma[+\sqrt{-1}\,\alpha_j\cos jt-\beta_j\sin jt], \qquad v_4=\Sigma[+\gamma_j\cos jt-\delta_j\sin jt],
\end{aligned}\right\} \quad (137)$$

where the a_j, b_j, c_j, d_j, α_j, β_j, γ_j, and δ_j are real constants and power series in e.

In the case of the periodic orbits we have simply

$$x_1=a_1^{(1)}\left[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2\right], \qquad y_1=a_1^{(1)}\left[e^{\sigma\sqrt{-1}t}v_1-e^{-\sigma\sqrt{-1}t}v_2\right]. \tag{138}$$

In the case of the periodic orbits it follows from equations (137) that the numerical coefficients of the cosine terms in x_1 are real, and that those of the sine terms are purely imaginary; and the opposite is true in y_1.

138. Coefficients of λ.—It is found from equations (13) that these coefficients are defined by

$$\left.\begin{aligned} &x_2''-2y_2'-[1+2A+6Ae\cos t+\cdots]x_2-[6Ae\sin t+\cdots]y_2=X_2,\\ &y_2''+2x_2'-[\ 6Ae\sin t+\cdots]x_2-[1-A-3Ae\cos t+\cdots]y_2=Y_2, \end{aligned}\right\} \tag{139}$$

$$\left.\begin{aligned} X_2&=\left[-3B-12Be\cos t+\cdots\right]x_1^2+\left[-24Be\sin t+\cdots\right]x_1y_1\\ &\qquad+\left[\tfrac{3}{2}B+6Be\cos t+\cdots\right]y_1^2,\\ Y_2&=\left[-12Be\sin t+\cdots\right]x_1^2+\left[3B+12Be\cos t+\cdots\right]x_1y_1\\ &\qquad+\left[9Be\sin t+\cdots\right]y_1^2,\\ B&=\pm\frac{1-\mu_0}{r_1^{(0)4}}\pm\frac{\mu_0}{r_2^{(0)4}}. \end{aligned}\right\} \tag{140}$$

The character of the solutions of the equations of the type to which (139) belongs was determined in §30, where it was shown that they consist of the complementary function plus terms of the same character as X_2 and Y_2. Hence the periodic solution of (139) is

$$\left.\begin{aligned} x_2&=a_1^{(2)}e^{\sigma\sqrt{-1}t}u_1+a_2^{(2)}e^{-\sigma\sqrt{-1}t}u_2+f_2,\\ y_2&=a_1^{(2)}e^{\sigma\sqrt{-1}t}v_1+a_2^{(2)}e^{-\sigma\sqrt{-1}t}v_2+g_2, \end{aligned}\right\} \tag{141}$$

where $a_1^{(2)}$ and $a_2^{(2)}$ are constants which are as yet undetermined, and where f_2 and g_2 are the particular solutions.

We shall need certain properties of f_2 and g_2. It is evident from (139) and (140) that they are homogeneous of the second degree in $a_1^{(1)}$ and in $e^{\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$. It follows from (137) and (138) that in x_1^2 and y_1^2 the coefficients of those cosine terms which are multiplied by $e^{2\sigma\sqrt{-1}t}$ and $e^{-2\sigma\sqrt{-1}t}$ are real and identical, while the coefficients of the sine terms are purely imaginary and differ only in sign. The terms which are independent of $e^{\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$ consist only of cosines whose coefficients are real. In the product x_1y_1 the coefficients of those cosine terms which are multiplied by $e^{2\sigma\sqrt{-1}t}$ are purely imaginary and differ from the coefficients of those cosine terms which are multiplied by $e^{-2\sigma\sqrt{-1}t}$ only in sign, while the coefficients of

the sine terms are real and identical. The terms which are independent of $e^{\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$ are only sine terms and are real. Therefore it follows from (140) that X_2 and Y_2 have the form

$$X_2 = X_2^{(2)} e^{2\sigma\sqrt{-1}t} + X_2^{(-2)} e^{-2\sigma\sqrt{-1}t} + X_2^{(0)},$$

$$Y_2 = Y_2^{(2)} e^{2\sigma\sqrt{-1}t} + Y_2^{(-2)} e^{-2\sigma\sqrt{-1}t} + Y_2^{(0)}.$$

In $X_2^{(2)}$ and $X_2^{(-2)}$ the coefficients of the cosine terms are real and identical, and the coefficients of the sine terms are purely imaginary and differ only in sign. In $X_2^{(0)}$ there are only cosine terms and their coefficients are real. In $Y_2^{(2)}$ and $Y_2^{(-2)}$ the coefficients of the cosine terms are purely imaginary and differ only in sign, and the coefficients of the sine terms are real and identical. In $Y_2^{(0)}$ there are only sine terms and the coefficients are real.

It follows from the properties of X_2 and Y_2 which have just been derived, and from the form of equations (139), that f_2 and g_2 have the form

$$\left.\begin{aligned} f_2 &= (a_1^{(1)})^2 f_2^{(2)} e^{2\sigma\sqrt{-1}t} + (a_1^{(1)})^2 f_2^{(-2)} e^{-2\sigma\sqrt{-1}t} + (a_1^{(1)})^2 f_2^{(0)}, \\ g_2 &= (a_1^{(1)})^2 g_2^{(2)} e^{2\sigma\sqrt{-1}t} + (a_1^{(1)})^2 g_2^{(-2)} e^{-2\sigma\sqrt{-1}t} + (a_1^{(1)})^2 g_2^{(0)}, \end{aligned}\right\} \tag{142}$$

where $f_2^{(2)}$, $f_2^{(-2)}$, $f_2^{(0)}$, $g_2^{(2)}$, $g_2^{(-2)}$, and $g_2^{(0)}$ are periodic with the period 2π. It follows from the properties of X_2 and Y_2 which have been found that equations (139) are not changed if in them the sign of $\sqrt{-1}$ is changed. Therefore this property is true of the particular solutions, and we have

$$f_2^{(2)}(-\sqrt{-1}) = f_2^{(-2)}(\sqrt{-1}), \qquad f_2^{(0)}(-\sqrt{-1}) = f_2^{(0)}(\sqrt{-1}),$$
$$g_2^{(2)}(-\sqrt{-1}) = g_2^{(-2)}(\sqrt{-1}), \qquad g_2^{(0)}(-\sqrt{-1}) = g_2^{(0)}(\sqrt{-1}).$$

It also follows from the properties of X_2 and Y_2 that if we change the sign of t and y_2, equations (139) are not altered. Therefore

$$f_2^{(2)}(-t) = +f_2^{(-2)}(t), \qquad f_2^{(0)}(-t) = +f_2^{(0)}(t),$$
$$g_2^{(2)}(-t) = -g_2^{(-2)}(t), \qquad g_2^{(0)}(-t) = -g_2^{(0)}(t).$$

It can be proved similarly, from a consideration of equations (139), that

$$f_2^{(2)}(-\sqrt{-1}, -t) = +f_2^{(2)}(\sqrt{-1}, t), \qquad g_2^{(2)}(-\sqrt{-1}, -t) = -g_2^{(2)}(\sqrt{-1}, t),$$
$$f_2^{(-2)}(-\sqrt{-1}, -t) = +f_2^{(-2)}(\sqrt{-1}, t), \qquad g_2^{(-2)}(-\sqrt{-1}, -t) = -g_2^{(-2)}(\sqrt{-1}, t),$$
$$f_2^{(0)}(-\sqrt{-1}, -t) = +f_2^{(0)}(\sqrt{-1}, t), \qquad g_2^{(0)}(-\sqrt{-1}, -t) = -g_2^{(0)}(\sqrt{-1}, t).$$

It follows from these three sets of relations that when $f_2^{(2)}, \ldots, g_2^{(0)}$ are written as Fourier series they have the form

$$\left.\begin{array}{ll} f_2^{(2)} = \Sigma[a_j\cos jt+\sqrt{-1}\, b_j \sin jt], & g_2^{(2)} = \Sigma[+\sqrt{-1}\, \alpha_j\cos jt + \beta_j \sin jt], \\ f_2^{(-2)} = \Sigma[a_j\cos jt-\sqrt{-1}\, b_j \sin jt], & g_2^{(-2)} = \Sigma[-\sqrt{-1}\, \alpha_j\cos jt + \beta_j \sin jt], \\ f_2^{(0)} = \Sigma\, c_j\cos jt, & g_2^{(0)} = \Sigma \gamma_j \sin jt, \end{array}\right\} \quad (143)$$

where the a_j, b_j, c_j, α_j, β_j, and γ_j are real constants and power series in e.

It is seen from equations (142) and (143) that $g_2(0)=0$. Therefore, since $y_2(0)=0$, we have $a_2^{(2)}=-a_1^{(2)}$, and equations (141) become

$$\left.\begin{array}{l} x_2=a_1^{(2)}\left[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2\right]+(a_1^{(1)})^2f_2^{(2)}e^{2\sigma\sqrt{-1}t}+(a_1^{(1)})^2f_2^{(-2)}e^{-2\sigma\sqrt{-1}t}+(a_1^{(1)})^2f_2^{(0)}, \\ y_2=a_1^{(2)}\left[e^{\sigma\sqrt{-1}t}v_1-e^{-\sigma\sqrt{-1}t}v_2\right]+(a_1^{(1)})^2g_2^{(2)}e^{2\sigma\sqrt{-1}t}+(a_1^{(1)})^2g_2^{(-2)}e^{-2\sigma\sqrt{-1}t}+(a_1^{(1)})^2g_2^{(0)}, \end{array}\right\} (144)$$

where both $a_1^{(1)}$ and $a_1^{(2)}$ are so far undetermined constants.

139. Coefficients of $\lambda^{3/2}$.—It is found from equations (12) and (13) that these terms are defined by

$$\left.\begin{array}{l} x_3''-2y_3'-\left[1+2A+6Ae\cos t+\cdots\right]x_3-\left[6Ae\sin t+\cdots\right]y_3=X_3, \\ y_3''+2x_3'-\left[6Ae\sin t+\cdots\right]x_3-\left[1-A-3Ae\cos t+\cdots\right]y_3=Y_3; \\[2ex] X_3=+\left[-2K-6Ke\cos t+\cdots\right]x_1+\left[-6Ke\sin t+\cdots\right]y_1 \\ \qquad +\left[-6B-24Be\cos t+\cdots\right]x_1x_2+\left[-24Be\sin t+\cdots\right](x_1y_2+x_2y_1) \\ \qquad +\left[3B+12Be\cos t+\cdots\right]y_1y_2+\left[4C+\cdots\right]x_1^3+\left[-6C+\cdots\right]x_1y_1^2; \\[2ex] Y_3=+\left[-6Ke\sin t+\cdots\right]x_1+\left[K+3Ke\cos t+\cdots\right]y_1 \\ \qquad +\left[-24Be\sin t+\cdots\right]x_1x_2+\left[3B+12Be\cos t+\cdots\right](x_1y_2+x_2y_1) \\ \qquad +\left[18Be\sin t+\cdots\right]y_1y_2+\left[-6C+\cdots\right]x_1^2y_1+\left[3C+\cdots\right]y_1^3; \\[2ex] K=\dfrac{1}{r_1^{(0)3}}-\dfrac{1}{r_2^{(0)3}}, \qquad C=\dfrac{1-\mu_0}{r_1^{(0)5}}+\dfrac{\mu_0}{r_2^{(0)5}}. \end{array}\right\} (145)$$

It follows from the results of §29 that in general the terms of the first degree in $e^{\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$ will introduce non-periodic terms into the solution. We must determine $a_1^{(1)}$, if possible, so as to make their coefficient vanish.

The general solution of the first two equations of (145), when X_3 and Y_3 are zero, is

$$
\left.\begin{aligned}
x_3 &= a_1^{(3)} e^{\sigma\sqrt{-1}t} u_1 + a_2^{(3)} e^{-\sigma\sqrt{-1}t} u_2 + a_3^{(3)} e^{\rho t} u_3 + a_4^{(3)} e^{-\rho t} u_4, \\
y_3 &= a_1^{(3)} e^{\sigma\sqrt{-1}t} v_1 + a_2^{(3)} e^{-\sigma\sqrt{-1}t} v_2 + a_3^{(3)} e^{\rho t} v_3 + a_4^{(3)} e^{-\rho t} v_4,
\end{aligned}\right\} \quad (146)
$$

where $a_1^{(3)}, \ldots, a_4^{(3)}$ are arbitrary constants. Now, on supposing they are variables and subjecting them to the conditions that equations (142) shall be satisfied when their right members are included, we get

$$
\left.\begin{aligned}
& e^{\sigma\sqrt{-1}t} u_1 (a_1^{(3)})' + e^{-\sigma\sqrt{-1}t} u_2 (a_2^{(3)})' + e^{\rho t} u_3 (a_3^{(3)})' + e^{-\rho t} u_4 (a_4^{(3)})' = 0, \\
& e^{\sigma\sqrt{-1}t} \left[\sigma\sqrt{-1}\, u_1 + u_1'\right] (a_1^{(3)})' + e^{-\sigma\sqrt{-1}t} \left[-\sigma\sqrt{-1}\, u_2 + u_2'\right] (a_2^{(3)})' \\
& \qquad + e^{\rho t} \left[\rho u_3 + u_3'\right] (a_3^{(3)})' + e^{-\rho t} \left[-\rho u_4 + u_4'\right] (a_4^{(3)})' = X_3, \\
& e^{\sigma\sqrt{-1}t} v_1 (a_1^{(3)})' + e^{-\sigma\sqrt{-1}t} v_2 (a_2^{(3)})' + e^{\rho t} v_3 (a_3^{(3)})' + e^{-\rho t} v_4 (a_4^{(3)})' = 0, \\
& e^{\sigma\sqrt{-1}t} \left[\sigma\sqrt{-1}\, v_1 + v_1'\right] (a_1^{(3)})' + e^{-\sigma\sqrt{-1}t} \left[-\sigma\sqrt{-1}\, v_2 + v_2'\right] (a_2^{(3)})' \\
& \qquad + e^{\rho t} \left[\rho v_3 + v_3'\right] (a_3^{(3)})' + e^{-\rho t} \left[-\rho v_4 + v_4'\right] (a_4^{(3)})' = Y_3,
\end{aligned}\right\} \quad (147)
$$

where $(a_1^{(3)})', \ldots, (a_4^{(3)})'$ are the derivatives of $a_1^{(3)}, \ldots, a_4^{(3)}$ with respect to t. The solutions of these equations for $(a_1^{(3)})', \ldots, (a_4^{(3)})'$ are

$$
\left.\begin{aligned}
\Delta (a_1^{(3)})' &= \left[D_{11} X_3 + D_{12} Y_3\right] e^{-\sigma\sqrt{-1}t}, & \Delta (a_3^{(3)})' &= \left[D_{31} X_3 + D_{32} Y_3\right] e^{-\rho t}, \\
\Delta (a_2^{(3)})' &= \left[D_{21} X_3 + D_{22} Y_3\right] e^{+\sigma\sqrt{-1}t}, & \Delta (a_4^{(3)})' &= \left[D_{41} X_3 + D_{42} Y_3\right] e^{+\rho t},
\end{aligned}\right\} \quad (148)
$$

where

$$
\Delta = \begin{vmatrix}
u_1, & u_2, & u_3, & u_4 \\
\sigma\sqrt{-1}\, u_1 + u_1', & -\sigma\sqrt{-1}\, u_2 + u_2', & \rho u_3 + u_3', & -\rho u_4 + u_4' \\
v_1, & v_2, & v_3, & v_4 \\
\sigma\sqrt{-1}\, v_1 + v_1', & -\sigma\sqrt{-1}\, v_2 + v_2', & \rho v_3 + v_3', & -\rho v_4 + v_4'
\end{vmatrix},
$$

$$
D_{11} = -\begin{vmatrix}
u_2, & u_3, & u_4 \\
v_2, & v_3, & v_4 \\
-\sigma\sqrt{-1}\, v_2 + v_2', & \rho v_3 + v_3', & -\rho v_4 + v_4'
\end{vmatrix},
$$

$$
D_{12} = +\begin{vmatrix}
u_2, & u_3, & u_4 \\
v_2, & v_3, & v_4 \\
-\sigma\sqrt{-1}\, u_2 + u_2', & \rho u_3 + u_3', & -\rho u_4 + u_4'
\end{vmatrix};
$$

D_{21} and D_{22} are obtained from D_{11} and D_{12}, respectively, by changing the subscript 2 to 1 and by changing the sign of $\sqrt{-1}$ and of the whole expression; D_{31} and D_{32} are obtained from D_{11} and D_{12}, respectively, by changing the subscript 3 to 1, ρ to $\sigma\sqrt{-1}$, and by changing the sign of the whole expression; and D_{41} and D_{42} are obtained from D_{11} and D_{12}, respectively, by changing the subscript 4 to 1, $-\rho$ to $\sigma\sqrt{-1}$, and by changing the sign of the whole expression. It follows from the discussion of § 18 that Δ is a constant, and in this case it is a power series in e.

In order that the solution shall be periodic it is necessary that the right members of (148) shall contain no constant terms. We shall show these conditions are sufficient. When they are satisfied the general term of the right member of either of the first two equations has the form

$$[a_{j,k} \cos jt + b_{j,k} \sin jt]\, e^{2k\sigma\sqrt{-1}t},$$

where j and k are integers distinct from zero and where $a_{j,k}$ and $b_{j,k}$ are constants. Consequently $a_1^{(3)}$ and $a_2^{(3)}$ are sums of terms of the type

$$\left[a_{j,k}(2k\sigma\sqrt{-1}\cos jt + j\sin jt) + b_{j,k}(-j\cos jt + 2k\sigma\sqrt{-1}\sin jt\right]\frac{e^{2k\sigma\sqrt{-1}t}}{j^2-4k^2\sigma^2}. \quad (149)$$

The right member of the third equation of (148) never has any terms which are independent of t, but contains terms of the type

$$\left[a_{j,k}\cos jt + b_{j,k}\sin jt\right] e^{[(2k+1)\sigma\sqrt{-1}-\rho]t},$$

where j and k are integers. There can be no exception to this form. Therefore $a_3^{(3)}$ is a sum of terms of the type

$$\left.\begin{aligned} &a_{j,k}\left\{\left[(2k+1)\sigma\sqrt{-1}-\rho\right]\cos jt + j\sin jt\right\}\frac{e^{(2k+1)\sigma\sqrt{-1}t}\, e^{-\rho t}}{j^2+[(2k+1)\sigma\sqrt{-1}-\rho]^2}, \\ &b_{j,k}\left\{j\cos jt - \left[(2k+1)\sigma\sqrt{-1}-\rho\right]\sin jt\right\}\frac{e^{(2k+1)\sigma\sqrt{-1}t}\, e^{-\rho t}}{j^2+[(2k+1)\sigma\sqrt{-1}-\rho]^2}. \end{aligned}\right\} \quad (150)$$

The type terms for $a_4^{(3)}$ differ from those for $a_3^{(3)}$ only in the sign of ρ. There is an additive constant of integration with each of the $a_i^{(3)}$. It follows, from the form of (146), (149), and (150), that if we put the constants of integration associated with $a_3^{(3)}$ and $a_4^{(3)}$ equal to zero, the resulting expressions for x_3 and y_3 are periodic with the period T. They may be written in the form

$$x_3 = a_1^{(3)} e^{\sigma\sqrt{-1}t} u_1 + a_2^{(3)} e^{-\sigma\sqrt{-1}t} u_2 + f_3, \quad y_3 = a_1^{(3)} e^{\sigma\sqrt{-1}t} v_1 + a_2^{(3)} e^{-\sigma\sqrt{-1}t} v_2 + g_3, \quad (151)$$

where $a_1^{(3)}$ and $a_2^{(3)}$ are constants which so far are undetermined.

It remains to show that $a_1^{(1)}$ can be so determined that the right members of the first two equations of (148) shall have no constant terms. Let us consider the first of these equations. We are to set equal to zero the constant part of the coefficient of $e^{\sigma\sqrt{-1}t}$ in $D_{11}X_3+D_{12}Y_3$. It follows, from the form of X_3 and Y_3, equations (145), that the term which must be made to vanish does not depend on $a_1^{(2)}$. It also follows that the conditional equation which must be imposed has the form

$$P_1 a_1^{(1)} - Q_1 (a_1^{(1)})^3 = 0, \tag{152}$$

where P_1 and Q_1 are power series in e, the former coming from those terms of X_3 and Y_3 which are linear in x_1 and y_1 and independent of x_2 and y_2, and the latter coming from those terms which are of the third degree in x_1 and y_1, or which involve x_2 and y_2.

The solutions of (152) are $a_1^{(1)}=0$, which leads us to the trivial result $x \equiv y \equiv 0$, and

$$a_1^{(1)} = \pm\sqrt{\frac{P_1}{Q_1}}. \tag{153}$$

The significance of the double sign was discussed in §§116–118 in connection with the existence of the solutions. The expressions for P_1 and Q_1 are power series in e and both of them contain terms independent of e, as was shown in Chapter VI in the discussion of the corresponding problem for $e=0$. Therefore $a_1^{(1)}$ is a power series in e having an absolute term.

It remains to be shown that this value of $a_1^{(1)}$ also satisfies the equation which is obtained when the constant term of the right member of the second equation of (148) is set equal to zero. We shall show that the constant part of the coefficient of $e^{\sigma\sqrt{-1}t}$ in $D_{11}X_3+D_{12}Y_3$ is identical with the constant part of the coefficient of $e^{-\sigma\sqrt{-1}t}$ in $D_{21}X_3+D_{22}Y_3$. Let us first consider the term $[-2K-6Ke\cos t+\cdots]x_1$ of X_3 which contributes to P_1 of equation (152). So far as this term is concerned, we have

$$\left.\begin{aligned}\Delta(a_1^{(3)})' &= D_{11}[-2K-6Ke\cos t+\cdots]\,x_1\,e^{-\sigma\sqrt{-1}t},\\ \Delta(a_2^{(3)})' &= D_{21}[-2K-6Ke\cos t+\cdots]\,x_1\,e^{+\sigma\sqrt{-1}t}.\end{aligned}\right\} \tag{154}$$

On referring to (138) and the values of D_{11} and D_{21}, we see that the constant parts of the right members of these two equations are respectively the constant parts of

$$-a_1^{(1)}\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ -\sigma\sqrt{-1}\,v_2+v_2', & \rho v_3+v_3', & -\rho v_4+v_4' \end{vmatrix}[-2K-6Ke\cos t+\cdots]\,u_1,$$

$$-a_1^{(1)}\begin{vmatrix} u_1, & u_3, & u_4 \\ v_1, & v_3, & v_4 \\ +\sigma\sqrt{-1}\,v_1+v_1', & \rho v_3+v_3', & -\rho v_4+v_4' \end{vmatrix}[-2K-6Ke\cos t+\cdots]\,u_2.$$

These expressions are equivalent to

$$-\frac{a_1^{(1)}}{2}\begin{vmatrix} u_1u_2 & , & u_3-u_4 & , & u_3+u_4 \\ u_1v_2 & , & v_3-v_4 & , & v_3+v_4 \\ -\sigma\sqrt{-1}\,u_1v_2+u_1v_2', & & \rho(v_3+v_4)+v_3'-v_4', & & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix} F_1(t),$$

$$-\frac{a_2^{(1)}}{2}\begin{vmatrix} u_1u_2 & , & u_3-u_4 & , & u_3+u_4 \\ u_2v_1 & , & v_3-v_4 & , & v_3+v_4 \\ +\sigma\sqrt{-1}\,u_2v_1+u_2v_1', & & \rho(v_3+v_4)+v_3'-v_4', & & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix} F_1(t),$$

where

$$F(t)=[-2K-6Ke\cos t+\cdots].$$

The parts of these expressions containing u_1u_2 as a factor are identical and need no further consideration. The parts multiplied by u_1v_2 and u_2v_1, so far as they appear in the second lines of the determinants, are respectively

$$+\frac{a_1^{(1)}}{2}u_1v_2\begin{vmatrix} u_3-u_4 & , & u_3+u_4 \\ \rho(v_3+v_4)+v_3'-v_4', & & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix}[-2K-6Ke\cos t+\cdots],$$

$$+\frac{a_1^{(1)}}{2}u_2v_1\begin{vmatrix} u_3-u_4 & , & u_3+u_4 \\ \rho(v_3+v_4)+v_3'-v_4', & & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix}[-2K-6Ke\cos t+\cdots].$$

It follows from (137) that u_3-u_4 is a sum of cosine terms, that u_3+u_4 is a sum of sine terms, that v_3+v_4 is a sum of cosine terms, that $v_3'-v_4'$ is a sum of cosine terms, that v_3-v_4 is a sum of sine terms, and that $v_3'+v_4'$ is a sum of sine terms. Therefore the determinant parts of these two expressions are sums of sine terms, which, multiplied by a cosine series on the right, are sums of sine terms. Hence, to get the constant parts of these expressions, we need only the sine terms of the products u_1v_2 and u_2v_1. It is seen at once from (137) that the sine terms of these products are identical, but that the cosine terms differ in sign. Therefore the constant terms coming from the parts of the two expressions which are multiplied by u_1v_2 and u_2v_1, so far as they come from the second lines of the determinants, are identical.

The parts of the expressions which contain u_1v_2 and u_2v_1, so far as they come from the third lines of the determinants, are respectively

$$+\frac{a_1^{(1)}}{2}\sigma\sqrt{-1}\,u_1v_2\begin{vmatrix} u_3-u_4, & u_3+u_4 \\ v_3-v_4, & v_3+v_4 \end{vmatrix}[-2K-6Ke\cos t+\cdots],$$

$$-\frac{a_1^{(1)}}{2}\sigma\sqrt{-1}\,u_2v_1\begin{vmatrix} u_3-u_4, & u_3+u_4 \\ v_3-v_3, & v_3+v_4 \end{vmatrix}[-2K-6Ke\cos t+\cdots].$$

The determinant is in this case a sum of cosine terms. Therefore we need only the cosine terms from $+u_1v_2$ and $-u_2v_1$. It is seen from (137) that

they are identical Therefore the constant parts of the two expressions, so far as they arise in this manner, are identical.

It remains to consider only the constant parts of the two functions

$$-\frac{a_1^{(1)}}{2}u_1v_2'\begin{vmatrix} u_3-u_4, & u_3+u_4 \\ v_3-v_4, & v_3+v_4 \end{vmatrix}[-2K-6Ke\cos t+\cdots],$$

$$-\frac{a_1^{(1)}}{2}u_2v_1'\begin{vmatrix} u_3-u_4, & u_3+u_4 \\ v_3-v_4, & v_3+v_4 \end{vmatrix}[-2K-6Ke\cos t+\cdots].$$

It follows, as before, that we need only the cosine terms of $-u_1v_2'$ and $-u_2v_1'$. We see from (137) that the coefficients of the cosine terms in these products are identical. Therefore the constant parts of the right members of equations (154) are identical.

The discussion for the other terms of X_3 and Y_3 which are linear in x_1 and y_1 is made in a similar manner, and it is thus proved that the P_1 which is obtained from the second equation of (148) is identical with the one which depends on the first.

It is now necessary to consider those terms of X_3 and Y_3 which are not linear in x_1 and y_1. Let us treat in detail the term in X_3 which contains x_1x_2 as a factor. So far as this term is concerned, the first two equations of (148) become

$$\left.\begin{aligned} \Delta(a_2^{(3)})' &= D_{11}[-6B-24Be\cos t+\cdots]\,x_1x_2e^{-\sigma\sqrt{-1}t}, \\ \Delta(a_1^{(3)})' &= D_{21}[-6B-24Be\cos t+\cdots]\,x_1x_2e^{+\sigma\sqrt{-1}t}. \end{aligned}\right\} \tag{155}$$

On referring to equations (138), (144), and the expressions for D_{11} and D_{21}, it is seen that the constant parts of the right members of these equations are respectively identical with the constant parts of

$$-(a_1^{(1)})^3\begin{vmatrix} u_2 & , u_3 & , u_4 \\ v_2 & , v_3 & , v_4 \\ -\sigma\sqrt{-1}\,v_2+v_2', & \rho v_3+v_3', & -\rho v_4+v_4' \end{vmatrix}[-6B-24Be\cos t+\cdots][u_1f_2^{(0)}-u_2f_2^{(2)}],$$

$$-(a_1^{(1)})^3\begin{vmatrix} u_1 & , u_3 & , u_4 \\ v_1 & , v_3 & , v_4 \\ +\sigma\sqrt{-1}\,v_1+v_1', & \rho v_3+v_3', & -\rho v_4+v_4' \end{vmatrix}[-6B-24Be\cos t+\cdots][u_2f_2^{(0)}-u_1f_2^{(-2)}].$$

Since $f_2^{(0)}$ is a cosine series, and the product of it and $[-6B-24Be\cos t+\cdots]$ is also a cosine series, the discussion for the terms multiplied by $u_1f_2^{(0)}$ and $u_2f_2^{(0)}$ does not differ from that given above for the terms multiplied by x_1.

We have now to find the constant parts of

$$\left.\begin{array}{l}
\dfrac{(a_1^{(1)})^3}{2}\begin{vmatrix} u_2 \quad , & u_3-u_4 \quad , & u_3+u_4 \\ v_2 \quad , & v_3-v_4 \quad , & v_3+v_4 \\ -\sigma\sqrt{-1}\,v_2+v_2', & \rho(v_3+v_4)+v_3'-v_4', & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix} F_2(t)\,u_2 f_2^{(2)}, \\[2em]
\dfrac{(a_1^{(1)})^3}{2}\begin{vmatrix} u_1 \quad , & u_3-u_4 \quad , & u_3+u_4 \\ v_1 \quad , & v_3-v_4 \quad , & v_3+v_4 \\ +\sigma\sqrt{-1}\,v_1+v_1', & \rho(v_3+v_4)+v_3'-v_4', & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix} F_2(t)\,u_1 f_2^{(-2)},
\end{array}\right\} \quad (156)$$

where $F_2(t)=[-6B-24Be\cos t+\cdots]$.

The factors by which $u_2^2 f_2^{(2)}$ and $u_1^2 f_2^{(-2)}$ are multiplied in these respective expressions are identical, and it follows from equation (137) that they are a sum of cosine terms having real coefficients. Consequently we need only the cosine terms of $u_2^2\, f_2^{(2)}$ and $u_1^2\, f_2^{(-2)}$ in order to obtain the constant parts of (156). Now it follows from (137) and (143) that the cosine terms of the products $u_2^2\, f_2^{(2)}$ and $u_1^2\, f_2^{(-2)}$ are identical. Therefore the constant parts of (156), which involve u_2^2 and u_1^2 as factors, are identical.

Now consider v_2 and v_1 in so far as they occur in the second lines of the determinants. It follows from (137) that the factors by which $-u_2 v_2 f_2^{(2)}$ and $-u_1 v_1 f_2^{(-2)}$ must be multiplied are sine series having real coefficients. Therefore we need only the sine terms in these products. It also follows from (137) that the expressions for $u_2 v_2$ and $u_1 v_1$ are respectively cosine terms with purely imaginary coefficients which differ only in sign, and sine terms with real coefficients which are identical. Therefore, in the products $u_2 v_2 f_2^{(2)}$ and $u_1 v_1 f_2^{(-2)}$ the coefficients of the sine terms are real and respectively equal.

There remain only the terms coming from the third line and first column of the determinants. We have first $-\sigma\sqrt{-1}\,u_2 v_2 f_2^{(2)}$ and $+\sigma\sqrt{-1}\,u_1 v_1 f_2^{(-2)}$. These expressions are multiplied into the same cosine series having real coefficients. Consequently we need compare only the coefficients of their cosine terms, which we find from (137) and (143) are real and respectively identical. Therefore the constant parts of the right members of (155) to which these terms give rise are identical.

Finally, there remain only the terms multiplied by $+u_2 v_2' f_2^{(2)}$ and by $+u_1 v_1' f_2^{(-2)}$ respectively. The term into which these factors are multiplied is a cosine series having real coefficients. It is seen from (137) and (143) that the coefficients of the cosine terms of $+u_2 v_2' f_2^{(2)}$ and $+u_1 v_1' f_2^{(-2)}$ are real and respectively identical. Therefore the constant parts of the right members of (155) are altogether identical.

The discussions for all the other terms of X_3 and Y_3 which involve x_2 or y_2 are made in a similar manner and lead to the same result. There remain only terms in X_3 and Y_3 which are of the third degree in x_1 and y_1.

Let us consider, for example, the term of X_3 which is multiplied by $x_1 y_1^2$. Then, so far as this term alone is concerned, the first two equations of (148) become

$$\left.\begin{aligned} \Delta (a_1^{(3)})' &= D_{11}[-6C+ \cdot \cdot \cdot] x_1 y_1^2 e^{-\sigma\sqrt{-1}t}, \\ \Delta (a_2^{(3)})' &= D_{21}[-6C+ \cdot \cdot \cdot] x_1 y_1^2 e^{+\sigma\sqrt{-1}t}. \end{aligned}\right\} \tag{157}$$

The constant parts of the right members of these equations are respectively the constant parts of

$$\left.\begin{aligned} &+(a_1^{(1)})^3 \begin{vmatrix} u_2 & , u_3 & , u_4 \\ v_2 & , v_3 & , v_4 \\ -\sigma\sqrt{-1}\,v_2+v_2', & \rho v_3+v_3', & -\rho v_4+v_4' \end{vmatrix} [-6C+\cdots][2u_1v_1v_2+u_2v_1^2], \\ &+(a_1^{(1)})^3 \begin{vmatrix} u_1 & , u_3 & , u_4 \\ v_1 & , v_3 & , v_4 \\ +\sigma\sqrt{-1}\,v_1+v_1', & \rho v_3+v_3', & -\rho v_4+v_4' \end{vmatrix} [-6C+\cdots][2u_2v_1v_2+u_1v_2^2]. \end{aligned}\right\} \tag{158}$$

Since $v_1 v_2$ is a cosine series having real coefficients, the discussion for the the terms multiplied by $2u_1v_1v_2$ and $2u_2v_1v_2$ does not differ from that given above for that part of X_3 which is multiplied simply by x_1.

If we refer to equations (137) and (138), we see that v_1^2 and v_2^2 have the properties of $f_2^{(2)}$ and $f_2^{(-2)}$, as regards the relations existing between their respective coefficients. Therefore the discussion of these terms of (158) is identical with that of (156), for which the proposition was established.

In a manner similar to this the identity of the constant parts of the right members of the first two equations of (148) can be established for all of the elements of which X_3 and Y_3 are composed.

140. General Proof that the Constant Parts of the Right Members of the First two Equations of (148) are Identical.—We shall treat first the parts which depend on X_3. We shall need the following properties of X_3:

(1) It is a polynomial in x_1, y_1, x_2, y_2.

(2) Those terms which are of even degree in y_1 and y_2 taken together are multiplied by cosine series having real coefficients.

(3) Those terms which are of odd degree in y_1 and y_2 taken together are multiplied by sine series having real coefficients.

(4) If the general term is $x_1^{j_1} x_2^{j_2} y_1^{k_1} y_2^{k_2}$, then $j_1+2j_2+k_1+2k_2$ is an odd integer (in the present case one or three).

The parts of the first two equations of (148) which depend on X_3 are

$$\Delta(a_1^{(3)})' = D_{11}\, X_3\, e^{-\sigma\sqrt{-1}t}, \qquad \Delta(a_2^{(3)})' = D_{21}\, X_3\, e^{\sigma\sqrt{-1}t}. \tag{159}$$

It is obvious from (137) and properties (2) and (3) that those parts of $X_3\, e^{-\sigma\sqrt{-1}t}$ and $X_3\, e^{\sigma\sqrt{-1}t}$ which are independent of the exponentials $e^{-\sigma\sqrt{-1}t}$ and $e^{\sigma\sqrt{-1}t}$ are sums of cosines having real coefficients and of sines having purely imaginary coefficients, and that the real coefficients in the two ex-

pressions differ respectively only in sign, while the imaginary coefficients are respectively identical. Hence, referring to the expressions for D_{11} and D_{21}, we may write these parts of equations (159) in the form

$$-\frac{1}{2}\begin{vmatrix} u_2 & , & u_3-u_4 & , & u_3+u_4 \\ v_2 & , & v_3-v_4 & , & v_3+v_4 \\ -\sigma\sqrt{-1}v_2+v_2', & & \rho(v_3+v_4)+v_3'-v_4', & & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix}\Sigma[A_j\cos jt + \sqrt{-1}B_j\sin jt],$$

$$-\frac{1}{2}\begin{vmatrix} u_1 & , & u_3-u_4 & , & u_3+u_4 \\ v_1 & , & v_3-v_4 & , & v_3+v_4 \\ +\sigma\sqrt{-1}v_1+v_1', & & \rho(v_3+v_4)+v_3'-v_4', & & \rho(v_3-v_4)+v_3'+v_4' \end{vmatrix}\Sigma[A_j\cos jt - \sqrt{-1}B_j\sin jt].$$

It easily follows from these expressions, as in the discussion in § 139, that their constant parts are real and identical.

Now consider the terms depending on Y_3, which has the properties (1) and (4) belonging to X_3, and

(2) Those terms which are of even degree in y_1 and y_2 taken together are multiplied by a sine series having real coefficients.

(3) Those terms which are of odd degree in y_1 and y_2 taken together are multiplied by a cosine series having real coefficients.

The parts of the first two equations of (148) which depend on Y_3 are

$$\Delta\,(a_1^{(3)})' = D_{12}\,Y_3\,e^{-\sigma\sqrt{-1}t}, \qquad \Delta\,(a_2^{(3)})' = D_{22}\,Y_2\,e^{\sigma\sqrt{-1}t}. \tag{160}$$

It follows from (137) and properties (2) and (3) that those parts of $Y_3\,e^{-\sigma\sqrt{-1}t}$ and $Y_3\,e^{\sigma\sqrt{-1}t}$ which are independent of the exponentials $e^{-\sigma\sqrt{-1}t}$ and $e^{\sigma\sqrt{-1}t}$ are sums of cosine terms having purely imaginary coefficients, and of sine terms having real coefficients, and that the purely imaginary coefficients are respectively identical while the real coefficients differ respectively only in sign. Hence, using the explicit values of D_{12} and D_{22}, these parts of the right members of (160) are found to have the form

$$\frac{1}{2}\begin{vmatrix} u_2 & , & u_3-u_4 & , & u_3+u_4 \\ v_2 & , & v_3-v_4 & , & v_3+v_4 \\ -\sigma\sqrt{-1}u_2+u_2', & & \rho(u_3+u_4)+u_3'-u_4', & & \rho(u_3-u_4)+u_3'+u_4' \end{vmatrix}\Sigma[\sqrt{-1}A_j\cos jt + B_j\sin jt],$$

$$\frac{1}{2}\begin{vmatrix} u_1 & , & u_3-u_4 & , & u_3+u_4 \\ v_1 & , & v_3-v_4 & , & v_3+v_4 \\ +\sigma\sqrt{-1}u_1+u_1', & & \rho(u_3+u_4)+u_3'-u_4', & & \rho(u_3-u_4)+u_3'+u_4' \end{vmatrix}\Sigma[\sqrt{-1}A_j\cos jt - B_j\sin jt].$$

It follows from (137) that the constant parts of these two expressions are real and identical. Therefore *the constant parts of the right members of the first two equations of* (148) *are identical, and when one of them is made to vanish by a special determination of* $a_1^{(1)}$ *the other one also vanishes.*

141. Form of the Periodic Solution of the Coefficients of $\lambda^{3/2}$.—It follows from the form of X_3 and Y_3, given in equations (145), that f_3 and g_3 of equation (151) have the form

$$\left.\begin{aligned} f_3 &= a_1^{(2)} f_3^{(2)} e^{2\sigma\sqrt{-1}t} + a_1^{(2)} f_3^{(-2)} e^{-2\sigma\sqrt{-1}t} + a_1^{(2)} f_3^{(0)} \\ &\quad + f_3^{(3)} e^{3\sigma\sqrt{-1}t} + f_3^{(1)} e^{\sigma\sqrt{-1}t} + f_3^{(-1)} e^{-\sigma\sqrt{-1}t} + f_3^{(-3)} e^{-3\sigma\sqrt{-1}t}, \\ g_3 &= a_1^{(2)} g_3^{(2)} e^{2\sigma\sqrt{-1}t} + a_1^{(2)} g_3^{(-2)} e^{-2\sigma\sqrt{-1}t} + a_1^{(2)} g_3^{(0)} \\ &\quad + g_3^{(3)} e^{3\sigma\sqrt{-1}t} + g_3^{(1)} e^{\sigma\sqrt{-1}t} + g_3^{(-1)} e^{-\sigma\sqrt{-1}t} + g_3^{(-3)} e^{-3\sigma\sqrt{-1}t}, \end{aligned}\right\} \quad (161)$$

where $f_3^{(2)}, \ldots, g_3^{(-3)}$ are known functions of t. We need certain properties of these functions. It follows from the properties of X_3 and Y_3 and of the left members of the differential equations, and from certain considerations of changes of sign of $\sqrt{-1}$, t, and y_3, in both the differential equations and the solutions, that

$$\begin{aligned} f_3^{(j)}(-\sqrt{-1}) &= f_3^{(-j)}(\sqrt{-1}), & f_3^{(j)}(-t) &= +f_3^{(-j)}(t), \\ g_3^{(j)}(-\sqrt{-1}) &= g_3^{(-j)}(\sqrt{-1}); & g_3^{(j)}(-t) &= -g_3^{(-j)}(t); \\ f_3^{(j)}(-\sqrt{-1}, -t) &= +f_3^{(j)}(\sqrt{-1}, t), \\ g_3^{(j)}(-\sqrt{-1}, -t) &= -g_3^{(j)}(\sqrt{-1}, t) & & (j=0,\ 1,\ 2,\ 3,\ -1,\ -2,\ -3). \end{aligned}$$

It follows from these relations that the $f_3^{(j)}$ and the $g_3^{(j)}$ have the form

$$\left.\begin{aligned} f_3^{(3)} &= \Sigma[a_j^{(3)} \cos jt + \sqrt{-1}\, b_j^{(3)} \sin jt], & g_3^{(3)} &= \Sigma[+\sqrt{-1}\,\alpha_j^{(3)} \cos jt + \beta_j^{(3)} \sin jt], \\ f_3^{(-3)} &= \Sigma[a_j^{(3)} \cos jt - \sqrt{-1}\, b_j^{(3)} \sin jt], & g_3^{(-3)} &= \Sigma[-\sqrt{-1}\,\alpha_j^{(3)} \cos jt + \beta_j^{(3)} \sin jt], \\ f_3^{(2)} &= \Sigma[a_j^{(2)} \cos jt + \sqrt{-1}\, b_j^{(2)} \sin jt], & g_3^{(2)} &= \Sigma[+\sqrt{-1}\,\alpha_j^{(2)} \cos jt + \beta_j^{(2)} \sin jt], \\ f_3^{(-2)} &= \Sigma[a_j^{(2)} \cos jt - \sqrt{-1}\, b_j^{(2)} \sin jt], & g_3^{(-2)} &= \Sigma[-\sqrt{-1}\,\alpha_j^{(2)} \cos jt + \beta_j^{(2)} \sin jt], \\ f_3^{(1)} &= \Sigma[a_j^{(1)} \cos jt + \sqrt{-1}\, b_j^{(1)} \sin jt], & g_3^{(1)} &= \Sigma[+\sqrt{-1}\,\alpha_j^{(1)} \cos jt + \beta_j^{(1)} \sin jt], \\ f_3^{(-1)} &= \Sigma[a_j^{(1)} \cos jt - \sqrt{-1}\, b_j^{(1)} \sin jt], & g_3^{(-1)} &= \Sigma[-\sqrt{-1}\,\alpha_j^{(1)} \cos jt + \beta_j^{(1)} \sin jt], \\ f_3^{(0)} &= \Sigma\, a_j^{(0)} \cos jt, & g_3^{(0)} &= \Sigma\, \beta_j^{(0)} \sin jt, \end{aligned}\right\} \quad (162)$$

where the $a_j^{(3)}, \ldots, \beta_j^{(0)}$ are real constants.

It follows from equations (161) and (162) that $g_3(0)=0$. Therefore, since $y_3(0)=0$, we have $a_2^{(3)} = -a_1^{(3)}$, and equations (151) become

$$\left.\begin{aligned} x_3 &= a_1^{(3)}[e^{\sigma\sqrt{-1}t} u_1 - e^{-\sigma\sqrt{-1}t} u_2] + a_1^{(2)}[f_3^{(2)} e^{2\sigma\sqrt{-1}t} + f_3^{(-2)} e^{-2\sigma\sqrt{-1}t} + f_3^{(0)}] \\ &\quad + f_3^{(3)} e^{3\sigma\sqrt{-1}t} + f_3^{(-3)} e^{-3\sigma\sqrt{-1}t} + f_3^{(1)} e^{\sigma\sqrt{-1}t} + f_3^{(-1)} e^{-\sigma\sqrt{-1}t}, \\ y_3 &= a_1^{(3)}[e^{\sigma\sqrt{-1}t} v_1 - e^{-\sigma\sqrt{-1}t} v_2] + a_1^{(2)}[g_3^{(2)} e^{2\sigma\sqrt{-1}t} + g_3^{(-2)} e^{-2\sigma\sqrt{-1}t} + g_3^{(0)}] \\ &\quad + g_3^{(3)} e^{3\sigma\sqrt{-1}t} + g_3^{(-3)} e^{-3\sigma\sqrt{-1}t} + g_3^{(1)} e^{\sigma\sqrt{-1}t} + g_3^{(-1)} e^{-\sigma\sqrt{-1}t}. \end{aligned}\right\} \quad (163)$$

142. Coefficients of λ^2.—It is seen from (13) that the coefficients of λ^2 are defined by the differential equations

$$\left.\begin{aligned} x_4'' - 2y_4' - [1 + 2A + 6Ae\cos t + \cdots]x_4 - [6Ae\sin t + \cdots]y_4 &= X_4, \\ y_4'' + 2x_4' - [6Ae\sin t + \cdots]\,x_4 - [1 - A - 3Ae\cos t + \cdots]y_4 &= Y_4, \end{aligned}\right\} \quad (164)$$

where

$$
\left.\begin{aligned}
X_4 = &+\left[-2K-6Ke\cos t+\cdots\right]x_2+\left[-6Ke\sin t+\cdots\right]y_2\\
&+\left[-3B-12Be\cos t+\cdots\right]\left[x_2^2+2x_1x_3\right]\\
&+\left[-24Be\sin t+\cdots\right]\left[x_2y_2+x_1y_3+x_3y_1\right]\\
&+\left[\tfrac{3}{2}B+6Be\cos t+\cdots\right]\left[y_2^2+2y_1y_3\right]+\overline{X}_4,\\
Y_4 = &+\left[-6Ke\sin t+\cdots\right]x_2+\left[K+3Ke\cos t+\cdots\right]y_2\\
&+\left[-12Be\sin t+\cdots\right]\left[x_2^2+2x_1x_3\right]\\
&+\left[3B+2Be\cos t+\cdots\right]\left[x_2y_2+x_1y_3+x_3y_1\right]\\
&+\left[9Be\sin t+\cdots\right]\left[y_2^2+2y_1y_3\right]+\overline{Y}_4,
\end{aligned}\right\}\qquad(165)
$$

where $\overline{X}_4$ and $\overline{Y}_4$ are independent of x_3 and y_3 and linear in x_2 and y_2. In $\overline{X}_4$ the terms which are of even degree in y_1 and y_2 are multiplied by cosine series having real coefficients, while those which are of odd degree in y_1 and y_2 are multiplied by sine series having real coefficients. In the case of $\overline{Y}_4$ the cosine series and sine series are interchanged. If $x_1^{j_1}\, x_2^{j_2}\, x_3^{j_3}\, y_1^{k_1}\, y_2^{k_2}\, y_3^{k_3}$ is the general term in X_4 or Y_4, then $j_1+2j_2+3j_3+k_1+2k_2+3k_3=4$ or 2. When the right members of (164) are set equal to zero, the general solution of the equations is

$$
\left.\begin{aligned}
x_4 &= a_1^{(4)}\, e^{\sigma\sqrt{-1}t}\, u_1+a_2^{(4)}\, e^{-\sigma\sqrt{-1}t}\, u_2+a_3^{(4)}\, e^{\rho t}\, u_3+a_4^{(4)}\, e^{-\rho t}\, u_4,\\
y_1 &= a_1^{(4)}\, e^{\sigma\sqrt{-1}t}\, v_1+a_2^{(4)}\, e^{-\sigma\sqrt{-1}t}\, v_2+a_3^{(4)}\, e^{\rho t}\, v_3+a_4^{(4)}\, e^{-\rho t}\, v_4,
\end{aligned}\right\}\qquad(166)
$$

where $a_1^{(4)}, \ldots, a_4^{(4)}$ are arbitrary constants. Now, on varying them and subjecting them to the conditions that (164), including the right members, shall be satisfied, we find

$$
\left.\begin{aligned}
\Delta\,(a_1^{(4)})' &= [D_{11}\,X_4+D_{12}\,Y_4]\,e^{-\sigma\sqrt{-1}t}, &\quad \Delta\,(a_3^{(4)})' &= [D_{31}\,X_4+D_{32}\,Y_4]\,e^{-\rho t},\\
\Delta\,(a_2^{(4)})' &= [D_{21}\,X_4+D_{22}\,Y_4]\,e^{+\sigma\sqrt{-1}t}, &\quad \Delta\,(a_4^{(4)})' &= [D_{41}\,X_4+D_{42}\,Y_4]\,e^{+\rho t},
\end{aligned}\right\}\qquad(167)
$$

where $D_{11}, \ldots, D_{42}$ are the same as in § 139.

Necessary conditions that the solution shall be periodic at this step are that the constant terms in the right members of the first two equations of (167) shall be zero. It follows from the form of X_4 and Y_4, as given in (165), and from their properties, that these constant terms are independent of $a_1^{(3)}$ and involve $a_1^{(2)}$ linearly. Therefore the condition that the constant term of the right member of the first equation shall be zero has the form

$$
P_2\, a_1^{(2)}+Q_2=0, \qquad (168)
$$

where P_2 and Q_2 are power series in e. It was shown in Chapter VI, in the treatment of the case where $e=0$, that P_2 has a term independent of e which is distinct from zero. Therefore for $|e|$ sufficiently small $a_1^{(2)}$ is uniquely determined by (168) as a power series in e.

The equation obtained by setting the constant part of the right member of the second equation of (167) equal to zero is of the same form as (168); it is, in fact, identical with (168), as will now be shown. It follows from the properties of X_4 and Y_4 that the parts of the right members of the first two equations of (167) which are independent of the exponentials $e^{-\sigma\sqrt{-1}t}$ and $e^{\sigma\sqrt{-1}t}$ have the form

$$\left.\begin{aligned}&+D_{11}\,\Sigma\,[A_j\cos jt+\sqrt{-1}\,B_j\sin jt]+D_{12}\,[+\sqrt{-1}\,C_j\cos jt+D_j\sin jt],\\&-D_{21}\,\Sigma\,[A_j\cos jt-\sqrt{-1}\,B_j\sin jt]-D_{22}\,[-\sqrt{-1}\,C_j\cos jt+D_j\sin jt].\end{aligned}\right\}\qquad(169)$$

These equations are of exactly the same form as those encountered in §140 in the preceding step of the integration, and the conclusion follows in the same manner. Consequently if $a_1^{(2)}$ is determined so as to satisfy (168), and if the additive constants arising with the integrals of the last two equations of (167) are taken equal to zero, then the solutions of (164) are periodic. It follows from the properties of X_4 and Y_4 that they have the form

$$\left.\begin{aligned}x_4&=a_1^{(4)}[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2]+a_1^{(3)}\bar{f}_4^{(2)}e^{2\sigma\sqrt{-1}t}+a_1^{(3)}\bar{f}_4^{(-2)}e^{-2\sigma\sqrt{-1}t}\\&\qquad+a_1^{(3)}\bar{f}_4^{(0)}+\sum_{j=-4}^{+4}f_4^{(j)}e^{j\sigma\sqrt{-1}t},\\y_4&=a_1^{(4)}[e^{\sigma\sqrt{-1}t}v_1-e^{-\sigma\sqrt{-1}t}v_2]+a_1^{(3)}\bar{g}_4^{(2)}e^{2\sigma\sqrt{-1}t}+a_1^{(3)}\bar{g}_4^{(-2)}e^{-2\sigma\sqrt{-1}t}\\&\qquad+a_1^{(3)}\bar{g}_4^{(0)}+\sum_{j=-4}^{+4}g_4^{(j)}e^{j\sigma\sqrt{-1}t},\end{aligned}\right\}\qquad(170)$$

where the $f_4^{(j)}$, $g_4^{(j)}$, $\bar{f}_4^{(j)}$, and $\bar{g}_4^{(j)}$ have properties exactly analogous to those of equations (162).

143. Induction to the General Term of the Solution.—We shall suppose the $x_1, \ldots, x_{n-1}$; $y_1, \ldots, y_{n-1}$ have been computed and that their coefficients have all been determined except $a_1^{(n-2)}$ and $a_1^{(n-1)}$, which enter in x_{n-2}, y_{n-2}, x_{n-1}, and y_{n-1} in the form

$$\left.\begin{aligned}x_{n-2}&=+a_1^{(n-2)}[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2]+\cdots,\\y_{n-2}&=+a_1^{(n-2)}[e^{\sigma\sqrt{-1}t}v_1-e^{-\sigma\sqrt{-1}t}v_2]+\cdots,\\x_{n-1}&=+a_1^{(n-1)}[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2]\\&\quad+a_1^{(n-2)}[\bar{f}_{n-1}^{(2)}e^{2\sigma\sqrt{-1}t}+\bar{f}_{n-1}^{(-2)}e^{-2\sigma\sqrt{-1}t}+\bar{f}_{n-1}^{(0)}]+\cdots,\\y_{n-1}&=+a_1^{(n-1)}[e^{\sigma\sqrt{-1}t}v_1-e^{-\sigma\sqrt{-1}t}v_2]\\&\quad+a_1^{(n-2)}[\bar{g}_{n-1}^{(2)}e^{2\sigma\sqrt{-1}t}+\bar{g}_{n-1}^{(-2)}e^{-2\sigma\sqrt{-1}t}+\bar{g}_{n-1}^{(0)}]+\cdots.\end{aligned}\right\}\qquad(171)$$

We shall suppose that x_p and y_p $(p=1, \ldots, n-1)$ have the properties

$$\left.\begin{aligned}
x_p &= \sum_{j=-p}^{+p} f_p^{(j)} e^{j\sigma\sqrt{-1}t}, & y_p &= \sum_{j=-p}^{+p} g_p^{(j)} e^{j\sigma\sqrt{-1}t},\\
f_p^{(j)} &= \Sigma[a_\nu^{(p,j)} \cos\nu t+\sqrt{-1}\, b_\nu^{(p,j)} \sin\nu t] & & (j\neq 0),\\
f_p^{(-j)} &= \Sigma[a_\nu^{(p,j)} \cos\nu t-\sqrt{-1}\, b_\nu^{(p,j)} \sin\nu t] & & (j\neq 0),\\
g_p^{(j)} &= \Sigma[+\sqrt{-1}\, \alpha_\nu^{(p,j)} \cos\nu t+\beta_\nu^{(p,j)} \sin\nu t] & & (j\neq 0),\\
g_p^{(-j)} &= \Sigma[-\sqrt{-1}\, \alpha_\nu^{(p,j)} \cos\nu t+\beta_\nu^{(p,j)} \sin\nu t] & & (j\neq 0),\\
f_p^{(0)} &= \Sigma\, a_\nu^{(p,0)} \cos\nu t, & g_p^{(0)} &= \Sigma\, \beta_\nu^{(p,0)} \sin\nu t.
\end{aligned}\right\} \quad (172)$$

The differential equations which define x_n and y_n are seen from (13) and (14) to be

$$\left.\begin{aligned}
&x_n''-2y_n'-[1+2A+6Ae\cos t+\cdots]x_n-[6Ae\sin t+\cdots]y_n=X_n,\\
&y_n''+2x_n'-[6Ae\sin t+\cdots]x_n+[1-A-3Ae\cos t+\cdots]y_n=Y_n,
\end{aligned}\right\} \quad (173)$$

where

$$\left.\begin{aligned}
X_n=&+[-2K-6Ke\cos t+\cdots]x_{n-2}+[-6Ke\sin t+\cdots]y_{n-2}\\
&+[-3B-12Be\cos t+\cdots][2x_1x_{n-1}+2x_2x_{n-2}]\\
&+[-24Be\sin t+\cdots][x_1y_{n-1}+x_2y_{n-2}+x_{n-2}y_2+x_{n-1}y_1]\\
&+[3B+12Be\cos t+\cdots][y_1y_{n-1}+y_2y_{n-2}]+\overline{X}_n,\\
Y_n=&+[-6Ke\sin t+\cdots]x_{n-2}+[K+3Ke\cos t+\cdots]y_{n-2}\\
&+[-12Be\sin t+\cdots][2x_1x_{n-1}+2x_2x_{n-2}]\\
&+[3B+2Be\cos t+\cdots][x_1y_{n-1}+x_2y_{n-2}+x_{n-2}y_2+x_{n-1}y_1]\\
&+[9Be\sin t+\cdots][2y_1y_{n-1}+2y_2y_{n-2}]+\overline{Y}_4.
\end{aligned}\right\} \quad (174)$$

The functions $\overline{X}_n$ and $\overline{Y}_n$ do not involve x_{n-1} or y_{n-1}, and are linear in x_{n-2} and y_{n-2}. In $\overline{X}_n$ the terms which are of even degree in $y_1, \ldots, y_{n-2}$ are multiplied by cosine series having real coefficients, while those which are of odd degree in $y_1, \ldots, y_{n-2}$ are multiplied by sine series having real coefficients. In the case of Y_n the cosine series and sine series are interchanged. If $x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} y_1^{k_1} \cdots y_{n-1}^{k_{n-1}}$ is the general term of X_n or Y_n, then

$$j_1+2j_2+\cdots+(n-1)j_{n-1}+k_1+2k_2+\cdots+(n-1)k_{n-1}=n \text{ or } n-2. \quad (175)$$

Necessary conditions that the solutions of (173) shall be periodic are that the right members of

$$\Delta(a_1^{(n)})'=[D_{11}X_n+D_{12}Y_n]\,e^{-\sigma\sqrt{-1}t}, \quad \Delta(a_2^{(n)})'=[D_{21}X_n+D_{22}Y_n]\,e^{\sigma\sqrt{-1}t} \quad (176)$$

shall contain no constant terms. It follows from (174) that these constant terms are independent of $a_1^{(n-1)}$ and involve $a_1^{(n-2)}$ linearly. The coefficient of $a_1^{(n-2)}$ is distinct from zero for $|e|$ sufficiently small, for in Chapter VI it

was seen to be distinct from zero for e equal to zero. Therefore $a_1^{(n-2)}$ is uniquely determined as a power series in e by setting the constant term of the right member of the first equation of (176) equal to zero.

It can be shown, precisely as in the discussion when $n=4$, that the constant parts of the right members of equations (176) are identical. Therefore $a_1^{(n-2)}$ is uniquely determined by the conditions that the solutions of (173) shall be periodic. It follows from the properties of $x_1, \ldots, x_{n-1}$; $y_1, \ldots, y_{n-1}$, and from (175), that when these conditions are satisfied the solution of (173) has the form

$$\left.\begin{aligned} x_n &= a_1^{(n)} e^{\sigma\sqrt{-1}t} u_1 + a_2^{(n)} e^{-\sigma\sqrt{-1}t} u_2 + a_3^{(n)} e^{\rho t} u_3 + a_4^{(n)} e^{-\rho t} u_4 \\ &\quad + a_1^{(n-1)} [\bar{f}_n^{(2)} e^{2\sigma\sqrt{-1}t} + \bar{f}_n^{(-2)} e^{-2\sigma\sqrt{-1}t} + \bar{f}_n^{(0)}] + \sum_{j=-n}^{+n} f_n^{(j)} e^{j\sigma\sqrt{-1}t}, \\ y_n &= a_1^{(n)} e^{\sigma\sqrt{-1}t} v_1 + a_2^{(n)} e^{-\sigma\sqrt{-1}t} v_2 + a_3^{(n)} e^{\rho t} v_3 + a_4^{(n)} e^{-\rho t} v_4 \\ &\quad + a_1^{(n-1)} [\bar{g}_n^{(2)} e^{2\sigma\sqrt{-1}t} + \bar{g}_n^{(-2)} e^{-2\sigma\sqrt{-1}t} + \bar{g}_n^{(0)}] + \sum_{j=-n}^{+n} g_n^{(j)} e^{j\sigma\sqrt{-1}t}, \end{aligned}\right\} \quad (177)$$

where $a_1^{(n)}, \ldots, a_4^{(n)}, a_1^{(n-1)}$ are undetermined, and where the $\bar{f}_n^{(2)}, \bar{f}_n^{(-2)}, \bar{f}_n^{(0)}, \bar{g}_n^{(2)}, \bar{g}_n^{(-2)}, \bar{g}_n^{(0)}, f_n^{(j)}$, and $g_n^{(j)}$ have the properties of (172).

In order that (177) shall be periodic it is necessary and sufficient to impose the conditions $a_3^{(n)} = a_4^{(n)} = 0$. Then it follows, from the properties of $v_1, v_2, \bar{g}_n^{(2)}, \bar{g}_n^{(-2)}, \bar{g}_n^{(0)}$ that $y_n(0) - a_1^{(n)} - a_2^{(n)} = 0$. Since $y_n(0) = 0$, it follows that $a_2^{(n)} = -a_1^{(n)}$, and equations (177) become

$$\left.\begin{aligned} x_n &= a_1^{(n)} [e^{\sigma\sqrt{-1}t} u_1 - e^{-\sigma\sqrt{-1}t} u_2] \\ &\quad + a^{(n-1)} [\bar{f}_n^{(2)} e^{2\sigma\sqrt{-1}t} + \bar{f}_n^{(-2)} e^{-2\sigma\sqrt{-1}t} + \bar{f}_n^{(0)}] + \sum_{j=-n}^{+n} f_n^{(j)} e^{j\sigma\sqrt{-1}t}, \\ y_n &= a_1^{(n)} [e^{\sigma\sqrt{-1}t} v_1 - e^{-\sigma\sqrt{-1}t} v_2] \\ &\quad + a_1^{(n-1)} [\bar{g}_n^{(2)} e^{2\sigma\sqrt{-1}t} + \bar{g}_n^{(-2)} e^{-2\sigma\sqrt{-1}t} + \bar{g}_n^{(0)}] + \sum_{j=-n}^{+n} g_n^{(j)} e^{j\sigma\sqrt{-1}t}, \end{aligned}\right\} \quad (178)$$

and equations (172) are satisfied for $p=n$.

In picking out the constant part of the right member of the first equation of (176), in general only those terms in X_n and Y_n which contain $e^{\sigma\sqrt{-1}t}$ as a factor to the first degree will be used. But because σ is a rational number there will eventually occur, in the higher powers of the exponentials, multiples of σ which are integers, and constant terms in the right member of the first of (176) may occur from these terms, but their presence does not prevent the determination of the constants so that the solutions shall be periodic. After such terms once appear, they occur in general at each succeeding step of the integration.

CHAPTER VIII.

THE STRAIGHT-LINE SOLUTIONS OF THE PROBLEM OF n BODIES.

144. Statement of Problem.—In his prize memoir* on the problem of three bodies, Lagrange showed that, for any three finite masses mutually attracting one another according to the Newtonian law, there are four distinct configurations such that, under proper initial projections, the *ratios of the mutual distances remain constant.* The bodies describe similar conic sections with respect to the center of mass of the system, the simplest case being that in which the orbits are circular. In three of the four solutions the masses lie always in a straight line, and in the fourth they remain at the vertices of an equilateral triangle. This memoir is one of the most elegant written by Lagrange, and its mathematical form does not seem capable of improvement. But the method which he employed can not be extended conveniently to the case of more than three bodies, and it has not led to practical results in applied celestial mechanics.

This chapter is devoted to the solution of two closely related problems:

I. The number of *straight-line* solutions is found for n arbitrary positive masses; that is, the ratios of the distances are determined so that under proper initial projections the bodies will always remain collinear. This is the generalization of Lagrange's straight-line solutions to the problem of n bodies. For each straight-line solution of n finite masses there are $n+1$ points of libration near which there are oscillating satellite orbits of the types treated in Chapters V—VII. Therefore the results of this chapter lead to generalizations of those of the preceding three chapters.

II. The problem is solved of determining, when possible, n masses such that, if they are placed at n arbitrary collinear points, they will, under proper initial projection, always remain in a straight line.

The first problem, in a somewhat different form, has been considered by Lehmann-Filhès.† The method of treatment adopted here‡, though originally developed independently, has considerable in common with that of Lehmann-Filhès, and the discussion completes in certain essential respects the demonstration of the German astronomer. It is shown that whatever real positive values the masses may have, and whatever the rate of their revolution, there are $\frac{1}{2}n!$ real collinear solutions.

*Lagrange's *Collected Works*, vol. VI, pp. 229–324. Tisserand's *Mécanique Céleste*, vol. I, chap. 8.

†*Astronomische Nachrichten*, vol. CXXVII (1891), No. 3033.

‡Read before the Chicago Section of the *American Mathematical Society*, December 28, 1900; abstract in *Bull. Am. Math. Soc.*, vol. VII (1900-1901), p. 249.

In the second problem it is proved that when the number of arbitrarily chosen real collinear points is even, the n masses are, in general, uniquely determined by the condition that it shall be possible to place them at these points and to give them initial projections so that they will always remain collinear and revolve in orbits of specified linear dimensions. Or, if it is preferred, the period of revolution can be taken as the arbitrary in place of the linear dimensions of the orbit. In general, the masses will not all be positive, and therefore the problem will not always have a physical interpretation. When the number of points is odd, it is not possible to determine the masses so as to satisfy the solution conditions unless the coördinates of the points themselves satisfy one algebraic equation. When they satisfy this condition, any one of the masses may be chosen arbitrarily, after which all of the others are, in general, uniquely determined.

I. DETERMINATION OF THE POSITIONS WHEN THE MASSES ARE GIVEN.

145. The Equations Defining the Solutions.—Let the origin of coördinates be taken at the center of gravity of the system, which will be supposed to be at rest. This point and the line of initial projection of any mass determine a plane. All the other masses must be projected in this plane, for otherwise they would not be collinear at the end of the first element of time. All the bodies being initially in a line and projected in the same plane, they will always remain in this plane. Consequently, if solutions exist in which the n masses are always in a straight line, the orbits are plane curves.

Let the plane of the motion be the $\xi\eta$–plane. Let the masses be denoted by $m_1, m_2, \ldots, m_n$, and their respective coördinates by (ξ_1, η_1), $(\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)$. Then, choosing the units so that the Gaussian constant is unity, the differential equations of motion are

$$\left.\begin{aligned} &\frac{d^2\xi_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \xi_i}, \qquad \frac{d^2\eta_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \eta_i} \qquad (i=1, \ldots, n), \\ &U = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{m_j m_k}{\rho_{jk}}, \qquad \rho_{jk} = \sqrt{(\xi_j-\xi_k)^2+(\eta_j-\eta_k)^2} \qquad (j \neq k). \end{aligned}\right\} \qquad (1)$$

Equations (1) always admit the integral of areas

$$\sum_{i=1}^{n} m_i\left(\xi_i\frac{d\eta_i}{dt} - \eta_i\frac{d\xi_i}{dt}\right) = \sum_{i=1}^{n} m_i r_i^2\frac{d\theta_i}{dt} = c, \qquad r_i = \sqrt{\xi_i^2+\eta_i^2}. \qquad (2)$$

In case the n bodies remain collinear, the line of the resultant acceleration to which each one is subject always passes through the origin. Therefore, in collinear solutions it follows from the law of areas that, for each body separately,

$$m_i r_i^2 \frac{d\theta_i}{dt} = c_i \qquad (i=1, \ldots, n).$$

But when the bodies remain collinear we have also

$$\frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = \cdots = \frac{d\theta_n}{dt};$$

from which it follows that

$$\frac{r_i}{r_j} = \sqrt{\frac{m_j c_i}{m_i c_j}} = a_{ij}, \tag{3}$$

where the a_{ij} are constants. That is, if any collinear solutions exist, the ratios of the distances of the bodies from the origin are constants, and it easily follows from this that the ratios of their mutual distances are also constants. They are therefore of the Lagrangian type.

If the n masses remain collinear, the ratios of their distances from the origin being therefore constants, the ratios of their coördinates are constants. Therefore in all collinear solutions

$$\xi_i = x_i \xi, \qquad \eta_i = x_i \eta \qquad (i=1, \ldots, n), \tag{4}$$

where the x_i are constants. Upon substituting in equations (1), we have, as necessary conditions for the existence of the collinear solutions,

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= -\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{x_i[(x_i - x_j)^2]^{3/2}} \frac{\xi}{r^3} \qquad (j \neq i;\ i=1, \ldots, n), \\ \frac{d^2\eta}{dt^2} &= -\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{x_i[(x_i - x_j)^2]^{3/2}} \frac{\eta}{r^3}, \qquad r = \sqrt{\xi^2 + \eta^2}. \end{aligned}\right\} \tag{5}$$

In order that ξ and η as defined by their initial values and equations (5) shall be the same for all values of i, the coefficients of ξ/r^3 and η/r^3 must be set equal to a constant independent of i. Letting $-\omega^2$ represent this constant and $r_{ij} = \sqrt{(x_i - x_j)^2}$, these conditions, which are sufficient as well as necessary for the existence of the collinear solutions, become

$$\left.\begin{array}{l} +\quad 0 \quad + \dfrac{m_2(x_1 - x_2)}{r_{12}^3} + \dfrac{m_3(x_1 - x_3)}{r_{13}^3} + \cdots + \dfrac{m_n(x_1 - x_n)}{r_{1n}^3} = \omega^2 x_1, \\ + \dfrac{m_1(x_2 - x_1)}{r_{21}^3} + \quad 0 \quad + \dfrac{m_3(x_2 - x_3)}{r_{23}^3} + \cdots + \dfrac{m_n(x_2 - x_n)}{r_{2n}^3} = \omega^2 x_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \dfrac{m_1(x_n - x_1)}{r_{n1}^3} + \dfrac{m_2(x_n - x_2)}{r_{n2}^3} + \dfrac{m_3(x_n - x_3)}{r_{n3}^3} + \cdots + \quad 0 \quad = \omega^2 x_n. \end{array}\right\} \tag{6}$$

Suppose the notation is so chosen that in any solution $x_1 < \cdots < x_n$; then the terms of the left member of the last equation are all positive. Since the origin is at the center of gravity, x_n is positive, and therefore ω^2 is positive in all real solutions. For every set of values of $x_1, \ldots, x_n$ satisfying equations (6) the solutions of (5) are the same for all values of i, and these solutions substituted in (4) give the coördinates in the collinear configurations.

Since equations (5) have the same form as the differential equations in the two-body problem, it follows that *in the collinear solutions the orbits are always similar conic sections.* In case the orbits are ellipses, the coefficient of $-\xi/r^3$ and $-\eta/r^3$ is the product of the cube of the major semi-axis of the orbit and the square of the mean angular speed of revolution. If the undetermined scale factor be so chosen that x_i is the major semi-axis of the orbit of m_i, the mean angular velocity of revolution of the system is ω.

The hypothesis is made that ω^2 and $m_1, \ldots, m_n$ are real positive numbers, and the problem is to find the number of real solutions of (6) for any value of n. For each of these solutions there is a six-fold infinity of collinear configurations, the six arbitraries being the two which define the plane of motion, the one which defines the orientation of the orbits in their plane, the one which determines the epochs at which the bodies pass their apses, the one which determines the scale of the system, and, finally, the eccentricity of the orbits.

146. Outline of the Method of Solution.—The method of solution involves a mathematical induction and consists of the following steps:

Assumption (A). It is assumed that for $n=\nu$ the number of real solutions of (6) for $x_1, \ldots, x_\nu$ is N_ν, whatever real positive values ω^2 and $m_1, \ldots, m_\nu$ may have. It is known from the work of Lagrange that when $\nu=3$ the number is $N_3=3=\frac{1}{2}\,3!$.

Theorem (B). If to the system $m_1, \ldots, m_\nu$ of positive masses an infinitesimal mass $m_{\nu+1}$ be added, then the whole number of real solutions is $(\nu+1)N_\nu$.

Theorem (C). As the infinitesimal mass $m_{\nu+1}$ increases continuously to any finite positive value whatever, the total number of real solutions remains precisely $(\nu+1)N_\nu$.

Conclusion (D). From successive applications of theorems (B) and (C) it follows that the number of real solutions of (6) for $n=\nu+\mu$ is

$$N_{\nu+\mu}=(\nu+\mu)(\nu+\mu-1)\cdots(\nu+2)(\nu+1)N_\nu.$$

Since $N_3=\frac{1}{2}\,3!$, it follows that $N_{3+\mu}=\frac{1}{2}(\mu+3)!$. Let $\mu+3=n$ and we have

$$N_n=\frac{1}{2}n!. \tag{7}$$

To complete the demonstration of this conclusion it remains only to prove theorems (B) and (C).

147. Proof of Theorem (B).—When there are ν finite bodies $m_1, \ldots, m_\nu$ and the infinitesimal body $m_{\nu+1}$, equations (6) become

$$\left.\begin{aligned}
\varphi_1 &\equiv -\omega^2 x_1 + 0 + \frac{m_2(x_1-x_2)}{r_{12}^3} + \cdots + \frac{m_\nu(x_1-x_\nu)}{r_{1\nu}^3} + \frac{m_{\nu+1}(x_1-x_{\nu+1})}{r_{1,\nu+1}^3} = 0,\\
\varphi_2 &\equiv -\omega^2 x_2 + \frac{m_1(x_2-x_1)}{r_{12}^3} + 0 + \cdots + \frac{m_\nu(x_2-x_\nu)}{r_{2\nu}^3} + \frac{m_{\nu+1}(x_2-x_{\nu+1})}{r_{2,\nu+1}^3} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\varphi_\nu &\equiv -\omega^2 x_\nu + \frac{m_1(x_\nu-x_1)}{r_{1\nu}^3} + \frac{m_2(x_\nu-x_2)}{r_{2\nu}^3} + \cdots + 0 + \frac{m_{\nu+1}(x_\nu-x_{\nu+1})}{r_{\nu,\nu+1}^3} = 0,\\
\varphi_{\nu+1} &\equiv -\omega^2 x_{\nu+1} + \frac{m_1(x_{\nu+1}-x_1)}{r_{1,\nu+1}^3} + \frac{m_2(x_{\nu+1}-x_2)}{r_{2,\nu+1}^3} + \cdots + \frac{m_\nu(x_{\nu+1}-x_\nu)}{r_{\nu,\nu+1}^3} + 0 = 0.
\end{aligned}\right\} \quad (8)$$

The last column of these equations is zero because $m_{\nu+1}=0$ (is infinitesimal). Consequently the first ν equations, which involve $x_1, \ldots, x_\nu$ alone as unknowns, are the equations defining the solutions when $n=\nu$. By (A), it is assumed that there are precisely N_ν real solutions of these equations. Let any one of these solutions be $x_1 = x_1^{(0)}, \ldots, x_\nu = x_\nu^{(0)}$. Then the last equation of (8) becomes

$$\varphi_{\nu+1} \equiv -\omega^2 x_{\nu+1} + \frac{m_1(x_{\nu+1}-x_1^{(0)})}{r_{1,\nu+1}^{(0)3}} + \frac{m_2(x_{\nu+1}-x_2^{(0)})}{r_{2,\nu+1}^{(0)3}} + \cdots + \frac{m_\nu(x_{\nu+1}-x_\nu^{(0)})}{r_{\nu,\nu+1}^{(0)3}} = 0. \quad (9)$$

The number of real solutions of this equation is required.

Consider $\varphi_{\nu+1}$ as a function of $x_{\nu+1}$. It is easily verified that

$$\left.\begin{aligned}
&\varphi_{\nu+1}(+\infty) = -\infty, \qquad \lim_{\epsilon=0} \varphi_{\nu+1}(x_j^{(0)}+\epsilon) = +\infty \qquad (j=1, \ldots, \nu),\\
&\varphi_{\nu+1}(-\infty) = +\infty, \qquad \lim_{\epsilon=0} \varphi_{\nu+1}(x_j^{(0)}-\epsilon) = -\infty \qquad (j=1, \ldots, \nu).
\end{aligned}\right\} \quad (10)$$

Since $\varphi_{\nu+1}$ is finite and continuous except at $x_{\nu+1}=x_1^{(0)}, \ldots, x_\nu^{(0)}, +\infty, -\infty$, it follows that there is an odd number of real solutions in each of the intervals $-\infty$ to $x_p^{(0)}$, where $x_p^{(0)}$ is the smallest $x_j^{(0)}$, $x_k^{(0)}$ to $x_l^{(0)}$, where $x_k^{(0)}$ and $x_l^{(0)}$ are any two $x_j^{(0)}$ which are adjacent, and $x_q^{(0)}$ to $+\infty$, where $x_q^{(0)}$ is the largest $x_j^{(0)}$. But we find from (9) that

$$\frac{\partial \varphi_{\nu+1}}{\partial x_{\nu+1}} \equiv -\omega^2 - \frac{2m_1}{r_{1,\nu+1}^{(0)2}} - \frac{2m_2}{r_{2,\nu+1}^{(0)2}} - \cdots - \frac{2m_\nu}{r_{\nu,\nu+1}^{(0)2}},$$

which is negative except at $x_{\nu+1}=x_1, \ldots, x_{\nu+1}=x_\nu$, where it is infinite. Therefore $\varphi_{\nu+1}$ is a decreasing monotonic function in each of the intervals, and consequently vanishes once, and but once, in each of them. Since

there are $\nu+1$ of these intervals, there are, for each real solution of the first ν equations of (8), precisely $\nu+1$ real solutions of the last equation of (8). Since the first ν equations have, by hypothesis (A), N_ν real solutions, equations (8) altogether have precisely $(\nu+1)N_\nu$ real solutions. This completes the demonstration of Theorem (B).

148. Proof of Theorem (C).—Let $x_j = x_j^{(0)}$ $(j=1, \ldots, \nu+1)$ be any one of the $(\nu+1)N_\nu$ real solutions of equations (8) which are known to exist when $m_{\nu+1}=0$. It will be shown that as $m_{\nu+1}$ increases continuously to any finite positive quantity whatever, the $x_j^{(0)}$ can be made to change continuously so as always to satisfy equations (8), and that they remain distinct, finite, and real. From this it will follow that there are at least $(\nu+1)N_\nu$ real solutions of (8) for every set of finite positive values of $m_1, \ldots, m_{\nu+1}$. It will also be shown that no new solutions can appear as $m_{\nu+1}$ increases from zero to any finite value. Consequently, it will follow that the number of real solutions of (8) is exactly $(\nu+1)N_\nu$ for all finite positive values of the masses $m_1, \ldots, m_{\nu+1}$.

The roots of algebraic equations are continuous functions of the coefficients of the equations so long as the roots are finite and the equations do not have indeterminate forms. Consequently, the $x_j^{(0)}$ are continuous functions of $m_{\nu+1}$ if no $x_i^{(0)}$ becomes infinite and if no $x_i^{(0)}=x_j^{(0)}$. The real roots of algebraic equations having real coefficients can disappear only by passing to infinity, or by an even number of real solutions becoming conjugate complex quantities in pairs. Therefore we have to determine (1) whether any finite $x_i^{(0)}$ can become equal to any $x_j^{(0)}$, (2) whether any $x_i^{(0)}$ can become infinite, and (3) whether any two real solutions can become complex for any finite positive values of $m_1, \ldots, m_{\nu+1}$.

(1). The masses $m_1, \ldots, m_{\nu+1}$, by hypothesis, are all positive. Let the notation be so chosen that for any values of $m_1, \ldots, m_{\nu+1}$ for which the $x_j^{(0)}$ are all distinct the inequalities $x_1^{(0)} < x_2^{(0)} < \cdots < x_\nu^{(0)} < x_{\nu+1}^{(0)}$ are satisfied. Suppose that as some mass is changed the difference $x_j^{(0)} - x_i^{(0)}$ approaches zero in such a way that $x_j^{(0)}$ and $x_i^{(0)}$ remain finite; that is, r_{ij}, which occurs only in the expressions φ_i and φ_j, approaches zero. Suppose $i<j$. Then the term involving r_{ij} becomes negatively infinite in φ_i and positively infinite in φ_j. Consider $\varphi_i = 0$. Another r_{ik} must approach zero in order to restore the finite value of the function φ_i, and the term involving r_{ik} must become positively infinite as r_{ik} approaches zero. Therefore $k<i$. But r_{ik} enters besides only in φ_k, and similar reasoning shows that r_{kl}, where $l<k$, must also approach zero. In this manner we are driven to the conclusion finally that an r_{pq}, where one of the subscripts is unity, approaches zero. Then consider $\varphi_1 = 0$. All its terms except $-\omega^2 x_1$ are negative, and since one of its r_{ij}, viz. r_{pq}, approaches zero the first equation of (8) can not be satisfied. Consequently the original assumption

that some r_{ij} can approach zero for finite values of $x_1^{(0)}, \ldots, x_{\nu+1}^{(0)}$ and finite positive values of $m_1 \ldots, m_{\nu+1}$ leads to an impossibility, and it is therefore false.

(2). On multiplying equations (8) by $m_1, m_2, \ldots, m_{\nu+1}$, respectively, and adding, it is found that

$$-\omega^2(m_1 x_1 + m_2 x_2 + \cdots + m_\nu x_\nu + m_{\nu+1} x_{\nu+1}) = 0.$$

It follows from this equation that no $x_i^{(0)}$ alone can become infinite, and that if one of them becomes negatively infinite then some other one must become positively infinite.

Suppose the notation is again so chosen that

$$x_1^{(0)} < x_2^{(0)} < \cdots < x_\nu^{(0)} < x_{\nu+1}^{(0)}.$$

Then, if any $x_i^{(0)}$ becomes negatively infinite $x_1^{(0)}$ must also become negatively infinite, and from the equation above it follows that $x_{\nu+1}$ must become positively infinite. Now suppose this occurs and consider $\varphi_1 = 0$. In order that this equation may remain satisfied, $x_2^{(0)}$ must also become negatively infinite in such a way that $x_1^{(0)} - x_2^{(0)}$ shall approach zero. But now it follows from $\varphi_2 = 0$, since $-\omega^2 x_2^{(0)}$ and $m_1(x_2^{(0)} - x_1^{(0)})/r_2^3$ are both positive, that $x_3^{(0)}$ must also become negatively infinite in such a way that $x_2^{(0)} - x_3^{(0)}$ shall approach zero. Then it follows similarly from $\varphi_3 = 0$ that $x_4^{(0)}$ must become negatively infinite in such a way that $x_3^{(0)} - x_4^{(0)}$ shall approach zero. This reasoning continues until it is found that $x_1^{(0)}, \ldots, x_{\nu+1}^{(0)}$ must all become negatively infinite. But $x_{\nu+1}^{(0)}$ at least must become positively infinite. Therefore $x_1^{(0)}$ can not become negatively infinite, and similarly $x_{\nu+1}^{(0)}$ can not become positively infinite. Hence no $x_i^{(0)}$ can become infinite.

In order to prove now that, as $m_{\nu+1}$ approaches zero, equations (8) and their solutions remain always determinate, and that there are accordingly no solutions besides those obtained in theorem (*B*), consider a solution $x_1, \ldots, x_{\nu+1}$, in which the x_j are all distinct for a set of positive values of $m_1, \ldots, m_{\nu+1}$, and then let m_i approach zero as a limit.

In the first place, if x_i approaches neither x_{i+1} nor x_{i-1} as a limit as m_i approaches zero as a limit, then by the reasoning of (1) and (2) above no x_j can approach any x_k as a limit.

In the second place, x_i can not approach x_{i+1} as a limit as m_i approaches zero as a limit unless x_{i-1} approaches x_{i+1} as a limit, for otherwise $\varphi_i = 0$ can not be satisfied. But if x_{i-1} approaches x_{i+1} as a limit as m_i approaches zero as a limit, then $\varphi_{i-1} = 0$ and $\varphi_{i+1} = 0$ can not be satisfied unless x_{i-2} and x_{i+2} respectively approach x_i as a limit. This shifts the difficulty to $\varphi_{i-2} = 0$ and $\varphi_{i+2} = 0$, and so on until $\varphi_1 = 0$ and $\varphi_{\nu+1} = 0$ are reached, which can not be satisfied under the hypotheses it has been necessary to make.

In the third place, x_i can not become positively infinite as m_i approaches zero, for then $\varphi_i = 0$ can not be satisfied unless x_{i-1} becomes infinite in such

a way that $x_i - x_{i-1}$ approaches zero. Continuing through $\varphi_{i-1} = 0, \ldots$ we are led to the conclusion finally that $x_1, \ldots, x_{\nu+1}$ all become positively infinite, but then the center of gravity equation can not be satisfied. Consequently the solutions all remain regular as m_i approaches zero as a limit.

(3). Since the solutions of (8) are continuous functions of $m_{\nu+1}$, it follows that no two solutions which are real for $m_{\nu+1}=0$ can ever become conjugate complex solutions for any real value of $m_{\nu+1}$ without having first become equal; and, conversely, no two solutions which are complex for $m_{\nu+1}=0$ can ever become real for any real value of $m_{\nu+1}$ without having first become equal. Consequently, if a multiple solution of (8) is impossible for every set of finite positive values of $m_1, \ldots, m_{\nu+1}$, it is impossible that any real solutions should disappear by becoming complex, or that any complex solutions should become real.

The conditions that $x = x^{(0)}$ shall be a multiple solution of $f(x) = 0$ are that $f(x^{(0)}) = 0$ and $\partial f(x^{(0)})/\partial x = 0$. The corresponding conditions that a set of simultaneous algebraic equations shall have a multiple solution are that a set of values of the variables shall satisfy the equations and that the Jacobian of the functions with respect to the dependent variables shall vanish for the same set of values. That is, the conditions that

$$x_j = x_j^{(0)} \qquad (j=1, \ldots, \nu+1),$$

shall be a multiple solution of (8) are that these values shall satisfy (8) and also the equation

$$\Delta \equiv \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1}, & \frac{\partial \varphi_1}{\partial x_2}, & \ldots, & \frac{\partial \varphi_1}{\partial x_{\nu+1}} \\ \frac{\partial \varphi_2}{\partial x_1}, & \frac{\partial \varphi_2}{\partial x_2}, & \ldots, & \frac{\partial \varphi_2}{\partial x_{\nu+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_{\nu+1}}{\partial x_1}, & \frac{\partial \varphi_{\nu+1}}{\partial x_2}, & \ldots, & \frac{\partial \varphi_{\nu+1}}{\partial x_{\nu+1}} \end{vmatrix} = 0. \tag{11}$$

Consider two solutions of a set of algebraic equations having real coefficients. As they change from real to conjugate complex quantities, or from conjugate complex to real quantities, for some value of a continuously varying parameter, then for this particular value of the parameter they are not only equal but they are *real*. Consequently, it is necessary to examine Δ only when all of its elements are real. It will now be shown that it can not vanish for any set of real values of the x_j when $m_1, \ldots, m_{\nu+1}$ are positive, and consequently that it can not vanish for any particular set which satisfies equations (8). When this is established, it will have been proved that all the solutions of (8) which are real for $m_{\nu+1}=0$ remain real when $m_{\nu+1}$ increases to any positive value, and that those which are complex remain complex.

From equations (8) and (11), it follows that

$$\Delta \equiv \begin{vmatrix} M_1\,, & \dfrac{2m_2}{r_{12}^3}\,, & \dots\,, & \dfrac{2m_{\nu+1}}{r_{1,\nu+1}^3} \\ \dfrac{2m_1}{r_{12}^3}\,, & M_2\,, & \dots\,, & \dfrac{2m_{\nu+1}}{r_{2,\nu+1}^3} \\ \dots & \dots & \dots & \dots \\ \dfrac{2m_1}{r_{1,\nu+1}^3}\,, & \dfrac{2m_2}{r_{2,\nu+1}^3}\,, & \dots\,, & M_{\nu+1} \end{vmatrix}, \tag{12}$$

where

$$\begin{aligned} M_1 &= -\omega^2 - 0 - \frac{2m_2}{r_{12}^3} - \dots - \frac{2m_{\nu+1}}{r_{1,\nu+1}^3}, \\ M_2 &= -\omega^2 - \frac{2m_1}{r_{12}^3} - 0 - \dots - \frac{2m_{\nu+1}}{r_{2,\nu+1}^3}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ M_{\nu+1} &= -\omega^2 - \frac{2m_1}{r_{1,\nu+1}^3} - \frac{2m_2}{r_{2,\nu+1}^3} - \dots - 0. \end{aligned}$$

If $m_{\nu+1}=0$ this determinant breaks up into the product of a determinant of the same type as (12) and a factor which is negative. Therefore, in examining whether or not Δ can vanish, it is sufficient to consider the general case in which all the m_i are positive.

Several properties of Δ are evident. (a) If the i^{th} row be multiplied by m_i ($i = 1, \dots, \nu+1$), the determinant becomes symmetrical. (b) The sum of the elements in each row is $-\omega^2$, from which it follows that the expansion of the determinant contains ω^2 as a factor. (c) The expansion of the determinant contains $(-1)^{\nu+1}\omega^{2(\nu+1)}$ as one of its terms, and since all the m_i are positive and all the x_i are real the sign of all the terms coming from the product of the elements of the main diagonal is $(-1)^{\nu+1}$.

The fact is that, when Δ is completely expanded, all those terms not having the sign $(-1)^{\nu+1}$ are canceled by terms coming from the product of the main diagonal elements, and since the term $(-1)^{\nu+1}\omega^{2(\nu+1)}$ is certainly present the determinant can in no case be zero. The following demonstration of this fact was invented in 1907 by Dr. T. H. Hildebrandt, now of the University of Michigan, as a class exercise.*

Since the determinant contains ω^2 as a factor, every term in its expansion must depend upon at least one of the elements of the main diagonal. Fasten the attention upon any term of the expansion. It can be supposed without loss of generality that it depends upon the first main diagonal element. In the expansion of the determinant this element is multiplied by its minor; consequently we must see if the minor can vanish. The minor is of the

*An earlier proof was devised by the author, and still another jointly by Professor N. B. McLean, of the University of Manitoba, and Mr. E. J. Moulton, now of Northwestern University.

same form as the original determinant, and the sum of the elements of its i^{th} row is $-\omega^2 - m_1/r_{1i}^3$. Consequently every term in the expansion of the minor which does not vanish will contain at least one of the $-\omega^2 - m_1/r_{1i}^3$ as a factor. But these elements appear only in the main diagonal of the minor. Hence all terms in the expansion of the minor which do not vanish depend upon at least one element of the main diagonal. In considering our particular term it may be supposed, without loss of generality, to depend upon the first main diagonal element of the minor. In the expansion of the original determinant the product of these two diagonal elements will be multiplied by the co-factor of the minor of the second order of which they are the main diagonal. This co-factor has the same properties as the first minor just considered, and in the same way it is proved that at least one of its diagonal elements must be involved in the term in question; that is, the term under consideration depends upon at least three elements of the main diagonal. On continuing in this manner it is proved that any term in the final expansion depends upon all the elements of the main diagonal, which are all of the same sign in every one of their terms. Consequently, all the terms which do not cancel out in the expansion of the determinant have the sign $(-1)^{\nu+1}$; and it has been seen that there is at least one such term, viz. $(-1)^{\nu+1}\,\omega^{2(\nu+1)}$. Therefore the determinant not only can never vanish, but it can never be less than $\omega^{2(\nu+1)}$ in numerical value.

Since Δ can never vanish for real distinct $x_j^{(c)}$ when all the m_j are real and either zero or positive, it follows that no real solutions can ever be lost or gained as the m_j vary so as not to become negative, and therefore that the number of real solutions of (8) is $(\nu+1)N_\nu = \frac{1}{2}(\nu+1)!$ for all positive finite values of $m_1, \ldots, m_{\nu+1}$, and ω^2.

149. Computation of the Solutions of Equations (6).—There are well-known methods of finding the roots of a single numerical algebraic equation of high degree, but they are not readily applicable to simultaneous equations of high degree. However, when the order of the masses has been chosen, equations (6) will become polynomials in $x_1, \ldots, x_n$ after they have been cleared of fractions. Then by rational processes $n-1$ of the x_i can be eliminated from these equations, giving a single equation in the remaining unknown. The solutions of this equation can be found by the usual methods and the results can be used to eliminate one unknown. By repeated application of this process to the successively reduced equations, the solutions can all be found. The one satisfying the conditions of reality of $x_1, \ldots, x_n$ and their order relation is the one desired.

The solutions of (6) can also be found by a method closely related to that by means of which their existence was proved above. Suppose for $m_i = m_i^{(0)}$ $(i=1, \ldots, n)$ a solution $x_i = x_i^{(0)}$ of equations (6) is known. The $m_i^{(0)}$ are supposed to be zero or positive. Suppose it is desired to find the corresponding solution, that is, the one in which the masses are arranged

on the line in the same order, for $m_i = m_i^{(0)} + \mu_i$. Let the corresponding set of the x_i satisfying (6) be $x_i = x_i^{(0)} + \xi_i$, where the ξ_i are functions of $\mu_1, \ldots, \mu_n$ to be determined. On substituting $x_i = x_i^{(0)} + \xi_i$ and $m_i = m_i^{(0)} + \mu_i$ in (6), making use of the notation of (8), expanding as power series in the ξ_i and μ_i (which is always possible, since it has been shown that no $x_i^{(0)}$ can become infinite and no $x_i^{(0)}$ can equal any $x_j^{(0)}$), and remembering that $x_i = x_i^{(0)}$ is a solution of (6) for $m_i = m_i^{(0)}$, the resulting equations are found to be

$$\left.\begin{array}{l}
\displaystyle\sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial x_j} \xi_j + \sum_{i=2}^{\infty} \frac{1}{i!}\left[\sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial x_j} \xi_j\right]^i = -\sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial m_j} \mu_j, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\displaystyle\sum_{j=1}^{n} \frac{\partial \varphi_n}{\partial x_j} \xi_j + \sum_{i=2}^{\infty} \frac{1}{i!}\left[\sum_{j=1}^{n} \frac{\partial \varphi_n}{\partial x_j} \xi_j\right]^i = -\sum_{j=1}^{n} \frac{\partial \varphi_n}{\partial m_j} \mu_j,
\end{array}\right\} \tag{13}$$

where the

$$\left[\sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial x_j} \xi_j\right]^i$$

are the symbolic powers used in connection with the power-series expansions of functions of several variables.

The determinant of the terms of the first degree in the ξ_j in equations (13) is the Δ of equation (11), which has been proved to be distinct from zero in this problem. Therefore equations (13) can be solved by the method explained in §1, and by §2 the solutions converge for $|\mu_i| > 0$, but sufficiently small. Suppose they converge if $|\mu_i| \leqq r$. Keeping the μ_i within this limit, a solution $x_i = x_i^{(1)}$ is computed. Then this can be used as a starting-point for a second application of the process, which can be repeated as many times as may be desired.

Hence, to find the solution in which the bodies $m_1, \ldots, m_n$ have any finite positive values and lie in a determined order on the line, we may start with m_1, m_2, and m_3 and solve the Lagrangian quintic* which defines their distribution on the line. Then an infinitesimal body m_4 is added and its position is found by solving the single equation (9), in which $\nu = 3$. This infinitesimal mass m_4 is made to increase, step by step, to the required finite value and the corresponding values of $x_1, \ldots, x_4$ are computed. It follows from the fact that the $\partial\varphi_i/\partial x_j$ are less than fixed finite quantities depending upon ω^2 and $m_1, \ldots, m_n$, while Δ is not less than ω^8 in numerical value, that any finite value of m_4 can be reached in this way by a finite number of steps. After the required value of m_4 has been reached, the process can be repeated for m_5, etc., to any finite number of bodies. Notwithstanding the fact that this would be very laborious if the number of bodies were large, we must regard the problem as completely solved both theoretically and practically.

*Tisserand's *Mécanique Céleste*, vol. I, p. 155, or Moulton's *Introduction to Celestial Mechanics*, p. 216.

II. DETERMINATION OF THE MASSES WHEN THE POSITIONS ARE GIVEN.

150. Determination of the Masses when n is Even.—Suppose ω and the n distinct points on a line, $x_1, \ldots, x_n$, are given, and consider the problem of determining $m_1, \ldots, m_n$ so that the circular solutions shall exist. There will be no loss of generality in selecting the notation so that $x_1 < x_2 < \cdots < x_n$. With this choice of notation equations (6), which are necessary and sufficient conditions for the solutions, become

$$\left.\begin{array}{l}
+\ 0 + \dfrac{m_2}{r_{12}^2} + \dfrac{m_3}{r_{13}^2} + \cdots + \dfrac{m_{n-1}}{r_{1,n-1}^2} + \dfrac{m_n}{r_{1n}^2} = -\omega^2 x_1\ , \\
-\dfrac{m_1}{r_{12}^2} + 0 + \dfrac{m_3}{r_{23}^2} + \cdots + \dfrac{m_{n-1}}{r_{2,n-1}^2} + \dfrac{m_n}{r_{2n}^2} = -\omega^2 x_2\ , \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-\dfrac{m_1}{r_{1,n-1}^2} - \dfrac{m_2}{r_{2,n-1}^2} - \dfrac{m_3}{r_{3,n-1}^2} - \cdots + 0 + \dfrac{m_n}{r_{n,n-1}^2} = -\omega^2 x_{n-1}\ , \\
-\dfrac{m_1}{r_{1n}^2} - \dfrac{m_2}{r_{2n}^2} - \dfrac{m_3}{r_{3n}^2} - \cdots - \dfrac{m_{n-1}}{r_{n,n-1}^2} + 0 = -\omega^2 x_n\ .
\end{array}\right\} \qquad (14)$$

The m_i enter these equations linearly and are therefore uniquely determined if the determinant

$$D = \begin{vmatrix}
+0, & +\dfrac{1}{r_{12}^2}, & +\dfrac{1}{r_{13}^2}, & \ldots, & +\dfrac{1}{r_{1,n-1}^2}, & +\dfrac{1}{r_{1n}^2} \\
-\dfrac{1}{r_{12}^2}, & +0, & +\dfrac{1}{r_{23}^2}, & \ldots, & +\dfrac{1}{r_{2,n-1}^2}, & +\dfrac{1}{r_{2n}^2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\dfrac{1}{r_{1n}^2}, & -\dfrac{1}{r_{2n}^2}, & -\dfrac{1}{r_{3n}^2}, & \ldots, & -\dfrac{1}{r_{n-1,n}^2}, & +0
\end{vmatrix} \qquad (15)$$

is distinct from zero. This is a skew-symmetrical determinant, and when n is even it is the square of an associated Pfaffian, and therefore is not in general zero. Therefore if n is even the masses are, in general, uniquely determined when ω and $x_1, \ldots, x_n$ are given, though it should be noted that they are not necessarily all positive.

151. Determination of the Masses when n is Odd.—In this case the skew-symmetric determinant is identically zero, but its first minors of the main diagonal elements, being skew-symmetrical determinants of even order, are in general all distinct from zero; consequently the x_i must satisfy one relation in order that equations (14) shall be consistent. To get this relation, take the right members to the left and add the equation

$m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = 0$, which is a consequence of (14), to the set of equations. There are then $n+1$ linear homogeneous equations in $-\omega^2, m_1, \ldots, m_n$. In order that they shall be consistent their eliminant

$$E \equiv \begin{vmatrix} 0, & +x_1, & +x_2, & \ldots, & +x_n \\ -x_1, & 0, & +\frac{1}{r_{12}^2}, & \ldots, & +\frac{1}{r_{1n}^2} \\ -x_2, & -\frac{1}{r_{12}^2}, & 0, & \ldots, & +\frac{1}{r_{2n}^2} \\ \cdots & & & & \\ -x_n, & -\frac{1}{r_{1n}^2}, & -\frac{1}{r_{2n}^2}, & \ldots, & 0 \end{vmatrix},$$

must vanish. This is also a skew-symmetrical determinant and is the square of the Pfaffian F, where

$$F \equiv \left| \begin{array}{llll} x_1, & x_2, & \ldots, & x_n \\ & \frac{1}{r_{12}^2}, & \ldots, & \frac{1}{r_{1n}^2} \\ & & & \cdots \\ & & & \frac{1}{r_{n-1,n}^2} \end{array} \right|. \tag{16}$$

Equation (16) can be found also by solving any $n-1$ equations of (14) for the corresponding m_i and substituting the solutions in the remaining one. The result is a sum of determinants which can be shown to be the expansion of F multiplied by the square root of the determinant of the coefficients of the $n-1$ masses m_j in the equations used.

When $F=0$ is satisfied by $x_1, \ldots, x_n$, equations (14) are consistent. Then, after any m_i has been chosen arbitrarily, the corresponding $n-1$ equations can in general be uniquely solved for the remaining m_j, and the unused equation will be satisfied because $F=0$.

152. Discussion of Case $n=3$.—When $n=3$ the determinant D becomes

$$D = \frac{1}{(r_{12}\, r_{23}\, r_{13})^2} - \frac{1}{(r_{12}\, r_{23}\, r_{13})^2} \equiv 0,$$

and the Pfaffian F is

$$F \equiv \frac{x_1}{r_{23}^2} - \frac{x_2}{r_{13}^2} + \frac{x_3}{r_{12}^2} = 0. \tag{17}$$

It will now be shown that when any two of x_1, x_2, x_3 are so chosen as to satisfy the conditions $x_1 < x_2 < x_3$, the third is uniquely determined by (17) and these inequalities. From the fact that in this case $r_{13} > r_{12}$, $r_{13} > r_{23}$, it follows that if x_2 is positive, then $-x_2/r_{13}^2 + x_3/r_{12}^2$ is positive, and therefore that x_1 must be negative in order that (17) may be satisfied. If x_2 is negative, x_1, being less, must also be negative. That is, x_1 *is necessarily negative; and similarly* x_3 *is necessarily positive.*

Suppose x_2 and x_3 are chosen and consider F as a function of x_1. Then it follows at once that

$$\lim_{x_1=-\infty} F(x_1) = -\infty, \quad \lim_{\epsilon=0} F(x_2-\epsilon) = +\infty, \quad \frac{\partial F}{\partial x_1} = \frac{1}{r_{23}^2} - \frac{2x_2}{r_{13}^3} + \frac{2x_3}{r_{12}^3}. \tag{18}$$

From the inequalities $x_2 < x_3$ and $r_{12} < r_{13}$, it follows that $\partial F/\partial x_1$ is positive for $-\infty < x_1 < x_2$. Therefore there is but one solution of (17) for $x_1 < x_2$ when x_2 and a positive x_3 are chosen. By symmetry, there is but one solution of (17) for $x_3 > x_2$ when x_2 and a negative x_1 are chosen.

If x_1 is negative and x_3 is positive, but both otherwise arbitrary, F considered as a function of x_2 gives

$$\lim_{\epsilon=0} F(x_1+\epsilon) = +\infty, \quad \lim_{\epsilon=0} F(x_3-\epsilon) = -\infty, \quad \frac{\partial F}{\partial x_2} = \frac{2x_1}{r_{23}^3} - \frac{1}{r_{13}^2} - \frac{2x_3}{r_{12}^3} < 0. \tag{19}$$

Therefore there is but one solution of equation (17) for x_2 which satisfies the inequalities $x_1 < x_2 < x_3$.

Suppose a negative x_1, a positive x_3, and m_2 are given arbitrarily and that x_2 is defined by (17). Then equations (14) give

$$m_1 = r_{13}^2\left[+\omega^2 x_3 - \frac{m_2}{r_{23}^2}\right], \qquad m_3 = r_{13}^2\left[-\omega^2 x_1 - \frac{m_2}{r_{12}^2}\right]. \tag{20}$$

If m_2 is negative, m_1 and m_3 are necessarily positive. If m_2 is positive and sufficiently small, both m_1 and m_3 are positive. As m_2 increases, m_1 and m_3 decrease. Suppose $x_3 > -x_1$. Then, for a certain positive value of m_2 the mass m_3 vanishes while m_2 is still positive. For a certain greater value of m_2, the mass m_1 is zero and m_3 is negative. For still greater values of m_2, both m_1 and m_3 are negative. From the fact that x_1 must be negative and x_3 positive, and from equations (20), it follows that not all three of the masses m_1, m_2, and m_3 can be negative simultaneously.

CHAPTER IX.

OSCILLATING SATELLITES NEAR THE LAGRANGIAN EQUILATERAL-TRIANGLE POINTS.

By THOMAS BUCK.

153. Introduction.—This chapter is devoted to an investigation of certain periodic orbits which an infinitesimal body may describe when attracted according to the Newtonian law by two finite bodies revolving in circles about their center of mass. It has been shown by Lagrange that three bodies placed at the vertices of an equilateral triangle can be given such initial projections that they will retain always the same configuration. The orbits here considered are in the vicinity of the equilateral-triangle points defined by the two finite bodies. The infinitesimal body is displaced from the vertex of the equilateral triangle, and its initial projection is determined so that its motion is periodic with respect to that of the finite bodies. The existence of the solution is established by the method of analytical continuation. The construction is made by the method of undetermined coefficients, using the properties obtained in the discussion of the existence. The solutions are given in the form of power series which converge for sufficiently small values of the parameter employed.

154. The Differential Equations.—The motion of the infinitesimal body will be referred to a rotating system of axes, the origin being at the center of mass, the $\xi\eta$-plane being the plane of the motion of the finite bodies, and the rate of rotation such that they remain on the ξ-axis. The masses of the finite bodies will be represented by μ and $1-\mu$ so taken that $\mu \overline{\gtrless} \frac{1}{2}$, their distance apart will be taken as the unit of distance, and the unit of time will be so chosen that the proportionality factor k^2 is unity. Then the equations of motion for the infinitesimal body are

$$\frac{d^2\xi}{dt^2} - 2\frac{d\eta}{dt} = \frac{\partial U}{\partial \xi}, \qquad \frac{d^2\eta}{dt^2} + 2\frac{d\xi}{dt} = \frac{\partial U}{\partial \eta}, \qquad \frac{d^2\zeta}{dt^2} = \frac{\partial U}{\partial \zeta}, \tag{1}$$

where

$$U = \frac{1}{2}(\xi^2 + \eta^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$

$$r_1 = \sqrt{(\xi+\mu)^2 + \eta^2 + \zeta^2},$$

$$r_2 = \sqrt{(\xi-1+\mu)^2 + \eta^2 + \zeta^2},$$

r_1 and r_2 being the distances from the infinitesimal body to the bodies $1-\mu$ and μ respectively.

The Lagrangian equilateral triangle-solutions are

$$\text{I.}\quad \xi_0=\frac{1}{2}-\mu, \qquad \eta_0=+\frac{1}{2}\sqrt{3}, \qquad \zeta_0=0.$$

$$\text{II.}\quad \xi_0=\frac{1}{2}-\mu, \qquad \eta_0=-\frac{1}{2}\sqrt{3}, \qquad \zeta_0=0.$$

The two points in the rotating plane defined by these solutions will be referred to as point I and point II respectively. The question of the existence of periodie solutions of equations (1) in the vicinity of these points is to be investigated. For this purpose the origin is transferred to the point in question by means of the transformation

$$\xi=\frac{1}{2}-\mu+x, \qquad \eta=\pm\frac{1}{2}\sqrt{3}+y, \qquad \zeta=z. \tag{2}$$

After the transformation is made the right members of the equations are expanded as power series in x, y, and z. The region of convergence of these series is determined by the singularities of the functions $1/r_1$ and $1/r_2$. When point I is considered, the region of convergence is given by the values of x, y, and z satisfying the inequalities

$$-1<x^2+y^2+z^2+x+\sqrt{3}\,y<+1, \qquad -1<x^2+y^2+z^2-x+\sqrt{3}\,y<+1.$$

This region consists of the common portion of two spheres, excluding their centers which are at the finite bodies, each of radius $\sqrt{2}$. For point II the region of convergence is defined by the inequalities

$$-1<x^2+y^2+z^2+x-\sqrt{3}\,y<+1, \qquad -1<x^2+y^2+z^2-x-\sqrt{3}\,y<+1.$$

Since the origin in this case is at the second point it follows that this region is the same as that found for the first point.

As the two cases differ only in the sign before the $\sqrt{3}$, it is necessary to consider in detail only one of them. The discussion will be given for point I with the understanding that by changing the sign of $\sqrt{3}$ the corresponding expressions for the point II are obtained.

Two parameters, ϵ and δ, are introduced as in Chapter V. Then, denoting derivation as to τ by accents, the differential equations become

$$\left.\begin{aligned} x''-2(1+\delta)y' &= (1+\delta)^2\,[X_1+X_2\,\epsilon+X_3\,\epsilon^2+\cdots], \\ y''+2(1+\delta)x' &= (1+\delta)^2\,[Y_1+Y_2\,\epsilon+Y_3\,\epsilon^2+\cdots], \\ z'' &= (1+\delta)^2\,[Z_1+Z_2\,\epsilon+Z_3\,\epsilon^2+\cdots], \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned}
&X_1=+\tfrac{3}{4}x+\tfrac{3}{4}\sqrt{3}\,(1-3\mu)y, \qquad Y_1=+\tfrac{3}{4}\sqrt{3}\,(1-2\mu)x+\tfrac{9}{4}y,\\
&X_2=+\tfrac{3}{16}[7(1-2\mu)\,x^2+2\sqrt{3}\,xy-11(1-2\mu)\,y^2+4(1-2\mu)z^2],\\
&Y_2=+\tfrac{3}{16}[\sqrt{3}x^2+22(1-2\mu)\,xy+3\sqrt{3}y^2-4\sqrt{3}z^2],\\
&Z_1=-z, \qquad Z_2=+\tfrac{3}{2}[(1-2\mu)xz+\sqrt{3}yz],\\
&X_3=+\tfrac{1}{32}[-37x^3+75\sqrt{3}(1-2\mu)x^2y+123xy^2+45\sqrt{3}(1-2\mu)y^3\\
&\qquad\qquad -12xz^2+6\sqrt{3}\,(1-2\mu)\,yz^2],\\
&Y_3=+\tfrac{1}{32}[-25\sqrt{3}\,(1-2\mu)\,x^3+123x^2y+135\sqrt{3}\,(1-2\mu)xy^2+3y^3\\
&\qquad\qquad -60\sqrt{3}\,(1-2\mu)\,xz^2+132yz^2],\\
&Z_3=-\tfrac{3}{8}\,[x^2z+11y^2z-4z^3+10\sqrt{3}\,(1-2\mu)\,xyz].
\end{aligned}\right\} \tag{4}$$

For sufficiently small values of x, y, z, and ϵ these series are all convergent.

Equations (1) admit the integral

$$\frac{1}{2}\left[\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2+\left(\frac{d\zeta}{dt}\right)^2\right]=U+\text{const.} \tag{5}$$

The corresponding integral of equations (3) can be expressed as a power series in ϵ. The terms independent of ϵ are

$$4[x'^2+y'^2+z'^2]-(1+\delta)^2[3x^2+9y^2-4z^2+6\sqrt{3}\,(1-2\mu)\,xy]=\text{const.} \tag{5'}$$

These terms will be found useful in the existence proofs which follow.

155. The Characteristic Exponents.—For $\epsilon=\delta=0$ equations (3) become

$$\left.\begin{aligned}
&x''-2y'-\tfrac{3}{4}\,x-\tfrac{3}{4}\,\sqrt{3}\,(1-2\mu)y=0,\\
&y''+2x'-\tfrac{3}{4}\,\sqrt{3}\,(1-2\mu)\,x-\tfrac{9}{4}\,y=0,\\
&z''+z=0.
\end{aligned}\right\} \tag{6}$$

The last equation, being independent of the first two, can be integrated immediately, giving

$$z=c_1\sin\tau+c_2\cos\tau.$$

To integrate the first two equations, let

$$x=Ke^{\lambda\tau}, \qquad y=Le^{\lambda\tau}.$$

On substituting in the first two equations of (6) and dividing out $e^{\lambda\tau}$, we have

$$\left.\begin{aligned}\left[\lambda^2-\tfrac{3}{4}\right]K-\left[2\lambda+\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\right]L=0,\\ \left[2\lambda-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\right]K+\left[\lambda^2-\tfrac{9}{4}\right]L=0.\end{aligned}\right\} \qquad (7)$$

In order that these equations may be satisfied by values of K and L different from zero, the determinant of the system must vanish. This gives for the determination of λ the characteristic equation

$$\lambda^4+\lambda^2+\tfrac{27}{4}\mu(1-\mu)=0. \qquad (8)$$

Each of the four values of λ satisfying this equation gives a particular solution of equations (6). The corresponding K and L must satisfy equations (7). Since these equations have a vanishing determinant the ratio only of the K and L is determined. In what follows K will be considered as arbitrary, and L will be determined in the form $L=bK$.

In order that a solution shall be periodic, the corresponding λ must be a purely imaginary quantity. Upon solving (8), we have

$$\lambda^2=\frac{-1\pm\sqrt{1-27\mu(1-\mu)}}{2}.$$

For small values of μ the roots of (8) are pure imaginaries; the limiting value of μ for which this is true is given by the equation

$$1-27\mu(1-\mu)=0.$$

The root of this equation which is less than $\frac{1}{2}$ is $\mu=.0385\ldots$. For $\mu\leqq.0385\ldots$ the values of λ are purely imaginary and the corresponding particular solutions are periodic. Let σ_1 and σ_2 be two numbers defined by

$$\sigma_1^2=\frac{1+\sqrt{1-27\mu(1-\mu)}}{2}, \qquad \sigma_2^2=\frac{1-\sqrt{1-27\mu(1-\mu)}}{2}.$$

It follows that σ_1 and σ_2 do not exceed unity for $\mu\leqq.0385\ldots$, and that $\sigma_1\geqq\sigma_2$. Then the roots of (8), which are the characteristic exponents of the problem, become $\pm\sigma_1\sqrt{-1}$ and $\pm\sigma_2\sqrt{-1}$.

156. The Generating Solutions.—The general solution of (6) is

$$
\begin{aligned}
x &= a_1 e^{\sigma_1\sqrt{-1}\tau} + a_2 e^{-\sigma_1\sqrt{-1}\tau} + a_3 e^{\sigma_2\sqrt{-1}\tau} + a_4 e^{-\sigma_2\sqrt{-1}\tau},\\
y &= b_1 a_1 e^{\sigma_1\sqrt{-1}\tau} + b_2 a_2 e^{-\sigma_1\sqrt{-1}\tau} + b_3 a_3 e^{\sigma_2\sqrt{-1}\tau} + b_4 a_4 e^{-\sigma_2\sqrt{-1}\tau},\\
z &= c_1 \sin\tau + c_2 \cos\tau .
\end{aligned}
$$

The quantities a_1, a_2, a_3, a_4, c_1, and c_2 are arbitrary, while b_1, b_2, b_3, and b_4 are determined by equations (7) when the proper values of λ are substituted. Thus it is found that

$$
b_1 = \frac{+8\sigma_1\sqrt{-1} - 3\sqrt{3}(1-2\mu)}{4\sigma_1^2+9}, \qquad b_3 = \frac{+8\sigma_2\sqrt{-1} - 3\sqrt{3}(1-2\mu)}{4\sigma_2^2+9},
$$

$$
b_2 = \frac{-8\sigma_1\sqrt{-1} - 3\sqrt{3}(1-2\mu)}{4\sigma_1^2+9}, \qquad b_4 = \frac{-8\sigma_2\sqrt{-1} - 3\sqrt{3}(1-2\mu)}{4\sigma_2^2+9},
$$

Various periodic solutions are obtained from this general solution by assigning suitable values to the arbitrary constants and to the quantity μ. For $\mu < .0385\ldots$ we have two distinct periodic solutions:

$$
\text{I.} \quad x = a_1 e^{\sigma_1\sqrt{-1}\tau} + a_2 e^{-\sigma_1\sqrt{-1}\tau}, \quad y = b_1 a_1 e^{\sigma_1\sqrt{-1}\tau} + b_2 a_2 e^{-\sigma_1\sqrt{-1}\tau}, \quad z = 0.
$$

$$
\text{II.} \quad x = a_3 e^{\sigma_2\sqrt{-1}\tau} + a_4 e^{-\sigma_2\sqrt{-1}\tau}, \quad y = b_3 a_3 e^{\sigma_2\sqrt{-1}\tau} + b_4 a_4 e^{-\sigma_2\sqrt{-1}\tau}, \quad z = 0.
$$

These equations represent ellipses in the xy-plane with centers at the origin. The major axes of the ellipses coincide and make an angle θ with the positive x-axis defined by

$$
\tan 2\theta = -\sqrt{3}\,(1-2\mu),
$$

with $\cos 2\theta$ positive. The major and minor axes of the second are greater and less respectively than those of the first. The periods are $2\pi/\sigma_1$ and $2\pi/\sigma_2$ respectively. If $\mu = .0385\ldots$ it follows that $\sigma_1 = \sigma_2$, and solutions I and II coincide.

For all values of μ we have the periodic solution

$$
\text{III.} \qquad x = 0, \qquad y = 0, \qquad z = c_1 \sin\tau + c_2 \cos\tau .
$$

This solution defines an oscillation on the z-axis with the period 2π.

It is possible to give μ values such that σ_1 and σ_2 are relatively commensurable. Let $m_2\sigma_1 = m_1\sigma_2$, where m_1 and m_2 are integers. Then, by using the definitions of σ_1 and σ_2, we find

$$
\sqrt{1-27\mu(1-\mu)} = \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2}.
$$

For $\mu \gtreqless .0385$ . . . the expression on the left takes all values on the interval from 0 to 1. By choosing m_1 and m_2 so that $0<(m_1^2-m_2^2)/(m_1^2+m_2^2)<1$, and solving the equation for μ, we have a value of μ making σ_1 and σ_2 commensurable. For such values of μ we have the additional periodic solution

$$\text{IV.}\quad \begin{cases} x= \quad a_1 e^{\sigma_1\sqrt{-1}\tau}+\quad a_2 e^{-\sigma_1\sqrt{-1}\tau}+\quad a_3 e^{\sigma_2\sqrt{-1}\tau}+\quad a_4 e^{-\sigma_2\sqrt{-1}\tau}, \\ y=b_1 a_1 e^{\sigma_1\sqrt{-1}\tau}+b_2 a_2 e^{-\sigma_1\sqrt{-1}\tau}+b_3 a_3 e^{\sigma_2\sqrt{-1}\tau}+b_4 a_4 e^{-\sigma_2\sqrt{-1}\tau}, \\ z=0, \end{cases}$$

the period of which is $2m_1\pi/\sigma_1=2m_2\pi/\sigma_2$.

By an argument precisely similar to the preceding it can be shown that for special values of μ, the characteristic exponents σ_1 and σ_2 separately may be commensurable with unity. We have then the periodic solutions

$$\text{V.}\quad \begin{cases} x= \quad a_1 e^{\sigma_1\sqrt{-1}\tau}+\quad a_2 e^{-\sigma_1\sqrt{-1}\tau}, \\ y=b_1 a_1 e^{\sigma_1\sqrt{-1}\tau}+b_2 a_2 e^{-\sigma_1\sqrt{-1}\tau}, \\ z=c_1 \sin\tau+c_2\cos\tau; \end{cases}$$

$$\text{VI.}\quad \begin{cases} x= \quad a_3 e^{\sigma_2\sqrt{-1}\tau}+\quad a_4 e^{-\sigma_2\sqrt{-1}\tau}, \\ y=b_3 a_3 e^{\sigma_2\sqrt{-1}\tau}+b_4 a_4 e^{-\sigma_2\sqrt{-1}\tau}, \\ z=c_1 \sin\tau+c_2\cos\tau. \end{cases}$$

The periods are $2m\pi=2m_1\pi/\sigma_1$ and $2n\pi=2n_2\pi/\sigma_2$ respectively.

Finally, σ_1 and σ_2 may be commensurable with unity for the same value of μ.

Let $a\sigma_1=b$ and $c\sigma_2=d$; then

$$\sqrt{1-27\mu(1-\mu)}=\frac{2b^2-a^2}{a^2}=\frac{c^2-2d^2}{c^2}.$$

From this relation it follows that $a^2d^2=c^2(a^2-b^2)$. If now we give a, b, c, and d such integral values that this relation is satisfied and such that $(2b^2-a^2)/a^2$ lies between 0 and 1, the corresponding value of μ will make σ_1 and σ_2 commensurable with unity. For example, such a choice is

$$a=c=13, \qquad b=12, \qquad d=5.$$

For such values of μ we have the periodic solution

$$\text{VII.}\quad \begin{cases} x= \quad a_1 e^{\sigma_1\sqrt{-1}\tau}+\quad a_2 e^{-\sigma_1\sqrt{-1}\tau}+\quad a_3 e^{\sigma_2\sqrt{-1}\tau}+\quad a_4 e^{-\sigma_2\sqrt{-1}\tau}, \\ y=b_1 a_1 e^{\sigma_1\sqrt{-1}\tau}+b_2 a_2 e^{-\sigma_1\sqrt{-1}\tau}+b_3 a_3 e^{\sigma_2\sqrt{-1}\tau}+b_4 a_4 e^{-\sigma_2\sqrt{-1}\tau}, \\ z=c_1 \sin\tau+c_2\cos\tau. \end{cases}$$

The period of this solution is $2m\pi=2m_1\pi/\sigma_1=2m_2\pi/\sigma_2$.

These periodic solutions are the generating solutions for the general problem. We shall now suppose that ϵ is not zero and consider the question of the existence of the continuations of these solutions with respect to the parameter ϵ. The period in the variable τ will in all cases be taken the same as that of the generating solution. The period in t of the solution is found from the relation $t=(1+\delta)\tau$.

157. General Periodicity Equations.—For $\epsilon=0$ the general solution of equations (3) is

$$\left.\begin{aligned} x&= a_1e^{\sigma_1(1+\delta)\sqrt{-1}\tau}+ a_2e^{-\sigma_1(1+\delta)\sqrt{-1}\tau}+ a_3e^{\sigma_2(1+\delta)\sqrt{-1}\tau}+ a_4e^{-\sigma_2(1+\delta)\sqrt{-1}\tau},\\ y&=b_1a_1e^{\sigma_1(1+\delta)\sqrt{-1}\tau}+b_2a_2e^{-\sigma_1(1+\delta)\sqrt{-1}\tau}+b_3a_3e^{\sigma_2(1+\delta)\sqrt{-1}\tau}+b_4a_4e^{-\sigma_2(1+\delta)\sqrt{-1}\tau},\\ z&=c_1\sin(1+\delta)\tau +c_2\cos(1+\delta)\tau.\end{aligned}\right\}\quad(9)$$

Normal variables are introduced by the transformation

$$\left.\begin{aligned} x &=x_1+x_2+x_3+x_4,\\ y &=b_1 x_1+b_2 x_2+b_3 x_3+b_4 x_4,\\ x' &=\sigma_1 (1+\delta)\sqrt{-1}\,(x_1-x_2)+\sigma_2 (1+\delta) \sqrt{-1}\,(x_3-x_4),\\ y' &=\sigma_1 (1+\delta)\sqrt{-1}\,(b_1 x_1-b_2 x_2)+\sigma_2 (1+\delta) \sqrt{-1}\,(b_3 x_3-b_4 x_4).\end{aligned}\right\}\quad(10)$$

The differential equations then become

$$\left.\begin{aligned} x_1'-\sigma_1(1+\delta)\sqrt{-1}x_1&= A_1(X_2\epsilon+X_3\epsilon^2+\cdots)+B_1(Y_2\epsilon+Y_3\epsilon^2+\cdots),\\ x_2'+\sigma_1(1+\delta)\sqrt{-1}x_2&= A_2(X_2\epsilon+X_3\epsilon^2+\cdots)+B_2(Y_2\epsilon+Y_3\epsilon^2+\cdots),\\ x_3'-\sigma_2(1+\delta)\sqrt{-1}x_3&= A_3(X_2\epsilon+X_3\epsilon^2+\cdots)+B_3(Y_2\epsilon+Y_3\epsilon^2+\cdots),\\ x_4'+\sigma_2(1+\delta)\sqrt{-1}x_4&= A_4(X_2\epsilon+X_3\epsilon^2+\cdots)+B_4(Y_2\epsilon+Y_3\epsilon^2+\cdots),\\ z''+(1+\delta)^2z &=(1+\delta)^2[Z_2\epsilon+Z_3\epsilon^2+\cdots],\end{aligned}\right\}\quad(11)$$

where

$$A_i= (1+\delta)^2\frac{\Delta_{3i}}{\Delta}, \qquad B_i= (1+\delta)^2\frac{\Delta_{4i}}{\Delta},$$

Δ being the determinant of the transformation (10), and Δ_{ji} the minor of an element in this determinant. The first subscript indicates the row and the second one the column.

For $\epsilon=\delta=0$ the initial values of the variables x_1, x_2, x_3, and x_4 are a_1, a_2, a_3, and a_4 respectively. In the general problem we take as the initial conditions

$$\left.\begin{array}{lll} x_1=a_1+\alpha_1\,, & x_3=a_3+\alpha_3\,, & z\;=0, \\ x_2=a_2+\alpha_2\,, & x_4=a_4+\alpha_4\,, & z'=c+\gamma. \end{array}\right\} \qquad (12)$$

Since there is a component of force always directed toward the xy-plane, it is clear that at some time z must be zero. Hence we have supposed that $z=0$ at $\tau=0$.

According to §§14 and 15 equations (11) can be integrated as power series in the parameters α_1, α_2, α_3, α_4, γ, δ, and ϵ, which converge for $|\alpha_1|$, . . . , $|\epsilon|$ sufficiently small, and for $0\leqq\tau\leqq T$, the value of T, which in this case is the period, being given in advance. These solutions have the form

$$\left.\begin{array}{l} x_1=(a_1+\alpha_1)\,e^{+\sigma_1(1+\delta)\sqrt{-1}\,\tau}+\epsilon p_1\,(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4,\,\gamma,\,\delta,\,\epsilon;\,\tau), \\ x_2=(a_2+\alpha_2)\,e^{-\sigma_1(1+\delta)\sqrt{-1}\,\tau}+\epsilon p_2\,(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4,\,\gamma,\,\delta,\,\epsilon;\,\tau), \\ x_3=(a_3+\alpha_3)\,e^{+\sigma_2(1+\delta)\sqrt{-1}\,\tau}+\epsilon p_3\,(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4,\,\gamma,\,\delta,\,\epsilon;\,\tau), \\ x_4=(a_4+\alpha_4)\,e^{-\sigma_2(1+\delta)\sqrt{-1}\,\tau}+\epsilon p_4\,(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4,\,\gamma,\,\delta,\,\epsilon;\,\tau), \\ z\;=(c+\gamma)\,\sin(1+\delta)\tau+\epsilon p_5\,(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4,\,\gamma,\,\delta,\,\epsilon;\,\tau), \\ z'=(1+\delta)(c+\gamma)\cos(1+\delta)\tau+\epsilon p_6\,(\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_4,\,\gamma,\,\delta,\,\epsilon;\,\tau). \end{array}\right\} \qquad (13)$$

The general periodicity equations for the period T are

$$\left.\begin{array}{ll} x_i(T)-x_i(0)=0 & (i=1,\,.\,.\,.\,,\,4), \\ z(T)-\;z(0)=0, & z'(T)-z'(0)=0. \end{array}\right\} \qquad (14)$$

These equations are sufficient conditions for the periodicity of the solution. On solving them for the arbitraries α_1, α_2, α_3, α_4, γ, δ as power series in ϵ, a determination of these quantities is obtained such that the corresponding solution is periodic. Hence, on substituting these series in (13), the resulting expressions for the x_i, z, and z' are periodic. These expressions can be rearranged as power series in ϵ which will converge for ϵ sufficiently small, and for all $0 \gtreqless \tau \gtreqless T$, provided the values of α_1, . . . , α_4, γ, δ obtained from (14) lie in the domain of convergence of (13). The convergence can be secured by imposing the condition that the expressions for the arbitraries obtained from (14) shall be power series in ϵ, which vanish with ϵ. The solutions are then analytical continuations of the generating solutions.

The periodicity equations will now be set up for each of the generating solutions, and the possibility of solving them for the arbitraries in the required form will be considered. The equations will be written for the point I only. The conclusions are the same in all cases for the point II.

158. The First Generating Solution.—The explicit form of equations (14) is now to be determined for $T=2\pi/\sigma_1$ and $a_3=a_4=c=0$. On account of the existence of the integral (5), one of the equations is redundant. For if we let

$$
\begin{array}{lll}
x_1=a_1e^{+\sigma_1\sqrt{-1}\tau}+y_1, & x_3=0+y_3, & z=0+w, \\
x_2=a_2e^{-\sigma_1\sqrt{-1}\tau}+y_2, & x_4=0+y_4, & z'=0+w',
\end{array}
$$

where $y_1(0)=y_2(0)=y_3(0)=y_4(0)=w(0)=w'(0)=0$, we find that the partial derivative of the integral (5′) with respect to y_2 is

$$\frac{64\sigma_1^2(2\sigma_1^2-1)\,a_1}{(4\sigma_1^2+9)}$$

for $\tau=2\pi/\sigma_1$, and $y_1=y_2=y_3=y_4=w=w'=0$. The integral can therefore be solved uniquely for y_2 in terms of y_1, y_3, y_4, w, and w'. If the latter quantities are periodic it follows that y_2 also is periodic. Therefore the second equation is redundant and can be suppressed. On computing the necessary terms of (13), it is found that the remaining equations have the form

$$
\left.
\begin{aligned}
&\delta\Big[\big(2\pi\sqrt{-1}\,a_1\big)+\cdots\Big]+\epsilon\Big[e_{11}\,\alpha_3+e_{12}\,\alpha_4+e_{13}\,\epsilon+\cdots\Big]=0,\\
&\alpha_3\Big[\Big(e^{\frac{2\pi\sigma_2\sqrt{-1}}{\sigma_1}}-1\Big)+\cdots\Big]+\epsilon\Big[e_{21}\,\alpha_3+e_{22}\,\alpha_4+e_{23}\,\epsilon+\cdots\Big]=0,\\
&\alpha_4\Big[\Big(e^{\frac{-2\pi\sigma_2\sqrt{-1}}{\sigma_1}}-1\Big)+\cdots\Big]+\epsilon\Big[e_{31}\,\alpha_3+e_{32}\,\alpha_4+e_{33}\,\epsilon+\cdots\Big]=0,\\
&\gamma\Big[\Big(\sin\frac{2\pi}{\sigma_1}\Big)+\cdots\Big]+\epsilon\Big[\cdots\cdots\cdots\cdots\Big]=0,\\
&\gamma\Big[\Big(\cos\frac{2\pi}{\sigma_1}-1\Big)+\cdots\Big]+\epsilon\Big[\cdots\cdots\cdots\cdots\Big]=0,
\end{aligned}
\right\}
\qquad (15)
$$

where the explicit computation shows that the e_{ij} are constants different from zero.

The right member of the z-equation in (11) carries the factor z. Consequently the solution carries the factor γ, and hence the last two equations of (15) have γ as a factor. If γ is not zero and is divided out, there remains a term in each equation which is independent of the arbitraries. These terms can vanish only if σ_1 is the reciprocal of an integer. If they do not vanish, it follows that the equations can not be satisfied by the vanishing of all the arbitraries, and consequently that solutions of the required form do not exist.

In order to satisfy equations (15) we must suppose, then, that $\gamma=0$. Hence the motion of the infinitesimal body will be entirely in the xy-plane. The first three equations are satisfied by $\alpha_3=\alpha_4=\delta=\epsilon=0$, and are not satisfied by $\alpha_3=\alpha_4=\delta=0$. The coefficient of δ in the first equation is distinct

from zero. If the coefficients of a_3 and a_4 in the second and third equations respectively are also distinct from zero, it follows that there is a unique solution for a_3, a_4, and δ as power series in ϵ, vanishing with ϵ. It is necessary, therefore, that σ_1 be such that the expressions

$$e^{\frac{2\pi\sigma_2\sqrt{-1}}{\sigma_1}}-1, \qquad e^{\frac{-2\pi\sigma_2\sqrt{-1}}{\sigma_1}}-1, \qquad \sin\frac{2\pi}{\sigma_1}, \qquad \cos\frac{2\pi}{\sigma_1}-1$$

shall not vanish. The first two vanish if $\sigma_2 = m\sigma_1$, where m is an integer. Since σ_1 and σ_2 are positive and $\sigma_1 \geqq \sigma_2$, this occurs only when $\sigma_1 = \sigma_2$. This case will be treated later in the discussion of the commensurable cases. It will be shown that orbits exist in this case also. The last two expressions vanish only if σ_1 is the reciprocal of an integer. Suppose, then, that $\sigma_1 = 1/m$. On solving the first three equations of (15) for a_3, a_4, δ and substituting in the last two, there remains, after dividing out ϵ^2 and γ, a term in each independent of the arbitraries. There can be, then, no solution of these equations in the required form. They can be satisfied only by putting $\gamma=0$ as before. Hence equations (15) have a unique solution of the same form in this case also.

The question of the existence of an orbit re-entering after m revolutions will now be considered. The period in this case is $2m\pi/\sigma_1$. The periodicity equations have a unique solution, as before, except when μ is such that $m\sigma_2=\sigma_1$. In this case the second and third equations do not admit solutions for a_3 and a_4. This case will be treated later in the discussion of the commensurable cases. It will be shown that orbits exist for these values of μ also. Hence there is a single orbit re-entering after m revolutions. But one such orbit is obtained by m repetitions of the orbits re-entering after one revolution. It follows, therefore, that this orbit is the only one.

The periodicity equations have now been satisfied with a_1 and a_2 still remaining arbitrary. Since we now have $z\equiv 0$, one relation between these arbitraries is obtained by fixing the origin of time. It will be supposed that $x'=0$ at $\tau=0$. This gives the relation

$$\sigma_1(a_1-a_2+\alpha_1-\alpha_2)+\sigma_2(\alpha_3-\alpha_4)=0.$$

The same choice of the origin of time in the generating solution gives $a_1=a_2$. We have then

$$\sigma_1(\alpha_1-\alpha_2)+\sigma_2(\alpha_3-\alpha_4)=0.$$

This equation may be regarded as determining α_2 in terms of the arbitrary α_1. There will then be in the final solution the two arbitraries α_1 and a_1 besides the parameter ϵ. Since α_1 and a_1 occur always in the combination $a_1+\alpha_1$ they can be replaced by a single arbitrary. When the solutions of the periodicity equations are substituted in (13), the desired continuation of of the first generating solution is obtained.

The discussion of the existence of the continuation of the second generating solution, which depends upon σ_2 as the first does on σ_1, differs from that just given only in notation, and will therefore be omitted. The orbits are in the xy-plane and have the period $2\pi/\sigma_2$.

159. The Third Generating Solution.—It can be proved from the integral that the last equation of (14) is redundant, and it will therefore be suppressed. The period is 2π and the periodicity conditions are

$$\left.\begin{aligned}
&\alpha_1(e^{+2\pi\sigma_1\sqrt{-1}}-1)+\alpha_1 e^{+2\pi\sigma_1\sqrt{-1}}(+2\pi\sigma_1\sqrt{-1}\,\delta+\cdots)+\epsilon(h_{10}+\cdots)=0,\\
&\alpha_2(e^{-2\pi\sigma_1\sqrt{-1}}-1)+\alpha_2 e^{-2\pi\sigma_1\sqrt{-1}}(-2\pi\sigma_1\sqrt{-1}\,\delta+\cdots)+\epsilon(h_{20}+\cdots)=0,\\
&\alpha_3(e^{+2\pi\sigma_2\sqrt{-1}}-1)+\alpha_3 e^{+2\pi\sigma_2\sqrt{-1}}(+2\pi\sigma_2\sqrt{-1}\,\delta+\cdots)+\epsilon(h_{30}+\cdots)=0,\\
&\alpha_4(e^{-2\pi\sigma_2\sqrt{-1}}-1)+\alpha_4 e^{-2\pi\sigma_2\sqrt{-1}}(-2\pi\sigma_2\sqrt{-1}\,\delta+\cdots)+\epsilon(h_{40}+\cdots)=0,\\
&\qquad\qquad(c+\gamma)[2\pi\delta+\cdots]+\epsilon(h_{50}+\cdots)=0,
\end{aligned}\right\}\quad(16)$$

where the h_{ij} are functions of μ, the explicit form of which will not be given.

The first four equations are satisfied by $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\epsilon=0$, and the determinant of the terms which are linear in $\alpha_1,\ldots,\alpha_4$ is distinct from zero, since σ_1 and σ_2 can not take integral values. Therefore these equations can be solved for α_1, α_2, α_3, and α_4 as power series in ϵ, δ, and γ, vanishing with ϵ. Since the right member of the z-equation in (11) carries the factor z, the last equation carries the factor $c+\gamma$. This factor is divided out and the series for α_1, α_2, α_3, and α_4 are substituted for these quantities. The equation is satisfied by $\delta=\epsilon=0$, and the coefficient of δ is not zero. Hence there is a unique solution for δ in the required form. This value of δ is substituted in the series already found for α_1, α_2, α_3, and α_4. The quantity γ remains arbitrary, but since it occurs always in the combination $c+\gamma$ it will be absorbed in the arbitrary constant c. The periodicity equations being satisfied in the required form, the existence of the orbits in question is established.

For an orbit re-entering after m revolutions the periodicity equations have a unique solution except when μ is such that $m\sigma_1=m_1$ or $m\sigma_2=m_2$, where m_1 and m_2 are integers. It will be shown later that the orbits exist uniquely in these cases also. Since these orbits include as a special case those re-entering after one revolution, it follows that no new orbits are obtained.

160. The Fourth Generating Solution.—In this case μ is restricted to those values for which $m_2\sigma_1=m_1\sigma_2$. Consequently the period is $2m_1\pi/\sigma_1=2m_2\pi/\sigma_2$. Just as in the case of the first generating solution, we must put $\gamma=0$ in order to satisfy the last two periodicity equations. The

second equation of (14) can be suppressed because of the integral. The required terms of the series for the x_i are found from equations (11), and the explicit forms of the periodicity conditions are

$$\left.\begin{aligned}
&(a_1+\alpha_1)\,\delta\Big[+2m_1\pi\sqrt{-1}+\cdots\Big]\\
&+(a_1+\alpha_1)\frac{2m_1\pi}{\sigma_1}\Big[\theta_{12}(a_1+\alpha_1)(a_2+\alpha_2)+\theta_{13}(a_3+\alpha_3)(a_4+\alpha_4)\Big]\epsilon^2+\cdots=0,\\
&(a_3+\alpha_3)\,\delta\Big[+2m_2\pi\sqrt{-1}+\cdots\Big]\\
&+(a_3+\alpha_3)\frac{2m_2\pi}{\sigma_2}\Big[\theta_{32}(a_1+\alpha_1)(a_2+\alpha_2)+\theta_{33}(a_3+\alpha_3)(a_4+\alpha_4)\Big]\epsilon^2+\cdots=0,\\
&(a_4+\alpha_4)\,\delta\Big[-2m_2\pi\sqrt{-1}+\cdots\Big]\\
&+(a_4+\alpha_4)\frac{2m_2\pi}{\sigma_2}\Big[\theta_{42}(a_1+\alpha_1)(a_2+\alpha_2)+\theta_{43}(a_3+\alpha_3)(a_4+\alpha_4)\Big]\epsilon^2+\cdots=0,
\end{aligned}\right\}\quad(17)$$

where the θ_{ij} are functions of μ which will not be given explicitly.

The first equation of (17) is solved for δ and the result substituted in the other two equations. After dividing them by ϵ^2, they have the form

$$\left.\begin{aligned}
&(a_3+\alpha_3)\Big[A_{11}(a_1+\alpha_1)(a_2+\alpha_2)+A_{12}(a_3+\alpha_3)(a_4+\alpha_4)\Big]+\epsilon\Big[\cdots\Big]=0,\\
&(a_4+\alpha_4)\Big[A_{21}(a_1+\alpha_1)(a_2+\alpha_2)+A_{22}(a_3+\alpha_3)(a_4+\alpha_4)\Big]+\epsilon\Big[\cdots\Big]=0,
\end{aligned}\right\}\quad(18)$$

where the A_{ij} are functions of μ. In order that solutions of the required form shall exist, it is necessary that

$$a_3[A_{11}a_1a_2+A_{12}a_3a_4]=0,\qquad a_4[A_{21}a_1a_2+A_{22}a_3a_4]=0.\qquad(19)$$

These equations are satisfied by $a_3=a_4=0$. When these conditions are imposed, equations (18) can be solved uniquely for α_3 and α_4 as power series in ϵ, vanishing with ϵ. The generating solution is reduced to that considered in § 158, and it is now possible to supply the proof for the exceptional cases which were not covered by the previous discussion.

For an orbit re-entering after one revolution the proof did not include the case when $\sigma_1=\sigma_2$. When $m_1=m_2=1$ the discussion just given supplies this deficiency. For an orbit re-entering after m revolutions, the case when $m\sigma_2=\sigma_1$ was not included. By putting $m_1=m$ and $m_2=1$, we have the desired proof.

The corresponding cases arising from the second generating solution can be treated by so combining the periodicity equations that the equations (19) carry the factors a_1 and a_2 respectively. The discussion is then the same as that just given.

In order that equations (19) may be satisfied by values of the a_i different from zero, it is necessary that the determinant of the A_{ij} should vanish. This determinant can be developed as a power series in $\sqrt{\mu}$. If it is identically zero in μ, each coefficient of this series must vanish. The coefficient

of $\sqrt{\mu}$ was computed and found to be different from zero. For the special values of μ under consideration here it may be possible to make this determinant vanish, but because of its complicated character this possibility has not been considered. The question of the existence of these orbits is thus left open, but it seems improbable that the necessary conditions can be satisfied.

161. The Fifth Generating Solution.—The values of μ in this case are such that the period of the generating solution is $2m\pi = 2m_1\pi/\sigma_1$. The last equation of (14) is suppressed. The remaining equations have the following form:

$$\left.\begin{aligned}
&(a_1+\alpha_1)\delta\left[+2m_1\pi\sqrt{-1}+\cdots\right]+\epsilon\left[a_{11}(a_1+\alpha_1)\alpha_3\right.\\
&\quad+a_{12}(a_1+\alpha_1)\alpha_4+a_{13}(a_2+\alpha_2)\alpha_3+a_{14}(a_2+\alpha_2)\alpha_4+a_{15}\alpha_3^2+\left.a_{16}\alpha_4^2\right]\\
&\quad+\epsilon^2\left[a_{21}(a_1+\alpha_1)^3+a_{22}(a_1+\alpha_1)^2(a_2+\alpha_2)+a_{23}(a_1+\alpha_1)(a_2+\alpha_2)^2\right.\\
&\quad+a_{24}(a_2+\alpha_2)^3+a_{25}(a_1+\alpha_1)(c+\gamma)^2+a_{26}(a_2+\alpha_2)(c+\gamma)^2+\left.\cdots\right]+\cdots=0,\\
&(a_2+\alpha_2)\delta\left[-2m_1\pi\sqrt{-1}+\cdots\right]+\epsilon\left[b_{11}(a_1+\alpha_1)\alpha_3\right.\\
&\quad+b_{12}(a_1+\alpha_1)\alpha_4+b_{13}(a_2+\alpha_2)\alpha_3+b_{14}(a_2+\alpha_2)\alpha_4+b_{15}\alpha_3^2+\left.b_{16}\alpha_4^2\right]\\
&\quad+\epsilon^2\left[b_{21}(a_1+\alpha_1)^3+b_{22}(a_1+\alpha_1)^2(a_2+\alpha_2)+b_{23}(a_1+\alpha_1)(a_2+\alpha_2)^2\right.\\
&\quad+b_{24}(a_2+\alpha_2)^3+b_{25}(a_1+\alpha_1)(c+\gamma)^2+b_{26}(a_2+\alpha_2)(c+\gamma)^2+\left.\cdots\right]+\cdots=0,\\
&\alpha_3\left[(e^{2m\sigma_2\pi\sqrt{-1}}-1)+\cdots\right]+\epsilon\left[c_{11}(a_1+\alpha_1)^2+c_{12}(a_1+\alpha_1)(a_2+\alpha_2)\right.\\
&\qquad\qquad+c_{22}(a_2+\alpha_2)^2+c_{23}(c+\gamma)^2+\alpha_3(\cdots)+\left.\alpha_4(\cdots)\right]+\cdots=0,\\
&\alpha_4\left[(e^{-2m\sigma_2\pi\sqrt{-1}}-1)+\cdots\right]+\epsilon\left[d_{11}(a_1+\alpha_1)^2+d_{12}(a_1+\alpha_1)(a_2+\alpha_2)\right.\\
&\qquad\qquad+d_{22}(a_2+\alpha_2)^2+d_{23}(c+\gamma)^2+\alpha_3(\cdots)+\left.\alpha_4(\cdots)\right]+\cdots=0,\\
&\delta\left[2m\pi+\cdots\right]+\epsilon\left[e_1\alpha_3+e_2\alpha_4+\delta(e_3+\cdots)\right]+\epsilon\left[e_{21}(a_1+\alpha_1)^2\right.\\
&\qquad+e_{22}(a_2+\alpha_2)^2+e_{23}(a_1+\alpha_1)(a_2+\alpha_2)+e_{24}(c+\gamma)^2\\
&\qquad\qquad+\alpha_3(\cdots)+\alpha_4(\cdots)+\left.\delta(\cdots)\right]+\cdots=0,
\end{aligned}\right\}\quad(20)$$

where the $a_{ij}, \ldots, e_{ij}$ are functions of μ which are readily determined. The last three equations are solved for α_3, α_4, and δ, and the results thus obtained are substituted in the first and second equations. After dividing by ϵ^2, these equations have the form

$$\left.\begin{aligned}
&A(a_1+\alpha_1)^2(a_2+\alpha_2)+B(a_1+\alpha_1)(c+\gamma)^2+\epsilon(\cdots)=0,\\
&C(a_1+\alpha_1)\,(a_2+\alpha_2)^2+D(a_2+\alpha_2)(c+\gamma)^2+\epsilon(\cdots)=0.
\end{aligned}\right\}\quad(21)$$

In order that solutions of (21) of the required form shall exist, it is necessary that

$$a_1[Aa_1a_2+Bc^2]=0, \qquad a_2[Ca_1a_2+Dc^2]=0. \tag{22}$$

These equations are satisfied by $a_1=a_2=0$. With a_1 and a_2 having this value, equations (21) then have a unique solution for α_1 and α_2. But the generating solution has reduced to that considered in §159. The orbit obtained is, therefore, the continuation of the third generating solution re-entering after m revolutions, and moreover the value of μ is such that $m\sigma_1=m_1$, where m_1 is an integer. Thus we have a proof of the existence of an orbit in one of the exceptional cases omitted in discussing the third solution.

In order that equations (22) may have a solution for which a_1, a_2, and c are different from zero, it is necessary that the determinant of the A, B, C, and D shall vanish. This determinant can be developed as a power series in $\sqrt{\mu}$. If it is identically zero, each coefficient in this development separately must vanish. The coefficient of $\sqrt{\mu}$ was computed and found to be different from zero. For special values of μ it may be possible to satisfy (22) by values of a_1, a_2, and c which are distinct from zero. Because of the complicated character of the coefficients, this possibility has not been established. As in the preceding case, the existence of orbits of this type seems improbable, but complete proof is lacking.

The discussion for the sixth generating solution differs only in notation from that just given. No new orbits are found, but a proof is obtained of the existence of the continuation of the third generating solution re-entering after m revolutions, when μ is such that $m\sigma_2=m_2$. This is another exceptional case not treated in the discussion of the third generating solution.

162. The Seventh Generating Solution.—The values of μ for this case are such that the period is $2m\pi=2m_1\pi/\sigma_1=2m_2\pi/\sigma_2$. As in the previous case, the last equation of (14) is suppressed. The remaining periodicity equations have the following form:

$$\left.\begin{aligned}
&(a_1+\alpha_1)\,\delta[+2m_1\pi\sqrt{-1}+\cdots]+2m\pi\,[\varphi_{11}(a_1+\alpha_1)^2(a_2+\alpha_2)\\
&\qquad+\varphi_{12}(a_1+\alpha_1)(a_3+\alpha_3)(a_4+\alpha_4)+\varphi_{13}(a_1+\alpha_1)(c+\gamma)^2]\epsilon^2+\cdots=0,\\
&(a_2+\alpha_2)\,\delta[-2m_1\pi\sqrt{-1}+\cdots]+2m\pi\,[\varphi_{21}(a_1+\alpha_1)(a_2+\alpha_2)^2\\
&\qquad+\varphi_{22}(a_2+\alpha_2)(a_3+\alpha_3)(a_4+\alpha_4)+\varphi_{23}(a_2+\alpha_2)(c+\gamma)^2]\epsilon^2+\cdots=0,\\
&(a_3+\alpha_3)\,\delta[+2m_2\pi\sqrt{-1}+\cdots]+2m\pi\,[\varphi_{31}(a_1+\alpha_1)(a_2+\alpha_2)(a_3+\alpha_3)\\
&\qquad+\varphi_{32}(a_3+\alpha_3)^2(a_4+\alpha_4)+\varphi_{33}(a_3+\alpha_3)(c+\gamma)^2]\epsilon^2+\cdots=0,\\
&(a_4+\alpha_4)\,\delta[-2m_2\pi\sqrt{-1}+\cdots]+2m\pi\,[\varphi_{41}(a_1+\alpha_1)(a_2+\alpha_2)(a_4+\alpha_4)\\
&\qquad+\varphi_{42}(a_3+\alpha_3)(a_4+\alpha_4)^2+\varphi_{43}(a_4+\alpha_4)(c+\gamma)^2]\epsilon^2+\cdots=0,\\
&2m\pi\delta+m\pi\,[\varphi_{51}(a_1+\alpha_1)(a_2+\alpha_2)+\varphi_{52}(a_3+\alpha_3)(a_4+\alpha_4)\\
&\qquad+\varphi_{53}(c+\gamma)^2]\epsilon^2+\cdots=0,
\end{aligned}\right\} \tag{23}$$

where the φ_{ij} are functions of μ.

The last equation of (23) is solved for δ, and the result obtained substituted in the first four. After dividing by ϵ^2, the equations have the form

$$\left.\begin{aligned}
&(a_1+\alpha_1)\left[\psi_{11}(a_1+\alpha_1)(a_2+\alpha_2)+\psi_{12}(a_3+\alpha_3)(a_4+\alpha_4)+\psi_{13}(c+\gamma)^2\right]+\epsilon\left[\cdot\cdot\cdot\right]=0,\\
&(a_2+\alpha_2)\left[\psi_{21}(a_1+\alpha_1)(a_2+\alpha_2)+\psi_{22}(a_3+\alpha_3)(a_4+\alpha_4)+\psi_{23}(c+\gamma)^2\right]+\epsilon\left[\cdot\cdot\cdot\right]=0,\\
&(a_3+\alpha_3)\left[\psi_{31}(a_1+\alpha_1)(a_2+\alpha_2)+\psi_{32}(a_3+\alpha_3)(a_4+\alpha_4)+\psi_{33}(c+\gamma)^2\right]+\epsilon\left[\cdot\cdot\cdot\right]=0,\\
&(a_4+\alpha_4)\left[\psi_{41}(a_1+\alpha_1)(a_2+\alpha_2)+\psi_{42}(a_3+\alpha_3)(a_4+\alpha_4)+\psi_{43}(c+\gamma)^2\right]+\epsilon\left[\cdot\cdot\cdot\right]=0,
\end{aligned}\right\}\quad(24)$$

where the ψ_{ij} are functions of μ. In order that solutions of these equations of the required form shall exist, it is necessary that

$$\left.\begin{aligned}
&a_1[\psi_{11}\,a_1a_2+\psi_{12}\,a_3a_4+\psi_{13}\,c^2]=0, \qquad a_3[\psi_{31}\,a_1a_2+\psi_{32}\,a_3a_4+\psi_{33}\,c^2]=0,\\
&a_2[\psi_{21}\,a_1a_2+\psi_{22}\,a_3a_4+\psi_{23}\,c^2]=0, \qquad a_4[\psi_{41}\,a_1a_2+\psi_{42}\,a_3a_4+\psi_{43}\,c^2]=0.
\end{aligned}\right\}\quad(25)$$

These equations are satisfied by $a_1=a_2=a_3=a_4=0$. With these values equations (24) can be solved in the required form for α_1, α_2, α_3, and α_4. The orbit obtained is the continuation of the third generating solution, and re-enters after m revolutions. Moreover, μ is such that $m\sigma_1=m_1$ and $m\sigma_2=m_2$, where m, m_1, and m_2 are integers. This is the only remaining exceptional case not considered in discussing the third generating solution. It has now been shown that the continuation of the third generating solution re-entering after m revolutions exists for all values of μ.

In order to obtain the continuation of the seventh generating solution, it must be possible to satisfy (25) by values of the a_i and c which are different from zero. On eliminating these quantities, two functions of the ψ_{ij} are obtained which must vanish if the non-vanishing solutions exist. These functions can be developed as power series in $\sqrt{\mu}$. If they are identically zero each coefficient must separately vanish. The coefficient of $\sqrt{\mu}$ was computed for one of the developments and found to be different from zero. It follows, then, that equations (25) can not in general be satisfied in the required way. For special values of μ this may be possible, but on account of the complicated character of the ψ_{ij} the possibility has not been proved.

163. Construction of the Solutions in the Plane.—In constructing the orbits in the plane it has been found convenient to use the normal variables which were introduced in the discussion of the existence. The differential equations are the first four of (11), and the solutions are given by (13), when the quantity γ has been put equal to zero. It has been shown that the quantities α_2, α_3, α_4, and δ can be determined as power series in ϵ,

vanishing with ϵ, so that the corresponding solution shall be periodic, while the quantity α_1 still remains arbitrary. When these series are substituted in (13), the expressions obtained for x_1, x_2, x_3, x_4 can be rearranged as power series in ϵ which converge for sufficiently small values of ϵ. We have then

$$\left.\begin{aligned} x_1 &= x_{10}+x_{11}\,\epsilon+x_{12}\,\epsilon^2+\cdots, & x_3 &= x_{30}+x_{31}\,\epsilon+x_{32}\,\epsilon^2+\cdots, \\ x_2 &= x_{20}+x_{21}\,\epsilon+x_{22}\,\epsilon^2+\cdots, & x_4 &= x_{40}+x_{41}\,\epsilon+x_{42}\,\epsilon^2+\cdots, \\ & & \delta &= \delta_1\,\epsilon+\ \delta_2\,\epsilon^2+\cdots, \end{aligned}\right\} \quad (26)$$

where the x_{ij} are functions of τ.

It has been shown that the series (26) have the following properties:

(*a*) They satisfy the differential equations identically in ϵ.
(*b*) Each x_{ij} is periodic with the period $2\pi/\sigma_1$ for one set of orbits, and with the period $2\pi/\sigma_2$ for the other set.
(*c*) We have supposed that $x'=0$ at $\tau=0$, and therefore it follows that

$$\sigma_1(x_1-x_2)+\sigma_2(x_3-x_4)=0 \text{ at } \tau=0.$$

(*d*) The arbitrary α_1 occurs always with the arbitrary a_1 in the combination $a_1+\alpha_1$; there will be no loss of generality if α_1 is specialized. It will be supposed, then, that α_1 is taken so that at $\tau=0$

$$x_1+x_2+x_3+x_4=a_1+a_2=a.$$

The differential equations in the normal variables are

$$\left.\begin{aligned} x_1'-\sigma_1(1+\delta)\sqrt{-1}\,x_1 &= A_1(X_2\,\epsilon+\cdots)+B_1(Y_2\,\epsilon+\cdots), \\ x_2'+\sigma_1(1+\delta)\sqrt{-1}\,x_2 &= A_2(X_2\,\epsilon+\cdots)+B_2(Y_2\,\epsilon+\cdots), \\ x_3'-\sigma_2(1+\delta)\sqrt{-1}\,x_3 &= A_3(X_2\,\epsilon+\cdots)+B_3(Y_2\,\epsilon+\cdots), \\ x_4'+\sigma_2(1+\delta)\sqrt{-1}\,x_4 &= A_4(X_2\,\epsilon+\cdots)+B_4(Y_2\,\epsilon+\cdots). \end{aligned}\right\} \quad (27)$$

The series (26) are now substituted in these equations, and the coefficients of the corresponding powers of ϵ are equated. The resulting equations are solved for the x_{ij}, and the periodicity conditions are imposed.

The construction will be made first for the orbits with the period $2\pi/\sigma_1$. The terms independent of ϵ are given by the equations

$$\begin{aligned} x_{10}'-\sigma_1\sqrt{-1}\,x_{10} &= 0, & x_{30}'-\sigma_2\sqrt{-1}\,x_{30} &= 0, \\ x_{20}'+\sigma_1\sqrt{-1}\,x_{20} &= 0, & x_{40}'+\sigma_2\sqrt{-1}\,x_{40} &= 0. \end{aligned}$$

The general solution of these equations is

$$x_{10}=a_{10}\,e^{\sigma_1\sqrt{-1}\,\tau},\qquad x_{20}=a_{20}\,e^{-\sigma_1\sqrt{-1}\,\tau},\qquad x_{30}=a_{30}\,e^{\sigma_2\sqrt{-1}\,\tau},\qquad x_{40}=a_{40}\,e^{-\sigma_2\sqrt{-1}\,\tau}.$$

On applying condition (*b*), it is found that $a_{30}=a_{40}=0$, and from (*c*) and (*d*) that $a_{10}=a_{20}=a/2$. The solution satisfying the conditions is, then,

$$x_{10}=\frac{a}{2}e^{\sigma_1\sqrt{-1}\,\tau},\qquad x_{20}=\frac{a}{2}e^{-\sigma_1\sqrt{-1}\,\tau},\qquad x_{30}=0,\qquad x_{40}=0;$$

which, expressed in terms of the original variables by (10), becomes

$$x_0=a\cos\sigma_1\tau,\qquad y_0=-\frac{3\sqrt{3}\,(1-2\mu)\,a}{4\sigma_1^2+9}\cos\sigma_1\tau-\frac{8\sigma_1 a}{4\sigma_1^2+9}\sin\sigma_1\tau. \tag{28}$$

The coefficients of the first power of ϵ are given by the equations

$$\begin{aligned}
x'_{11}-\sigma_1\sqrt{-1}\,x_{11}&=+\sigma_1\sqrt{-1}\,\delta_1 x_{10}+A_1\,X_2^{(0)}+B_1\,Y_2^{(0)},\\
x'_{21}+\sigma_1\sqrt{-1}\,x_{21}&=-\sigma_1\sqrt{-1}\,\delta_1 x_{20}+A_2\,X_2^{(0)}+B_2\,Y_2^{(0)},\\
x'_{31}-\sigma_2\sqrt{-1}\,x_{31}&=+\sigma_2\sqrt{-1}\,\delta_1 x_{30}+A_3\,X_2^{(0)}+B_3\,Y_2^{(0)},\\
x'_{41}+\sigma_2\sqrt{-1}\,x_{41}&=-\sigma_2\sqrt{-1}\,\delta_1 x_{40}+A_4\,X_2^{(0)}+B_4\,Y_2^{(0)},
\end{aligned}$$

where $X_2^{(0)}$ and $Y_2^{(0)}$ represent the expressions obtained by substituting x_0 and y_0 for x and y in X_2 and Y_2. In order that the solution of the first equation shall be periodic, the coefficient of $e^{\sigma_1\sqrt{-1}\,\tau}$ in the right member must vanish. Otherwise non-periodic terms of the type $\tau e^{\sigma_1\sqrt{-1}\,\tau}$ will be introduced. For the same reason the coefficients of $e^{-\sigma_1\sqrt{-1}\,\tau}$, $e^{\sigma_2\sqrt{-1}\,\tau}$, and $e^{-\sigma_2\sqrt{-1}\,\tau}$ in the right members of the second, third, and fourth equations respectively must vanish. All the terms of this type come from the first terms of the right members, since the other terms are of the second degree in x_{10}, x_{20}, x_{30}, and x_{40}. Since $x_{30}=x_{40}=0$, these conditions are satisfied in the third and fourth equations. Since we have at our disposal the undetermined quantity δ_1, the desired result is obtained in the first and second equations by putting $\delta_1=0$.

The equations are now integrated and conditions (*b*), (*c*), and (*d*) are imposed. The details of this work will not be given. The results expressed in the original variables are

$$\left.\begin{aligned}
x_1&=a_{10}+a_{11}\cos\sigma_1\tau+a'_{11}\sin\sigma_1\tau+a_{12}\cos 2\sigma_1\tau+a'_{12}\sin 2\sigma_1\tau,\\
y_1&=b_{10}+b_{11}\cos\sigma_1\tau+b'_{11}\sin\sigma_1\tau+b_{12}\cos 2\sigma_1\tau+b'_{12}\sin 2\sigma_1\tau,\\
\delta_1&=0,
\end{aligned}\right\} \tag{29}$$

where

$$a_{10}=-\frac{9A_{10}+3\sqrt{3}(1-2\mu)B_{10}}{27\mu(1-\mu)},\quad a_{11}=-(a_{10}+a_{12}),\quad a'_{11}=-2a'_{12},$$

$$a_{12}=-\frac{(16\sigma_1^2+9)A_{12}+3\sqrt{3}(1-2\mu)B_{12}+16\sigma_1 B'_{12}}{12\sigma_1^2(5\sigma_1^2-1)},$$

$$a'_{12}=-\frac{(16\sigma_1^2+9)A'_{12}-16\sigma_1 B_{12}+3\sqrt{3}(1-2\mu)B'_{12}}{12\sigma_1^2(5\sigma_1^2-1)},$$

$$b_{10}=+\frac{3\sqrt{3}(1-2\mu)A_{10}-3B_{10}}{27\mu(1-\mu)},\quad b_{11}=-\frac{3\sqrt{3}(1-2\mu)a_{11}+8\sigma_1 a'_{11}}{4\sigma_1^2+9},$$

$$b'_{11}=-\frac{8\sigma_1 a_{11}-3\sqrt{3}(1-2\mu)a'_{11}}{4\sigma_1^2+9},$$

$$b_{12}=+\frac{3\sqrt{3}(1-2\mu)A_{12}-16\sigma_1 A'_{12}-(16\sigma_1^2+3)B_{12}}{12\sigma_1^2(5\sigma_1^2-1)},$$

$$b'_{12}=+\frac{16\sigma_1 A_{12}+3\sqrt{3}(1-2\mu)A'_{12}-(16\sigma_1^2+3)B'_{12}}{12\sigma_1^2(5\sigma_1^2-1)},$$

$$A_{10}=+\frac{3a^2}{32}\left[7(1-2\mu)-2\sqrt{3}(b_1+b_2)-44(1-2\mu)b_1b_2\right],$$

$$A_{12}=+\frac{3a^2}{32}\left[7(1-2\mu)-2\sqrt{3}(b_1+b_2)-22(1-2\mu)(b_1^2+b_2^2)\right],$$

$$A'_{12}=+\frac{3\sigma_1 a^2}{4\sigma_1^2+9}\left[\sqrt{3}+11(1-2\mu)(b_1+b_2)\right],$$

$$B_{10}=-\frac{3a^2}{32}\left[\sqrt{3}+66(1-2\mu)(b_1+b_2)+12\sqrt{3}\,b_1b_2\right],$$

$$B_{12}=-\frac{3a^2}{32}\left[\sqrt{3}+66(1-2\mu)(b_1+b_2)+6\sqrt{3}(b_1^2+b_2^2)\right],$$

$$B'_{12}=+\frac{3\sigma_1 a^2}{4\sigma_1^2+9}\left[11(1-2\mu)+3\sqrt{3}(b_1+b_2)\right].$$

It will now be shown that this method of obtaining the coefficients of (26) is general. Suppose x_{ij} and δ_j ($i=1,\ldots,4$; $j=0,\ldots,n-1$) have been determined and that the x_{ij} are periodic. For the determination of x_{in} and δ_n we have equations of the following form:

$$\left.\begin{aligned}
x'_{1n}-\sigma_1\sqrt{-1}\,x_{1n}&=+\sigma_1\sqrt{-1}\,\delta_n x_{10}+\sum_{j=0}^{n+1}\left[\theta_{1j}e^{j\sigma_1\sqrt{-1}\tau}+\eta_{1j}e^{-j\sigma_1\sqrt{-1}\tau}\right],\\
x'_{2n}+\sigma_1\sqrt{-1}\,x_{2n}&=-\sigma_1\sqrt{-1}\,\delta_n x_{20}+\sum_{j=0}^{n+1}\left[\theta_{2j}e^{j\sigma_1\sqrt{-1}\tau}+\eta_{2j}e^{-j\sigma_1\sqrt{-1}\tau}\right],\\
x'_{3n}-\sigma_2\sqrt{-1}\,x_{3n}&=+\sigma_2\sqrt{-1}\,\delta_n x_{30}+\sum_{j=0}^{n+1}\left[\theta_{3j}e^{j\sigma_1\sqrt{-1}\tau}+\eta_{3j}e^{-j\sigma_1\sqrt{-1}\tau}\right],\\
x'_{4n}+\sigma_2\sqrt{-1}\,x_{4n}&=-\sigma_2\sqrt{-1}\,\delta_n x_{40}+\sum_{j=0}^{n+1}\left[\theta_{4j}e^{j\sigma_1\sqrt{-1}\tau}+\eta_{4j}e^{-j\sigma_1\sqrt{-1}\tau}\right],
\end{aligned}\right\}\quad(30)$$

where the θ_{ij} and η_{ij} are known constants. Since $x_{30}=x_{40}=0$, no non-periodic terms can enter the solutions of the last two equations. In order that the solutions of the first two equations shall be periodic, it is necessary that the coefficients of $e^{\sigma_1\sqrt{-1}\tau}$ and $e^{-\sigma_1\sqrt{-1}\tau}$ in the first and second respectively

shall vanish. This gives for the determination of δ_n, the only undetermined constant, two equations $\sigma_1\sqrt{-1}\,a\,\delta_n+2\theta_{11}=0$, $\sigma_1\sqrt{-1}\,a\,\delta_n-2\eta_{21}=0$. Since the existence proof has shown that δ is uniquely determined, it follows that these equations must give the same determination of δ_n.

An additional proof is obtained by means of the integral (5′). The terms of this integral which are independent of ϵ are first expressed in the normal variables. Then the variables are replaced by their expressions as power series in ϵ and the terms are rearranged so that the integral remains a power series in ϵ. Since the integral is an identity in τ and ϵ, it follows that the coefficient of each power of ϵ must reduce to a constant identically in τ. We will consider the coefficient of ϵ^n. When the expressions for the x_{ij} as functions of τ are substituted, this coefficient consists of a sum of linearly independent functions of τ. The coefficients of each of these functions must then vanish.

Let φ_1 and φ_2 denote the coefficients of $e^{\sigma_1\sqrt{-1}\tau}$ and $e^{-\sigma_1\sqrt{-1}\tau}$ in the first and second equations of (30) respectively. Then, on integrating these equations, the terms in x_{1n} and x_{2n} carrying φ_1 and φ_2 are found to be

$$x_{1n}=\varphi_1\,\tau\,e^{\sigma_1\sqrt{-1}\tau}+\cdots, \qquad x_{2n}=\varphi_2\,\tau\,e^{-\sigma_1\sqrt{-1}\tau}+\cdots.$$

The terms in the coefficient of ϵ^n in the integral which carry x_{1n} and x_{2n} are

$$\left.\begin{aligned}&-8\sigma_1^2\left[(x_{10}-x_{20})(x_{1n}-x_{2n})+(b_1x_{10}-b_2x_{20})(b_1x_{1n}-b_2x_{2n})\right]\\&-6(x_{10}+x_{20})(x_{1n}+x_{2n})-18(b_1x_{10}+b_2x_{20})(b_1x_{1n}+b_2x_{2n})\\&-6\sqrt{3}(1-2\mu)\left[(x_{10}+x_{20})(b_1x_{1n}+b_2x_{2n})+(x_{1n}+x_{2n})(b_1x_{10}+b_2x_{20})\right].\end{aligned}\right\}\tag{31}$$

When the expressions for x_{10}, x_{20}, x_{1n}, and x_{2n} are substituted, terms carrying τ, $\tau\,e^{2\sigma_1\sqrt{-1}\tau}$ and $\tau\,e^{-2\sigma_1\sqrt{-1}\tau}$ are obtained. All other terms entering this coefficient contain only x_{ij} $(i=1,\ldots,4;\ j=1,\ldots,n-1)$, and are consequently periodic. Hence the total coefficients of the above non-periodic terms are obtained from (31). Since the coefficients must vanish, relations are obtained which φ_1 and φ_2 must satisfy. The coefficients of $\tau\,e^{2\sigma_1\sqrt{-1}\tau}$ and $\tau\,e^{-2\sigma_1\sqrt{-1}\tau}$ vanish identically. The coefficient of τ gives the relation $32\sigma_1^2(4\sigma_1^2-1)(\varphi_1+\varphi_2)/(4\sigma_1^2+9)=0$. Since $\sigma_1^2>\frac{1}{2}$, the coefficient of $\varphi_1+\varphi_2$ does not vanish, and we have $\varphi_1+\varphi_2=0$. Both φ_1 and φ_2 carry δ_n linearly. Hence if δ_n is determined so that either of them vanishes, it follows that the other must vanish also. The determination of δ_n is therefore unique.

Equations (30) are now integrated. By means of conditions (*b*), (*c*), and (*d*), the new arbitrary constants are uniquely determined in terms of the original arbitrary a. The results when expressed in the original variables have the form

$$\left.\begin{aligned}&x_n=\sum_{j=0}^{n+1}[a_{nj}\cos j\tau+a'_{nj}\sin j\tau], \qquad y_n=\sum_{j=0}^{n+1}[b_{nj}\cos j\tau+b'_{nj}\sin j\tau],\\&\delta_n=-\frac{2\theta_{11}}{\sigma_1\sqrt{-1}\,a}=\frac{2\eta_{21}}{\sigma_1\sqrt{-1}\,a}.\end{aligned}\right\}\tag{32}$$

From the character of the differential equations it is readily shown that x_n and y_n carry the factor a^{n+1}, that δ_n carries the factor a^n, and that a enters in no other way. Recalling the transformation by which ϵ was introduced, we have the final series

$$x=x_0\epsilon+x_1\epsilon^2+\cdots, \quad y=y_0\epsilon+y_1\epsilon^2+\cdots, \quad \delta=\delta_1\epsilon+\delta_2\epsilon^2+\cdots, \tag{33}$$

the x_j, y_j, and δ_j being given by (32). From the way in which a enters the series, it is seen that the arbitraries a and ϵ occur always in the combination $a\epsilon$. Therefore we can put $a=1$ without loss of generality, and the final series contain the single arbitrary ϵ.

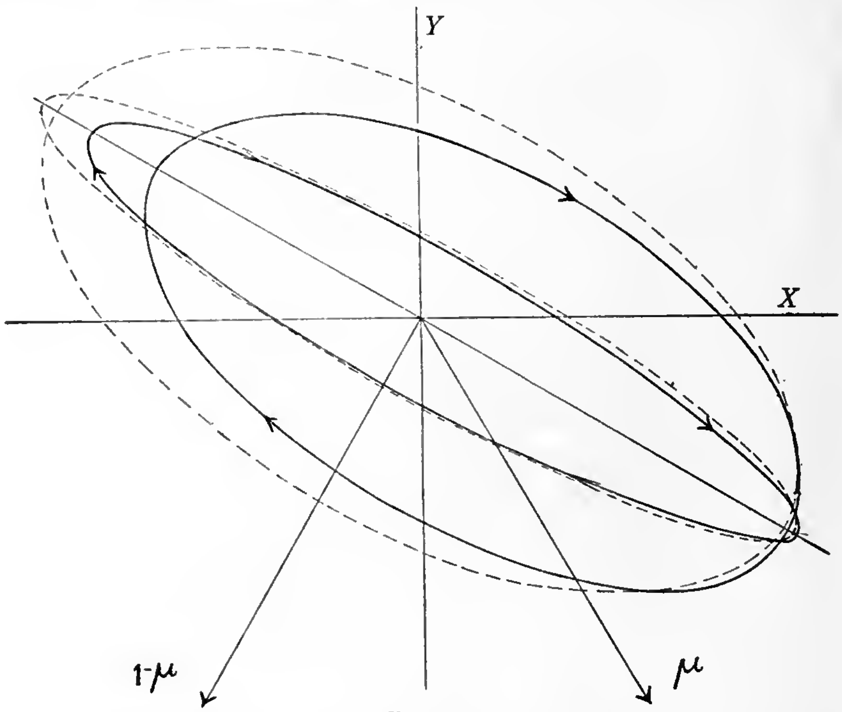

FIG. 4.

The construction for the other orbits in the plane differs only in notation from that just given. The corresponding expressions for the x_j, y_j, and δ_j can be obtained simply by replacing σ_1 by σ_2.

For very small values of ϵ the shape of the orbits deviates but slightly from that of the generating ellipses. For $\mu=0.01$ and $\epsilon=0.001$ two terms of (33) were computed for each set of orbits.

The curves found are given in Fig. 4, the dotted ellipses representing the generating solutions. The major semi-axis of the first generating solution is 0.00111, while that of the second is 0.00115. The finite bodies are at the distance unity in the directions indicated. The motion, as indicated by the arrows, is in the clockwise direction (the finite bodies revolve in the opposite direction), the starting-point being the point in the fourth quadrant where the tangent is parallel to the y-axis. The periods are $2\pi(1+\delta)/\sigma_1$ and $2\pi(1+\delta)/\sigma_2$ for the first and second solutions respectively.

164. Construction of the Solution with Period 2π.—The discussion of the existence has shown that the quantities α_1, α_2, α_3, α_4, and δ can be determined as power series in ϵ, vanishing with ϵ, so that x_1, x_2, x_3, x_4, and z will be periodic with the period 2π. When the power series obtained in this way are substituted in (13), we have, after re-arrangement, x_1, x_2, x_3, x_4, and z expressed as power series in ϵ, which converge for ϵ sufficiently small. By the use of equations (10), x and y can be expressed in the same way. Therefore the solution has the form

$$\left.\begin{aligned} x&=x_0+x_1\epsilon+x_2\epsilon^2+\cdots, & z&=z_0+z_1\epsilon+z_2\epsilon^2+\cdots,\\ y&=y_0+y_1\epsilon+y_2\epsilon^2+\cdots, & \delta&=0+\delta_1\epsilon+\delta_2\epsilon^2+\cdots. \end{aligned}\right\} \quad (34)$$

The series (34) have the following properties:

$$\left.\begin{aligned} &(a) \text{ They satisfy the differential equations identically in } \epsilon.\\ &(b) \text{ Each } x_j,\ y_j, \text{ and } z_j \text{ is periodic with the period } 2\pi.\\ &(c)\ z\,(0)\equiv 0;\ \text{ therefore } z_j(0)=0 \qquad\qquad (j=0,\ 1,\ 2,\ \ldots\ \infty).\\ &(d)\ z'\,(0)\equiv c;\ \text{ therefore } z_0'(0)=c,\quad z_j'(0)=0 \qquad (j=1,\ 2,\ \ldots\ \infty). \end{aligned}\right\} \quad (35)$$

The last property follows from the fact that the arbitrary γ occurs always with the arbitrary c in the form $c+\gamma$, and can be put equal to zero without loss of generality.

The differential equations are

$$\left.\begin{aligned} x''-2(1+\delta)y'-(1+\delta)^2\left[\tfrac{3}{4}x+\tfrac{3}{4}\sqrt{3}\,(1-2\mu)y\right]&=(1+\delta)^2\left[X_2\epsilon+X_3\epsilon^2+\cdots\right],\\ y''+2(1+\delta)x'-(2+\delta)^2\left[\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,x+\tfrac{9}{4}y\right]&=(1+\delta)^2\left[Y_2\epsilon+Y_3\epsilon^2+\cdots\right],\\ z''+(1+\delta)^2z&=(1+\delta)^2\left[Z_2\,\epsilon+Z_3\epsilon^2+\cdots\right]. \end{aligned}\right\} \quad (36)$$

The series (34) are to be substituted in these equations and the coefficients of the powers of ϵ equated. The x_j, y_j, and z_j are determined by solving the equations thus obtained and imposing on the results the conditions (35).

The terms independent of ϵ are given by the equations

$$\left.\begin{aligned} x_0''-2y_0'-\tfrac{3}{4}x_0-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,y_0&=0,\\ y_0''+2x_0'-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,x_0-\tfrac{9}{4}y_0&=0,\\ z_0''+z_0&=0. \end{aligned}\right\} \quad (37)$$

The general solution of these equations is

$$\left.\begin{aligned} x_0&=a_{01}\,e^{\sigma_1\sqrt{-1}\tau}+a_{01}'\,e^{-\sigma_1\sqrt{-1}\tau}+a_{02}\,e^{\sigma_2\sqrt{-1}\tau}+a_{02}'\,e^{-\sigma_2\sqrt{-1}\tau},\\ y_0&=b_{01}\,e^{\sigma_1\sqrt{-1}\tau}+b_{01}'\,e^{-\sigma_1\sqrt{-1}\tau}+b_{02}\,e^{\sigma_2\sqrt{-1}\tau}+b_{02}'\,e^{-\sigma_2\sqrt{-1}\tau},\\ z_0&=c_{01}\cos\tau+c_{01}'\sin\tau. \end{aligned}\right\} \quad (38)$$

By condition (*b*), we must have

$$a_{01}=a'_{01}=a_{02}=a'_{02}=b_{01}=b'_{01}=b_{02}=b'_{02}=0,$$

and conditions (*c*) and (*d*) give respectively

$$c_{01}=0, \qquad c'_{01}=c.$$

The solution satisfying the given conditions is, then,

$$x_0=0, \qquad y_0=0, \qquad z_0=c\sin\tau. \tag{39}$$

The coefficients of the first power of ϵ are given by the equations

$$\left.\begin{aligned} x_1''-2y_1'-\tfrac{3}{4}x_1-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,y_1&=\frac{3c^2}{8}\,(1-2\mu)(1-\cos2\tau),\\ y_1''+2x_1'-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,x_1-\tfrac{9}{4}y_1&=\frac{3\sqrt{3}\,c^2}{8}\,(1-\cos2\tau),\\ z_1''+z_1&=-2\delta_1c\sin\tau.\end{aligned}\right\} \tag{40}$$

In order that the last equation of this set shall have a periodic solution, it is necessary that the coefficient of $\sin\tau$ shall vanish. Hence we impose the condition $\delta_1=0$. Upon solving these equations and imposing the conditions (35), we find

$$\left.\begin{aligned} x_1&=\frac{8c^2(1-2\mu)}{73-9(1-2\mu)^2}\cos2\tau+\frac{8\sqrt{3}\,c^2}{73-9(1-2\mu)^2}\sin2\tau,\\ y_1&=-\frac{\sqrt{3}}{6}c^2+\frac{\sqrt{3}\,c^2[19-3(1-2\mu)^2]}{2[73-9(1-2\mu)^2]}\cos2\tau-\frac{8c^2(1-2\mu)}{73-9(1-2\mu)^2}\sin2\tau,\\ z_1&=0, \qquad \delta_1=0.\end{aligned}\right\} \tag{41}$$

From the coefficients of ϵ^2 we get

$$\left.\begin{aligned} &x_2''-2y_2'-\tfrac{3}{4}x_2-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,y_2=0,\\ &y_2''+2x_2'-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,x_2-\tfrac{9}{4}y_2=0,\\ &z_2''+z_2=-\left[2c\delta_2-\frac{24\mu(1-\mu)c^3}{73-9(1-2\mu)^2}\right]\sin\tau-\frac{24\mu(1-\mu)c^3}{73-9(1-2\mu)^2}\sin3\tau.\end{aligned}\right\} \tag{42}$$

In order that z_2 shall be periodic, the coefficient of $\sin\tau$ in the right member of the last equation must vanish. The relation obtained determines δ_2 uniquely. The solution of these equations satisfying the given conditions is

$$\left.\begin{aligned} &x_2=0, \qquad y_2=0,\\ &z_2=-\frac{9\mu(1-\mu)c^3}{73-9(1-2\mu)^2}\sin\tau+\frac{3\mu(1-\mu)c^2}{73-9(1-2\mu)^2}\sin3\tau,\\ &\delta_2=\frac{12\mu(1-\mu)c^2}{73-9(1-2\mu)^2}.\end{aligned}\right\} \tag{43}$$

For the general terms we proceed by induction. Suppose that x_j, y_j, z_j, and $\delta_j (j=0, 1, \ldots, n-1)$ have been determined, and that for j even it has been found that

$$x_j=0, \qquad y_j=0, \qquad z_j=\sum_{k=1}^{j/2}\left[c_k \cos(2k+1)\tau+c_k' \sin(2k+1)\tau\right];$$

while for j odd, it has been found that

$$x_j=\sum_{k=1}^{(j+1)/2}\left[a_k \cos 2k\tau+a_k' \sin 2k\tau\right],$$

$$y_j=\sum_{k=1}^{(j+1)/2}\left[b_k \cos 2k\tau+b_k' \sin 2k\tau\right],$$

$$z_j=0, \qquad \delta_j=0.$$

It can be readily shown that when n is even the coefficients of ϵ^n are given by the equations

$$\left.\begin{aligned}
&x_n''-2y_n'-\tfrac{3}{4}x_n-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,y_n=0,\\
&y_n''+2x_n'-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,x_n-\tfrac{9}{4}y_n=0,\\
&z_n''+z_n=-2c\delta_n\sin\tau+\sum_{j=0}^{n/2}\left[C_{2j+1}^{(n)}\cos(2j+1)\tau+C_{2j+1}'^{(n)}\sin(2j+1)\tau\right].
\end{aligned}\right\} \quad (44)$$

In order that the last equation shall have a periodic solution we must impose the conditions

$$-2c\delta_n+C_1'^{(n)}=0, \qquad C_1^{(n)}=0.$$

The first relation serves for the determination of δ_n. Since by the existence proof the periodic solution is known to exist it follows that the expression $C_1^{(n)}$ is zero.

An additional proof that $C_1^{(n)}$ is zero is obtained by considering the integral (5′). The series for x, y, z are substituted and the terms are re-arranged as a power series in ϵ. Each coefficient of this series must reduce to a constant identically in τ. Consider the coefficient of ϵ^n. The terms of this coefficient which carry z_n are

$$2\,z_0' z_n'+2\,z_0 z_n. \qquad (45)$$

Suppose $x_0, \ldots, x_n$; $y_0, \ldots, y_n$; $z_0, \ldots, z_{n-1}$; $\delta_1, \ldots, \delta_{n-1}$ have been determined and that the x_j, y_j, and z_j are periodic. The equation for the determination of z_n has the form

$$z_n''+z_n=\eta\sin\tau+C_1^{(n)}\cos\tau+\cdots$$

On integrating, the following non-periodic terms are obtained

$$z_n=-\frac{1}{2}\eta\tau\cos\tau-\frac{1}{2}C_1^{(n)}\,\tau\sin\tau.$$

When the expressions for x_j, y_j, and z_j as functions of τ are substituted in the coefficients of ϵ^n in the integral, the only non-periodic terms obtained come from z_n. They are of the form τ, $\tau\sin 2\tau$, and $\tau\cos 2\tau$. Since the coefficient of ϵ^n is a constant identically in τ, it follows that the coefficients of these non-periodic terms are zero. Those for $\tau\sin 2\tau$ and $\tau\cos 2\tau$ vanish identically. The coefficient of τ gives the relation $c\,C_1^{(n)}=0$. Since $c\neq 0$, it follows that $C_1^{(n)}=0$.

Upon integrating (44) and imposing conditions (35), we find

$$\left.\begin{aligned}
&x_n=0, \qquad y_n=0, \qquad \delta_n=\frac{C_1^{\prime(n)}}{2c},\\
&z_n=c_{n1}\cos\tau+c_{n1}'\sin\tau\\
&\quad-\sum_{j=1}^{n/2}\left[\frac{C_{2j+1}^{(n)}}{4j(j+1)}\cos(2j+1)\tau+\frac{C_{2j+1}^{\prime(n)}}{4j(j+1)}\sin(2j+1)\tau\right].
\end{aligned}\right\} \tag{46}$$

The quantities $C_{2j+1}^{(n)}$ and $C_{2j+1}^{\prime(n)}$ are known from the differential equations. The constants of integration c_{n1} and c_{n1}' are found by (*c*) and (*d*) of (35) to have the values

$$c_{n1}=\sum_{j=1}^{n/2}\frac{C_{2j+1}^{(n)}}{4j(j+1)}, \qquad c_{n1}^{\prime(n)}=\sum_{j=1}^{n/2}\frac{(2j+1)\,C_{2j+1}^{\prime(n)}}{4j(j+1)}.$$

When n is odd the equations obtained from the coefficients of ϵ^n are

$$\left.\begin{aligned}
x_n''-2y_n'-\tfrac{3}{4}x_n-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,y_n&=\sum_{j=0}^{(n+1)/2}\left[A_j^{(n)}\cos 2j\tau+A_j^{\prime(n)}\sin 2j\tau\right],\\
y_n''+2x_n'-\tfrac{3}{4}\sqrt{3}\,(1-2\mu)\,x_n-\tfrac{9}{4}y_n&=\sum_{j=0}^{(n+1)/2}\left[B_j^{(n)}\cos 2j\tau+B_j^{\prime(n)}\sin 2j\tau\right],\\
z_n''+z_n&=-2\delta_n c\sin\tau.
\end{aligned}\right\} \tag{47}$$

From the last equation it follows at once that $\delta_n=0$, for otherwise z_n will not be periodic. Integrating these equations and imposing the periodicity equations, we find

$$\left.\begin{aligned}
x_n&=\sum_{j=0}^{(n+1)/2}\left[\frac{-(16j^2+9)A_j^{\prime(n)}-16j\,B_j^{(n)}+3\sqrt{3}\,(1-2\mu)\,B_j^{\prime(n)}}{16j^2(4j^2-1)+27\mu(1-\mu)}\sin 2j\tau\right.\\
&\qquad\left.+\frac{-(16j^2+9)A_j^{(n)}+3\sqrt{3}\,(1-2\mu)\,B_j^{(n)}+16j\,B_j^{\prime(n)}}{16j^2(4j^2-1)+27\mu(1-\mu)}\cos 2j\tau\right],\\
y_n&=\sum_{j=0}^{(n+1)/2}\left[\frac{16j\,A_j^{(n)}+3\sqrt{3}\,(1-2\mu)\,A_j^{\prime(n)}-(16j^2+3)\,B_j^{\prime(n)}}{16j^2(4j^2-1)+27\mu(1-\mu)}\sin 2j\tau\right.\\
&\qquad\left.+\frac{3\sqrt{3}\,(1-2\mu)\,A_j^{(n)}-16j\,A_j^{\prime(n)}-(16j^2+3)\,B_j^{(n)}}{16j^2(4j^2-1)+27\mu(1-\mu)}\cos 2j\tau\right],\\
z_n&=0, \qquad \delta_n=0.
\end{aligned}\right\} \tag{48}$$

If we make use of the transformation by which the parameter ϵ was introduced, we have for the final series

$$\left.\begin{aligned} x&=x_0\epsilon+x_1\epsilon^2+x_2\epsilon^3+\cdots, & z&=z_0\epsilon+z_1\epsilon^2+z_2\epsilon^3+\cdots,\\ y&=y_0\epsilon+y_1\epsilon^2+y_2\epsilon^3+\cdots, & \delta&=\delta_1\epsilon+\delta_2\epsilon^2+\delta_3\epsilon^3+\cdots,\end{aligned}\right\} \quad (49)$$

the x_j, y_j, z_j, and δ_j being given by (46) and (48). It is not difficult to show that x_n, y_n, and z_n carry the factor c^{n+1}, and δ_n the factor c^n, and that c enters these expressions in no other way. Consequently in (49) the arbitraries c and ϵ occur always in the combination $c\epsilon$. Therefore we may put $c=1$ without loss of generality. The final series then contain only the arbitrary ϵ.

An approximate idea of the shape of the orbit can be obtained by considering the first two terms of (49). These terms were computed for $\mu=0.01$ and $\epsilon=0.5$. The projection on the xy-plane is an ellipse of small eccentricity whose center is on the negative y-axis and whose major axis cuts the positive x-axis. This projection is shown in Fig. 5. The projections on the xz and yz-planes are shown in Fig. 6 and Fig. 7 respectively. The orbit thus consists of two elongated loops, one above and the other below the xy-plane, the double point being in the fourth quadrant of the xy-plane.

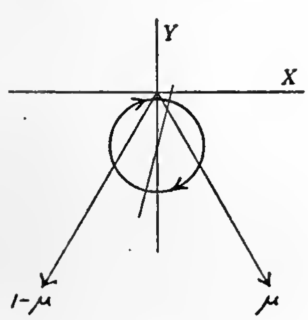

FIG. 5.

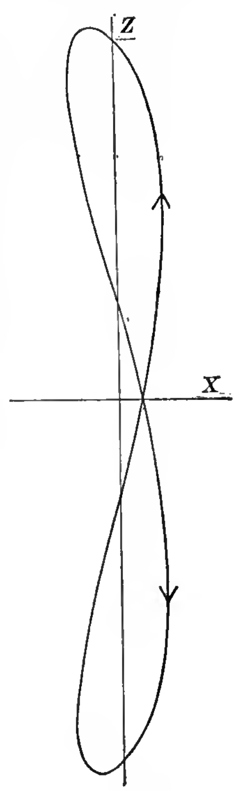

FIG. 6.

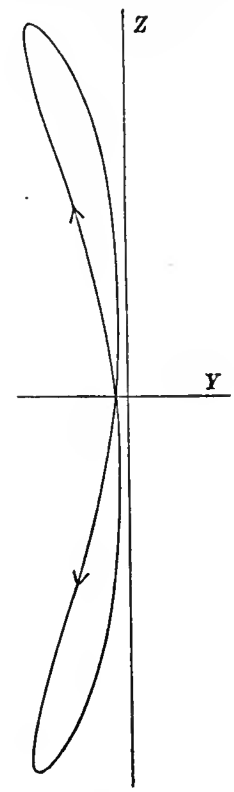

FIG. 7.

If τ is replaced by $\tau+\pi$ in the expressions for x_j, y_j, and z_j, then x_j and y_j remain unchanged while it is seen that z_j changes sign. It follows that the loops are symmetrical with respect to the xy-plane, and that each loop is described in half the period. For positive values of ϵ the upper loop is described first, and the motion is such that the projection of the infinitesimal body on the xy-plane moves in the clockwise direction. The period of the motion is given by the relation $T=2\pi(1+\delta)$.

Orbits about the Point II.—To each of the orbits about point I there corresponds an orbit about point II. The proofs of the existence of these orbits were omitted, since they are similar to those for the first point. The series for these orbits can be obtained easily from the corresponding series for the first set of orbits. The differential equations for the orbits about point II are obtained from those for the orbits about point I by changing the sign of $\sqrt{3}$. The periodicity conditions to be imposed are the same in both cases. It follows, therefore, that the solutions for the one case can be obtained from those for the other by changing the sign of $\sqrt{3}$. Therefore, in order to get the series for the orbits about the second point we make this change in the series already obtained for the first point. On referring to equations (1) and (2), it is seen that the differential equations for point I reduce to those for point II if the signs of y and τ are changed. Hence this transformation can be made geometrically by a reflection in the xz-plane and a reversal of the direction of motion. Thus, it is easy to get an idea of the shape of these orbits from those already discussed.

CHAPTER X.

ISOSCELES-TRIANGLE SOLUTIONS OF THE PROBLEM OF THREE BODIES.

BY DANIEL BUCHANAN.

165. Introduction.—This chapter treats of periodic solutions of the problem of three bodies, in which two of the masses are finite and equal. The third body is started at the initial time t_0 from the center of gravity of the equal masses, and the initial conditions are so chosen that it moves in a straight line and remains equidistant from the other bodies. In I the third body is assumed to be infinitesimal and the initial conditions are so chosen that the equal bodies move in a circle about the center of mass.* In II the third body is considered infinitesimal and the initial conditions are so chosen that the equal bodies move in ellipses with the center of mass as the common focus. In III the third body is considered finite and the solutions derived have the same period as those obtained in I, and reduce to those solutions when the third body becomes infinitesimal.

I. PERIODIC ORBITS WHEN THE FINITE BODIES MOVE IN A CIRCLE AND THE THIRD BODY IS INFINITESIMAL.

166. The Differential Equation of Motion.—Let m_1 and m_2 be two finite bodies of equal mass, and μ an infinitesimal body. Let the unit of mass be so chosen that $m_1 = m_2 = 1/2$; the linear unit so that the distance between m_1 and m_2 shall be unity; and the unit of time so that the Gaussian constant shall be unity. Let the origin of coördinates be taken at the center of mass, and the $\xi\eta$-plane as the plane of motion of the finite bodies. Let the coördinates of m_1, m_2, and μ be ξ_1, η_1, 0; ξ_2, η_2, 0; and 0, 0, ζ respectively. If m_1 and m_2 are started at the points 1/2, 0, 0 and $-1/2$, 0, 0, respectively, so that they move in a circle, then

$$\xi_1 = -\xi_2 = \frac{1}{2}\cos(t-t_0), \qquad \eta_1 = -\eta_2 = \frac{1}{2}\sin(t-t_0).$$

*When the finite bodies move in a circle, the motion of the infinitesimal body can be completely determined by means of elliptic integrals. The problem was first solved in this way by Pavanini in a memoir, "Sopra una Nuova Categoria di Soluzioni Periodiche nel Problema dei Tre Corpi," *Annali di Matematica*, Series III, vol. XIII (1907), pp. 179–202. The elliptic integrals obtained by Pavanini were later developed independently by MacMillan in an article, "An Integrable Case in the Restricted Problem of Three Bodies," *Astronomical Journal*, Nos. 625–626 (1911). MacMillan further developed the solution as a power series in a parameter, the coefficients of which are periodic functions of t. The solution obtained in I has a close relation to MacMillan's solution.

The differential equation for the motion of the infinitesimal body is

$$\zeta'' = -\frac{8\zeta}{1+4\zeta^2)^{3/2}}, \tag{1}$$

where the accents denote derivatives with respect to t. The integral of (1) is

$$(\zeta')^2 = \frac{4}{(1+4\zeta^2)^{1/2}} + C, \tag{2}$$

where C is the constant of integration. If C is positive, the particle μ recedes to infinity. If C is negative, the particle μ does not pass beyond a finite distance from the origin. From a consideration of (2) it can be shown that, if C is negative, the particle crosses the $\xi\eta$-plane. Hence the initial time t_0 can be chosen, without loss of generality, as the time when the particle is in the $\xi\eta$-plane. It can also be shown from (2) that if C is negative, the motion of the particle is periodic, and that the period can be expressed as a power series in the initial velocity of μ, whose limit is $2\pi/\sqrt{8}$ as the initial velocity approaches zero. We shall, however, prove the existence of a periodic solution of (1) by a different method.

167. Proof of Existence of a Periodic Solution of Equation (1).—In this proof it is convenient to make the transformation

$$t - t_0 = \sqrt{1/8}(1+\delta)\tau, \tag{3}$$

where δ is to be determined so that the solution of (1) shall be periodic in τ with the period 2π. At $\tau=0$ let

$$\zeta = 0, \qquad \dot{\zeta} = a, \tag{4}$$

where $\dot{\zeta} = d\zeta/d\tau$. Let us make the further transformation

$$\zeta = az; \tag{5}$$

then when (3) and (5) are substituted in equation (1), we obtain

$$\ddot{z} = -\frac{(1+\delta)z}{(1+4a^2z^2)^{3/2}}, \tag{6}$$

where $\ddot{z}$ is the second derivative of z with respect to τ. The initial conditions for z become, as a consequence of (4) and (5),

$$z(0) = 0, \qquad \dot{z}(0) = 1. \tag{7}$$

For $|a|$ sufficiently small the right member of (6) can be expanded into the series

$$\ddot{z} = -(1+\delta)z\left\{1 + \binom{-3/2}{1} 4a^2z^2 + \cdots + \binom{-3/2}{i}(4a^2z^2)^i + \cdots\right\}, \tag{8}$$

where

$$\binom{-3/2}{i} = \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2i+1}{2}\right)}{1 \cdot 2 \cdots i}.$$

Equation (8) can be integrated as a power series in a^2 and δ which converges for $0 \gtreqless \tau \leqq 2\pi$, provided $|a|$ and $|\delta|$ are sufficiently small. Let us write this solution in the form

$$z = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} z_{i,j}\, \delta^i a^{2j}. \tag{9}$$

The initial values of the $z_{i,j}$ as determined from (7) are

$$\left.\begin{array}{lll} z_{i,j}(0)=0, & (i, j=0, \ldots \infty), & \\ \dot{z}_{0,0}(0)=1, & \dot{z}_{i,j}(0)=0 & (i+j>0). \end{array}\right\} \tag{10}$$

Upon substituting (9) in (8) and equating the coefficients of the various powers of δ and a^2, we obtain differential equations from which the $z_{i,j}$ can be determined so that the initial conditions (10) shall be satisfied.

The differential equation for the term independent of δ and a^2 is

$$\ddot{z}_{0,0} + z_{0,0} = 0,$$

and the solution of it which satisfies (10) is

$$z_{0,0} = \sin\tau.$$

The differential equation for the term in δ alone is

$$\ddot{z}_{1,0} + z_{1,0} = -\sin\tau,$$

and the solution of it which satisfies (10) is

$$z_{1,0} = -\frac{1}{2}\sin\tau + \frac{\tau}{2}\cos\tau.$$

The solution of equation (8) is therefore

$$z = \sin\tau + \delta\left[\frac{\tau}{2}\cos\tau - \frac{1}{2}\sin\tau\right] + \text{terms of higher degree in } a^2 \text{ and } \delta. \tag{11}$$

With the initial conditions (7), the variable z is an odd function of τ, and therefore a sufficient condition that it shall be periodic with the period 2π in τ is $z(\pi)=0$. With the value of z given in (11), this condition becomes

$$0 = -\frac{\pi}{2}\delta + \text{terms of higher degree in } a^2 \text{ and } \delta. \tag{12}$$

Since the coefficient of δ is different from zero, this equation can be solved uniquely for δ as a power series in a^2, vanishing with a^2. Let us denote this solution for δ by

$$\delta = \sum_{j=1}^{\infty} \delta_{2j}\, a^{2j}. \tag{13}$$

When (13) is substituted in (9), we obtain

$$z = \sum_{j=0}^{\infty} z_{2j}\, a^{2j}, \tag{14}$$

which converges for $|a|$ sufficiently small. Since the periodicity condition has been satisfied, z is periodic in τ with the period 2π. Hence

$$z_{2j}(\tau+2\pi)\equiv z_{2j}(\tau) \qquad (j=0, \ldots \infty). \tag{15}$$

The initial values of z_{2j} as determined from (7) are

$$\left.\begin{aligned} &z_{2j}(0)=0 && && (j=0, \ldots \infty), \\ &\dot{z}_0(0)=1, && \dot{z}_{2j}(0)=0 && (j=1, \ldots \infty). \end{aligned}\right\} \tag{16}$$

168. Direct Construction of the Periodic Solution of Equation (1).—Let us substitute (13) and (14) in (8) and equate the coefficients of the various powers of a^2. Since the result is an identity in a^2, there is obtained a series of differential equations from which the coefficients of the solution (14) can be determined.

The differential equation for the term independent of a^2 is

$$\ddot{z}_0+z_0=0,$$

and the solution of it which satisfies (15) and (16) is

$$z_0=\sin\tau.$$

The differential equation for the term in a^2 is

$$\ddot{z}_2+z_2=-\delta_2 z_0+6z_0^3=\left(-\delta_2+\frac{9}{2}\right)\sin\tau-\frac{3}{2}\sin 3\tau.$$

The term $\sin\tau$ gives rise to a non-periodic term in the solution, and, in order that (15) shall be satisfied, its coefficient must be zero. Hence

$$\delta_2=\frac{9}{2},$$

and the solution for z_2 satisfying (16) is

$$z_2=\frac{3}{16}(\sin 3\tau-3\sin\tau).$$

The differential equation for the term in a^4 is

$$\ddot{z}_4+z_4=-\left(\delta_4+\frac{141}{32}\right)\sin\tau+6\sin 3\tau-\frac{87}{32}\sin 5\tau.$$

In order that (15) shall be satisfied, δ_4 must have the value

$$\delta_4=-\frac{141}{32},$$

and the solution for z_4 is found to be

$$z_4=\frac{1}{256}[431\sin\tau-192\sin 3\tau+29\sin 5\tau],$$

where the constants of integration have been determined so as to satisfy (16).

So far as computed, it has been found that the δ_{2j} are uniquely determined by the periodicity and the initial conditions, and that each z_{2j} is a sum of sines of odd multiples of τ, the highest multiple being $2j+1$. We shall now show by an induction to the general term that all the δ_{2j} are uniquely determined by the same conditions, and that all the z_{2j} have the properties which have been stated. Let us assume that $\delta_2, \ldots, \delta_{2j-2}$; $z_0, \ldots, z_{2j-2}$ have been uniquely determined, and that each $z_{2k}(k=0, \ldots, j-1)$ is a sum of sines of odd multiples of τ, the highest multiple being $2k+1$. From these assumptions and the differential equations it will be shown that δ_{2j} and z_{2j} are uniquely determined, and that z_{2j} is a sum of sines of odd multiples of τ, the highest multiple being $2j+1$.

Let us consider the term in a^{2j}. The differential equation is

$$\ddot{z}_{2j}+z_{2j}=-\delta_{2j}z_0+Z_{2j}, \tag{17}$$

where Z_{2j} is a known function of $z_{2i}(i=0, \ldots, j-1)$ and δ_{2k} $(k=1, \ldots, j-1)$. The general term in Z_{2j} has the form

$$T_{2j}=z_{\lambda_1}^{\mu_1} \cdots z_{\lambda_k}^{\mu_k}\delta_p^q,$$

where $\lambda_1, \ldots, \lambda_k$; $\mu_1, \ldots, \mu_k$; p, and q are positive integers (or zero) having the following properties:

(a) $\mu_1+\cdots+\mu_k$ is an odd integer,
(b) $\mu_1\lambda_1+\cdots+\mu_k\lambda_k+\mu_1+\cdots+\mu_k-1+qp=2j$,
(c) q is 0 or 1.

Since each $z_0, \ldots, z_{2j-2}$ is a sum of sines of odd multiples of τ, it follows from (a) that Z_{2j} is a sum of sines of odd multiples of τ. The highest multiple is

$$N_{2j}=\mu_1(\lambda_1+1)+\cdots+\mu_k(\lambda_k+1)=2j+1-qp.$$

The highest value of N_{2j} is obtained when $q=0$ and is, therefore, $2j+1$. Hence (17) has the form

$$\ddot{z}_{2j}+z_{2j}=[-\delta_{2j}+a_1^{(2j)}]\sin\tau+a_3^{(2j)}\sin 3\tau+\cdots+a_{2j+1}^{(2j)}\sin(2j+1)\tau, \tag{18}$$

where $a_1^{(2j)}, \ldots, a_{2j+1}^{(2j)}$ are known constants. From (15) it follows that

$$\delta_{2j}=a_1^{(2j)}.$$

The solution of (18) satisfying (16) is therefore

$$z_{2j}=A_1^{(2j)}\sin\tau+A_3^{(2j)}\sin 3\tau+\cdots+A_{2j+1}^{(2j)}\sin(2j+1)\tau,$$

where

$$A_1^{(2j)}=-\sum_{k=1}^{j}(2k+1)A_{2k+1}^{(2j)},$$

$$A_{2k+1}^{(2j)}=+\frac{a_{2k+1}^{(2j)}}{1-(2k+1)^2} \qquad (k=1, \ldots, j).$$

Hence the periodic solution of (1) in terms of the variable τ is

$$\left.\begin{aligned}\zeta=\psi=a\sin\tau+\tfrac{3}{16}a^3[\sin 3\tau-3\sin\tau]+\frac{a^5}{256}[431\sin\tau-192\sin 3\tau+29\sin 5\tau]\\ +\cdots+a^{2j+1}[A_1^{(2j)}\sin\tau+A_3^{(2j)}\sin 3\tau+\cdots+A_{2j+1}^{(2j)}\sin(2j+1)\tau]+\cdots\end{aligned}\right\}\quad(19)$$

It is a power series in odd powers of a with sums of sines of odd multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k+1} is $2k+1$. In the sequel we shall call such a series a *triply odd power series.* The period in τ is 2π, and in t it is

$$\frac{2\pi}{\sqrt{8}}\left[1+\tfrac{9}{2}a^2-\tfrac{141}{32}a^4+\tfrac{35}{2}a^6+\cdots\right]^{\frac{1}{2}}.$$

II. SYMMETRICAL PERIODIC ORBITS WHEN THE FINITE BODIES MOVE IN ELLIPSES AND THE THIRD BODY IS INFINITESIMAL.

169. The Differential Equation.—Let m_1 and m_2 represent the two finite bodies and μ the infinitesimal body. Let the system of coördinates be chosen as in §166. Let the unit of mass be so chosen that $m_1=m_2=1/2$, and then let the linear and time units be so determined that the mean distance from m_1 to m_2 and the gravitational constant are each unity. With these units the mean angular motion of the bodies also is unity.

Let μ be started from the center of gravity of m_1 and m_2 perpendicularly to the plane of their motion when they are at apsides of their orbits, which can be assumed to lie on the ξ-axis. From the symmetry of the motion with these initial conditions, it follows that

$$\xi_2=-\xi_1,\qquad \eta_2=-\eta_1.$$

Let the motion of the finite bodies be referred to a system of axes rotating about the ζ-axis with the uniform velocity unity. The coördinates referred to the rotating axes are defined by

$$x_i=\xi_i\cos t+\eta_i\sin t,\qquad y_i=-\xi_i\sin t+\eta_i\cos t,\qquad z=\zeta\qquad (i=1,2).$$

The x_i and y_i are determined by the conditions that m_1 and m_2 shall move in ellipses and be at apsides at $t=t_0$, which in this case is put equal to zero. Then it follows, from the properties of elliptic motion, that

$$x_1=-x_2=r\cos(v-t),\qquad y_1=-y_2=r\sin(v-t),\qquad (20)$$

where

$$r=m\left[1-e\cos t+\frac{e^2}{2}(1-\cos 2t)\cdots\right],$$

$$v=t+2e\sin t+\tfrac{5}{4}e^2\sin 2t+\cdots,$$

$$m=m_1=m_2=\tfrac{1}{2},\qquad e=\text{eccentricity of ellipses.}$$

The differential equation for the motion of μ is

$$z'' = -\frac{mz}{r_1^3} - \frac{mz}{r_2^3} = -\frac{2mz}{r_1^3}, \tag{21}$$

where

$$r_1 = r_2 = \sqrt{x_1^2 + y_1^2 + z^2} = \sqrt{x_2^2 + y_2^2 + z^2}.$$

When we substitute the values of x_i and y_i from (20), equation (21) becomes

$$z'' = \frac{-2mz}{\left[m^2\left\{1 - 2e\cos t + \frac{e^2}{2}(3 - \cos 2t) + \cdots\right\} + z^2\right]^{3/2}}. \tag{22}$$

Where m occurs in the denominator we shall substitute its value $1/2$, but in the numerator we shall make the substitution $m = m_0 + \lambda$, and consider λ as a variable parameter while m and m_0 both remain fixed. In order to obtain the solution of the physical problem we must put $\lambda = m - m_0$ in the final results. With these substitutions, (22) becomes

$$z'' = -(m_0 + \lambda) \sum_{j=0}^{\infty} E_{2j+1}\, z^{2j+1}, \tag{23}$$

where

$$E_1 = +16\left[1 + 3e\cos t + \frac{3e^2}{2}(1 + 3\cos 2t) + \cdots\right],$$

$$E_3 = -96\left[1 + 5e\cos t + \cdots\right],$$

and where each E_{2j+1} is a power series in e with cosines of integral multiples of t in the coefficients, the highest multiple being the same as the exponent of the eccentricity e.

170. Determination of the Period by a Necessary Condition for a Periodic Solution of (23).—If the motion is periodic, let the period be denoted by T. Since the period of motion of the finite bodies is 2π, we must have

$$T = 2\nu\pi, \tag{24}$$

where ν is an integer which denotes the number of revolutions made by the finite bodies in the period T.

Let us take the initial conditions

$$z(0) = 0, \qquad z'(0) = a. \tag{25}$$

With these initial conditions it can be shown from (23) that z is an odd function of t. Hence if μ is started from the $\xi\eta$-plane when m_1 and m_2 are at apsides of their orbits, a necessary and sufficient condition that z shall be periodic with the period T is

$$z(T/2) = 0. \tag{26}$$

In order to determine the period T from the condition (26), we integrate equation (23) as a power series in α and λ, but only in so far as the term of the first degree in α is concerned. The differential equation for this term is

$$z''_{1,0}+m_0E_1z_{1,0}=0. \tag{27}$$

This equation belongs to the class of differential equations with periodic coefficients which was treated in Chapter III, where it was found that the character of the solutions depends upon whether or not $4\sqrt{m_0}$ is an integer. Since m_0 depends upon the way in which m is separated into $m_0+\lambda$, and since $|\lambda|$ must be taken small in order that certain solutions appearing in the sequel shall be convergent, the value of m_0 is in the vicinity of 1/2. We may therefore regard $4\sqrt{m_0}$ as not an integer, and when it is not an integer the general solution of (27) is

$$z_{1,0}=A_1^{(0)}e^{\sigma\sqrt{-1}t}u_1+A_2^{(0)}e^{-\sigma\sqrt{-1}t}u_2, \tag{28}$$

where u_1 and u_2 are conjugate complex functions of the form

$$\begin{aligned}
u_1&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}[+\sqrt{-1}\,a_k^{(n)}\sin kt+b_k^{(n)}(\cos kt-1)]\,e^n,\\
u_2&=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}[-\sqrt{-1}\,a_k^{(n)}\sin kt+b_k^{(n)}(\cos kt-1)]\,e^n,\\
\sigma&=4\sqrt{m_0}+\frac{3\sqrt{m_0}(1-16m_0)}{1-64m_0}e^2+\cdots\cdot
\end{aligned}$$

The $A_1^{(0)}$ and $A_2^{(0)}$ are constants of integration; the $a_k^{(n)}$ and $b_k^{(n)}$ are real constants which depend upon the coefficients of the various powers of e in E_1; and σ is a power series in e with real constant coefficients, determined by the condition that u_1 and u_2 shall be periodic with the period 2π.

From (25) we have

$$A_1^{(0)}+A_2^{(0)}=0,\qquad [\sigma\sqrt{-1}+u_1'(0)]\,A_1^{(0)}+[-\sigma\sqrt{-1}+u_2'(0)]\,A_2^{(0)}=\alpha. \tag{29}$$

Since $u_2'(0)=-u_1'(0)$, the determinant of the coefficients of $A_1^{(0)}$ and $A_2^{(0)}$ in (29) is

$$\Delta=-2[\sigma\sqrt{-1}+u_1'(0)]=\sqrt{-1}\,\Delta_1, \tag{30}$$

where Δ_1 is real, and it is different from zero because Δ is the determinant of a fundamental set of solutions at $\tau=0$. The solutions of (29) for $A_1^{(0)}$ and $A_2^{(0)}$ are

$$A_1^{(0)}=-A_2^{(0)}=\frac{\sqrt{-1}\,\alpha}{\Delta_1}.$$

Since $A_1^{(0)}$ vanishes with α, and conversely, it is convenient to integrate (23) as a power series in $A_1^{(0)}$ and λ.

The form of the solution of (23) arranged as a power series in $A_1^{(0)}$ is

$$z = A_1^{(0)}[e^{\sigma\sqrt{-1}t}u_1 - e^{-\sigma\sqrt{-1}t}u_2] + A_1^{(0)}P(A_1^{(0)}, \lambda, e; t),$$

where P is a power series in $A_1^{(0)}$, λ, and e. Upon imposing the condition (26) that z shall be periodic with the period T, we obtain

$$0 = A_1^{(0)}u_1\left(\frac{T}{2}\right)\left[e^{\sigma\sqrt{-1}\frac{T}{2}} - e^{-\sigma\sqrt{-1}\frac{T}{2}}\right] + A_1^{(0)}P\left(A_1^{(0)}, \lambda, e; \frac{T}{2}\right). \tag{31}$$

This equation is satisfied by $A_1^{(0)} = 0$, but this value of $A_1^{(0)}$ leads to the trivial solution $z \equiv 0$. In order, then, that (31) shall have a solution for $A_1^{(0)}$ which is different from zero, the coefficient of $A_1^{(0)}$ must be zero. Now

$$u_1\left(\frac{T}{2}\right) = 1, \text{ or } 1 + \text{a power series in } e,$$

according as ν in (24) is even or odd respectively. Hence $u_1(T/2) \neq 0$ for $|e|$ sufficiently small, and we have

$$e^{\sigma\sqrt{-1}\frac{T}{2}} - e^{-\sigma\sqrt{-1}\frac{T}{2}} = 0. \tag{32}$$

In order that this condition for the existence of a periodic solution may be satisfied, T must have the value

$$T = \frac{2N\pi}{\sigma},$$

where N is an integer which denotes the number of oscillations made by the infinitesimal body in the period T. Then it follows from (24) that

$$N = \nu\sigma. \tag{33}$$

If σ is a rational fraction, ν can be so chosen that N will be an integer. Inasmuch as σ is a continuous function of e, it can be made a rational fraction by a proper choice of e less than any value of $|e|$ which will insure the convergence of the power series. The numerical values of N and ν can be obtained when σ has been determined as a rational fraction.

171. Existence of Symmetrical Periodic Orbits.—Let us consider the terms in (23) of higher degree in $A_1^{(0)}$ and λ. The differential equation which defines the term in $A_1^{(0)}\lambda$ is

$$z_{1,1}'' + m_0 E_1 z_{1,1} = Z_1 = -\lambda A_1^{(0)} E_1 (e^{\sigma\sqrt{-1}t}u_1 - e^{-\sigma\sqrt{-1}t}u_2). \tag{34}$$

The complementary function of (34) is the same as that of (27), viz.,

$$z_{1,1} = a_1^{(1)} e^{\sigma\sqrt{-1}t}u_1 + a_2^{(1)} e^{-\sigma\sqrt{-1}t}u_2 .$$

On using the method of variation of parameters, we obtain

$$\left.\begin{aligned} (a_1^{(1)})' e^{\sigma\sqrt{-1}t}u_1 + (a_2^{(1)})' e^{-\sigma\sqrt{-1}t}u_2 = 0, \\ (a_1^{(1)})'(\sigma\sqrt{-1}\,u_1 + u_1')e^{\sigma\sqrt{-1}t} + (a_2^{(1)})'(-\sigma\sqrt{-1}\,u_2 + u_2')e^{-\sigma\sqrt{-1}t} = Z_1. \end{aligned}\right\} \tag{35}$$

The determinant of the coefficients of $(a_1^{(1)})'$ and $(a_2^{(1)})'$ is a constant, by §18, and is the same as (30), viz., $\Delta=\sqrt{-1}\ \Delta_1\neq 0$. Therefore

$$\left.\begin{aligned}(a_1^{(1)})'&=+\frac{\sqrt{-1}\,e^{-\sigma\sqrt{-1}t}u_2Z_1}{\Delta_1}=-\frac{\sqrt{-1}\,\lambda A_1^{(0)}E_1}{\Delta_1}(u_1u_2-e^{-2\sigma\sqrt{-1}t}u_2^2),\\(a_2^{(1)})'&=-\frac{\sqrt{-1}\,e^{+\sigma\sqrt{-1}t}u_1Z_1}{\Delta_1}=-\frac{\sqrt{-1}\,\lambda A_1^{(0)}E_1}{\Delta_1}(u_1u_2-e^{+2\sigma\sqrt{-1}t}u_1^2).\end{aligned}\right\}\qquad(36)$$

The integration of (36) gives non-periodic terms as well as periodic terms having the period T. We shall be concerned only with the non-periodic terms. Let the constant part of $-\sqrt{-1}\ E_1u_1u_2/\Delta_1$ be denoted by P_1; it is a power series in e with constant coefficients which are purely imaginary, the absolute term of which is found to be $2\sqrt{-1}/\sqrt{m_0}$. Hence

$$a_1^{(1)}=A_1^{(1)}+\lambda A_1^{(0)}[P_1t+\text{periodic terms}],$$
$$a_2^{(1)}=A_2^{(1)}+\lambda A_1^{(0)}[P_1t+\text{periodic terms}],$$

where $A_1^{(1)}$ and $A_2^{(1)}$ are constants of integration which are to be so determined that $z_{1,1}(0)=z'_{1,1}(0)=0$. Then

$$z_{1,1}=\lambda A_1^{(0)}P_1t[e^{\sigma\sqrt{-1}t}u_1+e^{-\sigma\sqrt{-1}t}u_2]+\text{periodic terms.}\qquad(37)$$

It is necessary to obtain in addition to this only the term in $(A_1^{(0)})^3$. This term is obtained from the differential equation

$$z''_{3,0}+m_0E_1z_{3,0}=Z_3=-m_0E_3z_0^3.\qquad(38)$$

On forming the equations analogous to (36), we have

$$\left.\begin{aligned}(a_1^{(2)})'&=+\frac{\sqrt{-1}\,e^{-\sigma\sqrt{-1}t}u_2Z_3}{\Delta_1}=+\frac{\sqrt{-1}}{\Delta_1}(A_1^{(0)})^3m_0E_3[3u_1^2u_2^2+\cdots],\\(a_2^{(2)})'&=-\frac{\sqrt{-1}\,e^{+\sigma\sqrt{-1}t}u_1Z_3}{\Delta_1}=+\frac{\sqrt{-1}}{\Delta_1}(A_1^{(0)})^3m_0E_3[3u_1^2u_2^2+\cdots].\end{aligned}\right\}\qquad(39)$$

The terms not written in (39) carry the exponentials $e^{\pm 2j\sigma\sqrt{-1}t}$ $(j=1,2)$ as a factor multiplied by the fourth power of u_1 and u_2 considered together. The integration of these terms gives periodic terms with the period T. Let the constant part of $3\sqrt{-1}m_0E_3u_1^2u_2^2/\Delta_1$ be denoted by P_2, a power series in e with constant coefficients which are purely imaginary, the absolute term of which is found by computation to be $36\sqrt{-1}\sqrt{m_0}$. Then upon integrating (39), we obtain

$$\left.\begin{aligned}a_1^{(2)}&=A_1^{(2)}+(A_1^{(0)})^3[P_2t+\text{periodic terms}],\\a_2^{(2)}&=A_2^{(2)}+(A_1^{(0)})^3[P_2t+\text{periodic terms}],\end{aligned}\right\}\qquad(40)$$

where $A_1^{(2)}$ and $A_2^{(2)}$ are constants of integration which are to be determined so that $z_{3,0}(0)=z'_{3,0}(0)=0$. Hence

$$z_{3,0}=(A_1^{(0)})^3P_2t[e^{\sigma\sqrt{-1}t}u_1+e^{-\sigma\sqrt{-1}t}u_2]+\text{periodic terms.}\qquad(41)$$

Now imposing the condition (26) that z shall be periodic with the period T, we obtain from (37) and (41)

$$0=\frac{T}{2}\left[\lambda A_1^{(0)}P_1+(A_1^{(0)})^3P_2\right]\left[e^{\sigma\sqrt{-1}\frac{T}{2}}u_1\left(\frac{T}{2}\right)+e^{-\sigma\sqrt{-1}\frac{T}{2}}u_2\left(\frac{T}{2}\right)\right]+\cdots \tag{42}$$

The expression $[e^{\sigma\sqrt{-1}T/2}\,u_1(T/2)+e^{-\sigma\sqrt{-1}T/2}\,u_2(T/2)]$ is different from zero for $e=0$, and therefore remains different from zero for $|e|$ sufficiently small. Equation (42) is satisfied by $A_1^{(0)}=0$, and hence the right side carries $A_1^{(0)}$ as a factor. In order to find a solution of (23) other than $z\equiv 0$, it is necessary to consider $A_1^{(0)}\neq 0$; therefore the factor $A_1^{(0)}$ can be divided out of (42). There remains a power series in λ and $A_1^{(0)}$ and, since P_1 and P_2 are different from zero for $|e|$ sufficiently small, the terms of lowest degree are λ and $(A_1^{(0)})^2$. There are no terms in $A_1^{(0)}e$ and e alone, and the coefficient of $(A_1^{(0)})^2$ has a term independent of e, viz., $36\sqrt{-1}\sqrt{m_0}$. Hence, after $A_1^{(0)}$ is divided out, equation (42) can be solved for $A_1^{(0)}$ as a power series in $\pm\lambda^{\frac{1}{2}}$, the coefficients being power series in integral powers of e.

Two periodic solutions of (23) therefore exist having the period T. They have the form

$$z=\pm\lambda^{\frac{1}{2}}Q(\pm\lambda^{\frac{1}{2}};\,t),$$

where Q is a power series in $\pm\lambda^{\frac{1}{2}}$ whose coefficients are power series in e. In the practical construction of the solutions it can be shown that z is a power series in odd powers of $\lambda^{\frac{1}{2}}$. This fact follows also from the dynamical nature of the problem, since the motion of μ is obviously symmetrical with respect to the xy-plane. The two solutions are therefore of the form

$$z=\sum_{j=0}^{\infty} z_{2j+1}\,\lambda^{\frac{2j+1}{2}}, \tag{43}$$

where each z_{2j+1} is periodic with the period T.

In §§117–118 it is shown by a discussion, which is applicable in this problem, that if ν is even in (24), the orbits obtained by taking the two signs before $\lambda^{\frac{1}{2}}$ are geometrically the same, but in the one the infinitesimal body is half a period ahead of its position in the other. If ν is odd, the orbits for $+\lambda^{\frac{1}{2}}$ and $-\lambda^{\frac{1}{2}}$ are geometrically distinct.

By an argument similar to that in §115, it can be shown that it is possible to choose $\lambda>0$ so that the solutions (43) will converge for all $0\leqq t\leqq T$.

172. Direct Construction of Symmetrical Periodic Solutions of (23).—Let us substitute (43) in (23) and equate the coefficients of the various powers of $\lambda^{\frac{1}{2}}$. The constants of integration occurring at each step are determined by the conditions that the orbits shall be symmetrical and periodic with

the period T. The condition to be imposed in order that the orbits shall be symmetrical is $z(0)=0$, from which it follows that

$$z_{2j+1}(0)=0 \qquad (j=0, \ldots \infty). \tag{44}$$

It is necessary to consider the terms up to $\lambda^{5/2}$ before the induction to the general term can be made.

The differential equation for the term in $\lambda^{\frac{1}{2}}$ is

$$z_1''+m_0 E_1 z_1=0,$$

and the solution of this equation is (28). When (44) is imposed

$$z_1=A_1^{(1)}\left[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2\right],$$

where $A_1^{(1)}$ is an undetermined constant.

The differential equation for the term in $\lambda^{3/2}$ is

$$z_3''+m_0 E_1 z_3=Z_3=-E_1 z_1-m_0 E_3 z_1^3. \tag{45}$$

When expressed in terms of t, the right side of (45) has the form

$$Z_3=A_1^{(1)}\theta_3^{(1)}+(A_1^{(1)})^3\theta_3^{(3)}, \tag{46}$$

where the $\theta_3^{(2i+1)}(i=0,\ 1)$ are homogeneous and of degree $2i+1$ in $e^{+\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$. The undetermined constant $A_1^{(1)}$ is written explicitly so far as it occurs. In $\theta_3^{(2i+1)}(i=0,\ 1)$ the coefficient of $[e^{\sigma\sqrt{-1}t}]^{j_1}[-e^{-\sigma\sqrt{-1}t}]^{j_2}$ differs from the coefficient of $[e^{\sigma\sqrt{-1}t}]^{j_2}[-e^{-\sigma\sqrt{-1}t}]^{j_1}$ only in the sign of $\sqrt{-1}$, j_1 and j_2 being positive integers (or zero) such that $j_1+j_2=2i+1$. These coefficients are power series in e with $\sqrt{-1}\sin jt$ and $\cos jt$ in the coefficients, k being the highest multiple of t in the coefficient of e^k. If the exponentials in $\theta_3^{(2i+1)}$ are expressed in trigonometric form, it is observed that the $\theta_3^{(2i+1)}$ are power series in e in which the coefficient of e^k has the form

$$\sqrt{-1}\sum_{j=0}^{k} c_j^{(2i+1)}\sin[(2i+1)\sigma+j]t,$$

where the $c_j^{(2i+1)}$ are real constants. Hence the $\theta_j^{(2i+1)}$ are purely imaginary.

In order that z_3 shall be periodic, the constant parts of the coefficients of $e^{+\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$ in $u_2 Z_3$ and $u_1 Z_3$ respectively must be zero. From the form of u_1, u_2, and Z_3 it follows that, when we equate to zero the constant parts of these coefficients, we obtain only the one equation

$$-\Delta A_1^{(1)}\left[P_1+(A_1^{(1)})^2 P_2\right]=0, \tag{47}$$

where P_1 and P_2 are the power series which appear in (42). Equation (47) is satisfied by $A_1^{(1)}=0$, but this value of $A_1^{(1)}$ leads to the solution $z\equiv 0$ and is excluded. The solutions of (47) for $A_1^{(1)}$ which are different from zero are

$$A_1^{(1)}=\pm\sqrt{-1}\,p_1, \tag{48}$$

where p_1 is a power series in e with constant coefficients which are real since P_1 and P_2 are both purely imaginary and their absolute terms have the same sign. The absolute term of p_1 is found by computation to be $1/\sqrt{18m_0}$. Since $A_1^{(1)}$ is purely imaginary the expression for z_1 is real. When the sign of $\sqrt{-1}$ is chosen in (48), the periodic solution of (23) which satisfies the initial condition $z(0)=0$ is unique.

The general solution of (45) is

$$z_3 = A_1^{(3)} e^{\sigma\sqrt{-1}t} u_1 + A_2^{(3)} e^{-\sigma\sqrt{-1}t} u_2 + \sqrt{-1}\,[\varphi_3^{(1)} + \varphi_3^{(3)}], \tag{49}$$

where $A_1^{(3)}$ and $A_2^{(3)}$ are the constants of integration. The particular integrals $\varphi_3^{(1)}$ and $\varphi_3^{(3)}$ are respectively of the same form as $\theta_3^{(1)}$ and $\theta_3^{(3)}$ in (46). From the form of $\varphi_3^{(1)}$ and $\varphi_3^{(3)}$ it follows that

$$\varphi_3^{(1)}(0) = \varphi_3^{(3)}(0) = 0,$$

and imposing the condition (44) on (49), we obtain

$$A_1^{(3)} + A_2^{(3)} = 0.$$

The solution (49) therefore becomes

$$z_3 = A_1^{(3)}\,[e^{\sigma\sqrt{-1}t} u_1 - e^{-\sigma\sqrt{-1}t} u_2] + \sqrt{-1}\,[\varphi_3^{(1)} + \varphi_3^{(3)}], \tag{50}$$

where $A_1^{(3)}$ remains undetermined at this step. If $A_1^{(3)}$ is found to be purely imaginary this solution for z_3 is real.

From a consideration of the terms in $\lambda^{5/2}$, and then by an induction to the general term, we shall show that $A_1^{(3)}$ and all the remaining constants of integration are uniquely determined, after the choice of the sign in (48) has been made, by the conditions that the orbits shall be symmetrical and periodic with the period T.

The differential equation for the term in $\lambda^{5/2}$ is

$$z_5'' + m_0 E_1 z_5 = Z_5 = -[E_1 z_3 + 3m_0 E_3 z_1^2 z_3] - [E_3 z_1^3 + m_0 E_5 z_1^5]. \tag{51}$$

In order that z_5 shall be periodic the constant parts of the coefficients of $e^{+\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$ in $u_2 Z_5$ and $u_1 Z_5$ respectively must be zero. From the form of u_1, u_2, and Z_5 it follows that, when we equate to zero the constant parts of these coefficients, we obtain only the one equation

$$A_1^{(3)} P_1^{(3)} + \sqrt{-1}\,P_3 = 0, \tag{52}$$

where $P_1^{(3)}$ and P_3 are power series in e with real constant coefficients which are unique after the sign of $\sqrt{-1}$ in (48) has been chosen. The absolute term of $P_1^{(3)}$ is found to be 32, and therefore the solution of (52) is

$$A_1^{(3)} = \sqrt{-1}\,p_3, \tag{53}$$

where p_3 is a power series in e with real constant coefficients.

With $A_1^{(3)}$ determined as in (53), the general solution of (51) is periodic and has the form

$$z_5=A_1^{(5)}e^{\sigma\sqrt{-1}t}u_1+A_2^{(5)}e^{-\sigma\sqrt{-1}t}u_2+\sqrt{-1}[\varphi_5^{(1)}+\varphi_5^{(3)}+\varphi_5^{(5)}], \tag{54}$$

where $A_1^{(5)}$ and $A_2^{(5)}$ are the constants of integration and where the $\varphi_5^{(2i+1)}$ $(i=0, 1, 2)$ are of the same form as the $\theta_3^{(2i+1)}$ $(i=0, 1, 2)$, respectively. It follows from the form of $\varphi_5^{(1)}$, $\varphi_5^{(3)}$, and $\varphi_5^{(5)}$ that, when the condition (44) is imposed, (54) becomes

$$z_5=A_1^{(5)}[e^{+\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2]+\sqrt{-1}[\varphi_5^{(1)}+\varphi_5^{(3)}+\varphi_5^{(5)}], \tag{55}$$

where $A_1^{(5)}$ remains as yet undetermined. This solution for z_5 is real if $A_1^{(5)}$ is purely imaginary.

We shall now make the induction to the general term. Let us suppose that $A_1^{(1)}, \ldots, A_1^{(2n-3)}$ have all been uniquely determined as power series in e with constant coefficients which are purely imaginary. Let us also suppose that $z_1, \ldots, z_{2n-1}$ have been uniquely determined and that they are of the form

$$z_{2k+1}=A_1^{(2k+1)}[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2]+\sqrt{-1}\sum_{i=0}^{k}\varphi_{2k+1}^{(2i+1)} \qquad (k=1, \ldots, n-1), \tag{56}$$

where the $\varphi_{2k+1}^{(2i+1)}$ are of the same form as the $\theta_3^{(2i+1)}$ in (46) $i=0, \ldots, k$. It will be shown that $A_1^{(2n-1)}$ is purely imaginary and is uniquely determined by the condition that z_{2n+1} shall be periodic; also when the condition $z_{2n+1}(0)=0$ has been imposed, that z_{2n+1} has the form

$$z_{2n+1}=A_1^{(2n+1)}[e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2]+\sqrt{-1}\sum_{i=0}^{n}\varphi_{2n+1}^{(2i+1)},$$

where the $\varphi_{2n+1}^{(2i+1)}$ are of the same form as the $\varphi_{2k+1}^{(2i+1)}$ in (56).

Let us consider the term in $\lambda^{(2n+1)/2}$. The differential equation for this term is

$$z_{2n+1}''+m_0E_1z_{2n+1}=Z_{2n+1}=-[E_1z_{2n-1}+3m_0E_3z_1^2z_{2n-1}]+\cdots \tag{57}$$

The part of Z_{2n+1} not written explicitly involves $z_1, \ldots, z_{2n-3}$ to odd degrees when considered together. The only undetermined constant which enters Z_{2n+1} is $A_1^{(2n-1)}$, and it has the same coefficient in Z_{2n+1} that $A_1^{(3)}$ has in Z_5. In order that z_{2n+1} shall be periodic, the constant parts of the coefficients of $e^{+\sigma\sqrt{-1}t}$ and $e^{-\sigma\sqrt{-1}t}$ in u_2Z_{2n+1} and u_1Z_{2n+1} respectively must be zero. Now since Z_{2n+1} is similar in form to Z_5, we obtain only one equation when the constant parts of these coefficients are equated to zero. The form of the equation is

$$A_1^{(2n-1)}P_1^{(3)}+\sqrt{-1}P_{2n-1}=0, \tag{58}$$

where P_{2n-1} is a power series in e with real constant coefficients. The solution of this equation for $A_1^{(2n-1)}$ is

$$A_1^{(2n-1)}=\sqrt{-1}\,p_{2n-1}, \tag{59}$$

where p_{2n-1} is a power series in e with real constant coefficients.

In general, there are no other terms in $u_2 Z_{2n+1}$ and $u_1 Z_{2n+1}$ which yield non-periodic terms in z_{2n+1}. But since $\sigma = N/\nu$, N and ν being integers, there are values of n for which other non-periodic terms than those already discussed can occur. It follows from the properties of Z_{2n+1} that $u_2 Z_{2n+1}$ contains the term

$$\left.\begin{aligned} K\,(A_1^{(1)})^{2n+1}\, e^{(2n+1)\sigma\sqrt{-1}t}[a_0+a_1\cos t+\cdots+a_k\cos kt+\cdots \\ +\sqrt{-1}\,b_1\sin t+\cdots+\sqrt{-1}\,b_k\sin kt+\cdots], \end{aligned}\right\} \quad (60)$$

where K is a constant. Now

$$e^{(2n+1)\sigma\sqrt{-1}t} = e^{\sigma\sqrt{-1}t}[\cos 2n\sigma t+\sqrt{-1}\,\sin 2n\sigma t].$$

Consequently these non-periodic terms arise if $k=2n\sigma$, k an integer, or if $k\nu = 2nN$. This relation is satisfied if $2n$ becomes a multiple of ν. Suppose ν is odd. Since ν and N are taken relatively prime, the smallest values of n and k for which the non-periodic terms in question can arise are $n=\nu$ and $k=2N$. If ν is even, N is odd; and the smallest values of n and k are $n=\nu/2$ and $k=N$. The terms in which these non-periodic terms first arise are multiplied by $\lambda^{(2\nu+1)/2}e^{2N}$ or $\lambda^{\nu+1}e^{N}$, according as ν is odd or even. After these terms first appear they in general occur similarly at all subsequent steps. When they are present, the equation analogous to (58), in so far as the terms in $u_2 Z_{2n+1}$ are concerned, is

$$A_1^{(2n-1)}P_1^{(3)}+\sqrt{-1}\,P_{2n-1}+K_{2n-1}\,(A_1^{(1)})^{2n+1}=0, \quad (61)$$

where K_{2n-1} is a constant multiplied by e^{2N} or e^{N} according as ν is odd or even. The terms in $u_1 Z_{2n+1}$ corresponding to (60) differ from (60) only in the sign of K and $\sqrt{-1}$. Non-periodic terms arise from these terms in the same way as from (60). The equation analogous to (58), in so far as the terms in $u_1 Z_{2n+1}$ are concerned, is the same as (61). This equation can be solved uniquely for $A_1^{(2n-1)}$ and the solution is of the same form as (59). Hence in all cases $A_1^{(2n-1)}$ can be determined by the symmetrical and the periodicity conditions.

With $A_1^{(2n-1)}$ determined as in (59), the solution of (57) is periodic. The general solution of (57) is

$$z_{2n+1}=A_1^{(2n+1)}\,e^{\sigma\sqrt{-1}t}u_1+A_2^{(2n+1)}\,e^{-\sigma\sqrt{-1}t}u_2+\sqrt{-1}\,\sum_{i=0}^{n}\varphi_{2n+1}^{(2i+1)}.$$

From the form of $\varphi_{2n+1}^{(2i+1)}$ it follows that

$$\varphi_{2n+1}^{(2i+1)}(0)=0,$$

and when the condition (44) is imposed, z_{2n+1} becomes

$$z_{2n+1}=A_1^{(2n+1)}\,(e^{\sigma\sqrt{-1}t}u_1-e^{-\sigma\sqrt{-1}t}u_2)+\sqrt{-1}\,\sum_{i=0}^{n}\varphi_{2n+1}^{(2i+1)},$$

where $A_1^{(2n+1)}$ remains undetermined at this step. This solution is real if $A_1^{(2n+1)}$ is purely imaginary. This completes the induction.

III. PERIODIC ORBITS WHEN THE THREE BODIES ARE FINITE.

173. The Differential Equations.—We shall now consider the question of the existence of orbits which are periodic when μ is finite, and which have the same period as those obtained in I. The question is one of determining initial conditions for m_1, m_2, and μ so that the motion of the system shall be periodic when μ is finite, and shall have the same period as when μ is infinitesimal.

The origin of coördinates will be taken at the center of mass of the system. The plane passing through the center of mass and perpendicular to the initial motion of μ will be taken as the $\xi\eta$-plane. Let the coördinates of m_1, m_2, and μ be ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; and ξ, η, ζ respectively. Let the values of ξ, η, ζ, ξ', η', η_1, η_2, ζ_1, and ζ_2 be zero at $t=t_0$. Further, let

$$\xi_1(t_0) = -\xi_2(t_0) = \frac{1}{2}, \qquad \xi_1'(t_0) = -\xi_2'(t_0), \qquad \eta_1'(t_0) = -\eta_2'(t_0).$$

Under these symmetrical initial conditions

$$\xi_1 \equiv -\xi_2, \qquad \eta_1 \equiv -\eta_2, \qquad \zeta_1 \equiv \zeta_2. \tag{62}$$

On making use of (62) in the center of gravity equations, which are

$$m_1\xi_1 + m_2\xi_2 + \mu\xi = 0, \qquad m_1\eta_1 + m_2\eta_2 + \mu\eta = 0, \qquad m_1\zeta_1 + m_2\zeta_2 + \mu\zeta = 0,$$

we have

$$\xi \equiv \eta \equiv 0, \qquad \zeta_1 + \mu\zeta = 0. \tag{63}$$

Hence μ always remains on the ζ-axis.

With the units chosen as in § 166, the differential equations are

$$\left.\begin{aligned} \xi_1'' &= -\frac{1}{8}\frac{\xi_1}{r^3} - \frac{\mu\xi_1}{[r^2+(1+\mu)^2\zeta^2]^{3/2}}, \\ \eta_1'' &= -\frac{1}{8}\frac{\eta_1}{r^3} - \frac{\mu\eta_1}{[r^2+(1+\mu)^2\zeta^2]^{3/2}}, \\ \zeta'' &= -\frac{(1+\mu)\zeta}{[r^2+(1+\mu)^2\zeta^2]^{3/2}}, \end{aligned}\right\} \tag{64}$$

where $r^2 = \xi_1^2 + \eta_1^2$. Let us transform (64) by the substitutions

$$\xi_1 = r\cos v, \qquad \eta_1 = r\sin v, \qquad t - t_0 = \sqrt{1/8}\,(1+\delta)\,\tau, \tag{65}$$

where δ has the value determined in I. Then equations (64) become

$$\left.\begin{aligned} \ddot{r} - r\dot{v}^2 + \frac{(1+\delta)}{64r^2} &= -\frac{\mu(1+\delta)r}{8[r^2+(1+\mu)^2\zeta^2]^{3/2}}, \\ r\ddot{v} + 2\dot{r}\dot{v} &= 0, \\ \ddot{\zeta} &= -\frac{(1+\mu)(1+\delta)\zeta}{8[r^2+(1+\mu)^2\zeta^2]^{3/2}}. \end{aligned}\right\} \tag{66}$$

For $\mu=0$ these equations admit the solutions

$$v=\sqrt{1/8(1+\delta)}\tau, \qquad r=\frac{1}{2}, \qquad \zeta=\psi,$$

where ψ is the function defined in (19) and is periodic in τ with the period 2π.

Now let

$$r=\frac{1}{2}(1+p), \qquad v=\sqrt{1/8(1+\delta)}\,\tau+u, \qquad \zeta=\psi+w, \tag{67}$$

where p, u, and w vanish with $\mu=0$. When equations (67) are substituted in (66), the differential equations for p, u, and w are found to be

$$\left.\begin{aligned}
&\ddot{p}-(1+p)(\sqrt{1/8(1+\delta)}+\dot{u})^2+\frac{1+\delta}{8(1+p)^2}=-\frac{\mu(1+\delta)(1+p)}{[(1+p)^2+4(1+\mu)^2(\psi+w)^2]^{3/2}},\\
&(1+p)\ddot{u}+2\dot{p}(\sqrt{1/8(1+\delta)}+\dot{u})=0,\\
&\ddot{w}=-\frac{(1+\delta)(1+\mu)(\psi+w)}{[(1+p)^2+4(1+\mu)^2(\psi+w)^2]^{3/2}}+\frac{(1+\delta)\psi}{[1+4\psi^2]^{3/2}}.
\end{aligned}\right\} \tag{68}$$

The second equation of (68) admits the integral

$$\dot{u}=\frac{d}{(1+p)^2}-\sqrt{1/8(1+\delta)}, \tag{69}$$

where d is an arbitrary constant. Since $\dot{u}$ and p vanish with μ, we substitute

$$d=d_0+\lambda\mu,$$

where $d_0=\sqrt{1/8(1+\delta)}$, and λ is an undetermined constant. On substituting (69) in (68), we obtain

$$\left.\begin{aligned}
\ddot{p}+\frac{(1+\delta)p}{8(1+p)^3}&=\frac{2d_0\lambda\mu+\lambda^2\mu^2}{(1+p)^3}-\frac{\mu(1+\delta)(1+p)}{[(1+p)^2+4(1+\mu)^2(\psi+w)^2]^{3/2}},\\
\ddot{w}&=-\frac{(1+\delta)(1+\mu)(\psi+w)}{[(1+p)^2+4(1+\mu)^2(\psi+w)^2]^{3/2}}+\frac{(1+\delta)\psi}{[1+4\psi^2]^{3/2}}.
\end{aligned}\right\} \tag{70}$$

174. Proof of Existence of Periodic Solutions of Equations (70).—For $\mu=0$ equations (70) admit the periodic solutions $p=\dot{p}=w=\dot{w}=0$. It will now be proved that if $|\mu|$ is not zero, but sufficiently small, equations (70) admit solutions expansible as converging power series in μ, which vanish with μ and which are periodic in τ with the period 2π.

Let us take the initial conditions

$$p(0)=\alpha_1, \qquad \dot{p}(0)=0, \qquad w(0)=0, \qquad \dot{w}(0)=\alpha_2. \tag{71}$$

With these initial conditions it can be shown from the properties of (70), by the usual method, that p is even in τ and that w is odd in τ. Therefore

$$p(\pi)=p(-\pi), \qquad \dot{w}(\pi)=\dot{w}(-\pi),$$

and if the conditions

$$\dot{p}(\pi)=w(\pi)=0, \tag{72}$$

are satisfied, p and w will be periodic in τ with the period 2π.

Equations (70) will now be integrated as power series in α_1, α_2, and μ in so far as the α_1 and α_2 enter linearly in the solutions. If the terms of the solutions in which the α_1 and α_2 enter linearly are denoted by p_1 and w_1, then the differential equations defining p_1 and w_1 are

$$\left.\begin{aligned} &\ddot{p}_1+\tfrac{1}{8}(1+\delta)p_1=P_1=2\sqrt{1/8(1+\delta)}\lambda\mu-\frac{\mu(1+\delta)}{(1+4\psi^2)^{3/2}},\\ &\ddot{w}_1+(1+\delta)\Big[1+\sum_{j=1}^{\infty}\binom{-3/2}{j}(2j+1)(4\psi^2)^j\Big]w_1=W_1\\ &\quad=-\frac{\mu(1+\delta)\psi}{(1+4\psi^2)^{3/2}}-(1+\delta)\psi(2p_1+8\mu\psi^2)\sum_{j=1}^{\infty}\binom{-3/2}{j}j(4\psi^2)^{j-1}. \end{aligned}\right\} \quad (73)$$

The first equation of (73) is independent of the second equation. The complementary function of the first equation is

$$p_1=A_1\cos\sqrt{1/8(1+\delta)}\,\tau+B_1\sin\sqrt{1/8(1+\delta)}\,\tau,$$

where A_1 and B_1 are constants of integration. The function P_1 involves ψ to even degrees, and is therefore a power series in a^2 with cosines of even multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k} is $2k$. In the sequel, such a power series is called a *triply even power series*. The particular integral arising from P_1 is a triply even power series unless $\sqrt{1/8(1+\delta)}$ is an even integer. If $\sqrt{1/8(1+\delta)}$ is an even integer the left side of the first equation of (73) has the same period as certain terms of the right side, and the solution will therefore contain non-periodic terms. When $\sqrt{1/8(1+\delta)}$ is an even integer, the period of the motion of m_1 and m_2 is an even integral multiple of the period of the oscillations of μ. The mutual attractions of the three bodies will then have a cumulative effect and produce non-periodic motion. We therefore exclude from our consideration those values of a for which $\sqrt{1/8(1+\delta)}$ is an even integer. With this restriction upon a, the solution of the p_1-equation satisfying the initial conditions (71) is

$$p_1=[\alpha_1-\mu C_1(0)]\cos\sqrt{1/8(1+\delta)}\,\tau+\mu C_1(\tau), \quad (74)$$

where $C_1(\tau)$ is a triply even power series, and it contains λ as an undetermined constant.

When (74) is substituted in the w_1-equation, all the terms of W_1 are known. With the left side simplified and $W_1=0$, the equation becomes

$$\ddot{w}_1+\Big[1+\sum_{j=1}^{\infty}\theta_{2j}a^{2j}\Big]w_1=0, \quad (75)$$

where

$$\theta_2=-\frac{9}{2}+9\cos 2\tau, \qquad \theta_4=\frac{687}{32}-48\cos 2\tau+\frac{177}{8}\cos 4\tau,$$

and where each θ_{2k} is a sum of cosines of even multiples of τ, the highest multiple being $2k$.

Equation (75) is one of the *equations of variation*, and the expression $\psi(\tau)$ or $\psi\{(t-t_0)/\sqrt{1/8(1+\delta)}\}$, obtained in (19), is the *generating solution*. Two arbitrary constants, viz., t_0 and a, appear in its generating solution, and according to §§ 32 and 33 the two fundamental solutions of (75) are obtained by taking the first partial derivatives of ψ with respect to these constants. One solution is therefore

$$w_{11}=\frac{\partial}{\partial t_0}\psi\{(t-t_0)/\sqrt{1/8(1+\delta)}\}=-\frac{\partial}{\partial\tau}\psi(\tau),$$

and it is periodic in τ with the period 2π. This solution contains the factor $-a$, and since it is multiplied later by an undetermined constant the factor $-a$ may be absorbed by the undetermined constant. This solution can then be expressed as (see page 330 for ψ)

$$\overline{w}_{11}=\varphi=\sum_{j=0}^{\infty}\varphi_{2j}a^{2j}=\cos\tau+\frac{9}{16}a^2(\cos 3\tau-\cos\tau)+\cdots, \tag{76}$$

where $a\varphi=\partial\psi/\partial\tau$. Therefore the φ_{2j} are sums of cosines of odd multiples of τ, the highest multiple being $2j+1$. The initial values of this solution are

$$\overline{w}_{11}(0)=1, \qquad \dot{\overline{w}}_{11}(0)=0. \tag{77}$$

The other solution of (75) is obtained by differentiating the generating solution with respect to the constant a; hence this solution is

$$w_{12}=\frac{\partial\psi}{\partial a}=\left(\frac{\partial\psi}{\partial a}\right)+\frac{\partial\psi}{\partial\tau}\frac{\partial\tau}{\partial a}, \tag{78}$$

where $\left(\frac{\partial\psi}{\partial a}\right)$ denotes that the differentiation is performed only in so far as a occurs explicitly. Now

$$\left(\frac{\partial\psi}{\partial a}\right)=\sin\tau+\frac{9}{16}a^2(\sin 3\tau-3\sin\tau)+\cdots,$$

$$\frac{\partial\tau}{\partial a}=\frac{\partial\tau}{\partial\delta}\frac{\partial\delta}{\partial a}=\tau\left[-\frac{9}{2}a+\frac{465}{16}a^3+\cdots\right],$$

$$\frac{\partial\psi}{\partial\tau}=a\varphi,$$

and therefore the solution (78) is

$$w_{12}=\left[\sin\tau+\frac{9}{16}a^2(\sin 3\tau-3\sin\tau)+\cdots\right]+\tau\varphi\left[-\frac{9}{2}a^2+\frac{465}{16}a^4\cdots\right].$$

The initial values of this solution are

$$w_{12}(0)=0, \qquad \dot{w}_{12}(0)=B=1-\frac{9}{2}a^2+\frac{465}{16}a^4+\cdots.$$

Since it is more convenient for computation to have a solution $\overline{w}_{12}$ in which the initial values are

$$\overline{w}_{12}(0)=0, \qquad \dot{\overline{w}}_{12}(0)=1, \tag{79}$$

we take as the second solution of (75)

$$\overline{w}_{12}=\frac{w_{12}}{B}=\chi+A\tau\varphi, \tag{80}$$

where χ and A are found by computation to be

$$\chi=\sum_{j=0}^{\infty}\chi_{2j}a^{2j}=\sin\tau+\frac{9}{16}a^2(5\sin\tau+\sin 3\tau)+\cdots,$$
$$A=-\frac{9}{2}a^2+\frac{141}{16}a^4+\cdots.$$

From the way in which χ has been derived, viz.,

$$\chi=\frac{1}{B}\left(\frac{\partial\psi}{\partial a}\right),$$

it follows that each χ_{2j} is a sum of sines of odd multiples of τ and that the highest multiple is $2j+1$. Further, since

$$\varphi=\frac{1}{a}\frac{\partial\psi}{\partial\tau},$$

it follows from the character of ψ that the coefficients of the cosines and sines of the highest multiples of τ in φ_{2j} and χ_{2j} respectively are equal numerically and have the same sign.

The solutions (76) and (80) constitute a fundamental set of solutions, since their determinant is unity, and hence the general solution of (75) is

$$w_1=n_1^{(1)}\varphi+n_2^{(1)}[\chi+A\tau\varphi], \tag{81}$$

where $n_1^{(1)}$ and $n_2^{(1)}$ are constants of integration.

When (74) is substituted in (73), W_1 becomes an odd power series in a with two types of terms in the coefficients.

(1) There are terms not multiplied by $\cos\sqrt{1/8(1+\delta)}\,\tau$ which enter through p_1 and they form a triply odd power series. They have μ as a factor and will be denoted by μM_1.

(2) The remaining part of W_1 consists of terms which are multiplied by $\cos\sqrt{1/8(1+\delta)}\,\tau$. As we have already excluded those values of a for which $\sqrt{1/8(1+\delta)}$ is an even integer, and as we subsequently exclude those values of a for which $\sqrt{1/8(1+\delta)}$ is an odd integer, these terms in W_1 do not have the period 2π. In the direct construction of the solutions such terms do not appear in the right members of the w_j-equations. They appear as the complementary functions of the p_j-equations and, since they do not have the period 2π, they are excluded by assigning zero values to the constants of integration. We can, therefore, disregard the terms in W_1 which do not have the period 2π and consider W_1 to have the form μM_1.

By varying the parameters $n_1^{(1)}$ and $n_2^{(1)}$ in (81), we obtain

$$\dot{n}_1^{(1)}\varphi+\dot{n}_2^{(1)}[\chi+A\tau\varphi]=0, \qquad \dot{n}_1^{(1)}\dot{\varphi}+\dot{n}_2^{(1)}[\dot{\chi}+A(\tau\dot{\varphi}+\varphi)]=\mu M_1. \tag{82}$$

The determinant of the coefficients of $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$ in (82) is a constant [§18], and from (77) and (79) it is seen that the value is unity. Equations (82) can therefore be solved for $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$, and the solutions are

$$\dot{n}_1^{(1)}=-\mu M_1[\chi+A\tau\varphi], \qquad \dot{n}_2^{(1)}=\mu M_1\varphi. \tag{83}$$

Upon integrating these equations, we obtain

$$n_1^{(1)}=\eta_1^{(1)}-\mu[aM_1^{(0)}\tau+M_1^{(1)}+A\tau R_1], \qquad n_2^{(1)}=\eta_2^{(1)}+\mu R_1. \tag{84}$$

The $\eta_1^{(1)}$ and $\eta_2^{(1)}$ are the constants of integration. The $M_1^{(0)}$ is a power series in a^2 with constant coefficients. The $M_1^{(1)}$ is a power series in odd powers of a with sines of even multiples of τ in the coefficients, the highest multiple of τ in the coefficient of a^{2k+1} being $2k+2$. The R_1 has the same form as the $M_1^{(1)}$ except that it has cosines instead of sines. Since the coefficients of $a^{2j}\cos(2j+1)\tau$ and $a^{2j}\sin(2j+1)\tau$ occurring in φ and χ respectively are equal, so also the coefficients of $a^{2j+1}\sin(2j+2)\tau$ and $a^{2j+1}\cos(2j+2)\tau$ in $M_1^{(1)}$ and R_1 respectively are equal. When (84) is substituted in (81) the solution of the second equation in (73) is found to be

$$w_1=\eta_1^{(1)}\varphi+\eta_2^{(1)}[\chi+A\tau\varphi]+\mu[S_1-aM_1^{(0)}\tau\varphi], \tag{85}$$

where S_1 is a power series in odd powers of a with sines of odd multiples of τ in the coefficients. From the form of the φ, χ, $M_1^{(1)}$, and R_1 it follows that S_1 is a triply odd power series. The expressions S_1 and $M_1^{(0)}$ carry λ as an undetermined constant. When the constants of integration are chosen so that (71) shall be satisfied, the solution (85) becomes

$$w_1=[\alpha_2-\mu\{\dot{S}_1(0)-aM_1^{(0)}\}][\chi+A\tau\varphi]+\mu[S_1-aM_1^{(0)}\tau\varphi]. \tag{86}$$

Now let us impose the periodicity conditions (72) on the solutions (74) and (86). In so far as the linear terms in α_1 and α_2 are concerned, we get

$$\left.\begin{aligned} \dot{p}(\pi)=0=&-\sqrt{1/8(1+\delta)}\,[\alpha_1-\mu C_1(0)]\sin\sqrt{1/8(1+\delta)}\,\pi \\ &+\text{terms in } \mu\alpha_2 \text{ and higher degree terms in } \alpha_1,\ \mu\alpha_2, \text{ and } \mu, \\ w(\pi)=0=&-\alpha_2 A\pi \\ &+\text{terms in } \alpha_1,\ \mu \text{ and higher degree terms in } \mu,\ \alpha_1, \text{ and } \alpha_2. \end{aligned}\right\} \tag{87}$$

These equations are satisfied by $\alpha_1=\alpha_2=\mu=0$, and λ arbitrary. The determinant of the coefficients of the linear terms in α_1 and α_2 is

$$D=A\pi\sqrt{1/8(1+\delta)}\sin\sqrt{1/8(1+\delta)}\,\pi.$$

We now exclude those values of a for which $\sqrt{1/8(1+\delta)}$ is an odd integer, and as we have already excluded those values of a for which $\sqrt{1/8(1+\delta)}$ is an even integer, $\sin\sqrt{1/8(1+\delta)}\,\pi$ is not zero and D can vanish only when a is zero. In order to obtain solutions which are not identically zero, a must be distinct from zero. Hence the determinant D is distinct from zero and, by the theory of implicit functions, equations (87) can be solved for α_1 and α_2 as power series in μ, vanishing with μ. The coefficients of the various powers of μ are power series in a and contain additional terms in $1/a^k$. The λ enters the coefficients of the solutions as an arbitrary, but the solutions are unique if λ is assigned. Hence periodic solutions of (70) exist and are of the form

$$p=\sum_{i=1}^{\infty} p_i\mu^i, \qquad w=\sum_{i=1}^{\infty} w_i\mu^i, \tag{88}$$

where each p_i and w_i is periodic in τ with the period 2π.

175. Proof that all the Periodic Orbits are Symmetrical.—Let us suppose that the condition is no longer imposed that the equal bodies shall be at apsides of their orbits when the third body crosses the $\xi\eta$-plane, and let us consider the question of the existence of periodic solutions of (70), with the period 2π in τ, when the initial conditions are

$$p(0)=\alpha_1, \qquad \dot{p}(0)=\alpha_2, \qquad w(0)=0, \qquad \dot{w}(0)=\alpha_3. \tag{89}$$

Sufficient conditions that p and w shall be periodic with the period 2π are

$$p(2\pi)-p(0)=0, \quad \dot{p}(2\pi)-\dot{p}(0)=0, \quad w(2\pi)-w(0)=0, \quad \dot{w}(2\pi)-\dot{w}(0)=0. \tag{90}$$

These four conditions can not be satisfied by the three constants α_i unless one condition is a consequence of the other three. We now show that the last condition can be suppressed when the first three have been imposed.

The original differential equations (64) admit the integral

$$(\xi_1')^2+(\eta_1')^2+\mu(1+\mu)\,(\zeta')^2=\frac{1}{4r}+\frac{2\mu}{[r^2+(1+\mu)^2\zeta^2]^{\frac{1}{2}}}+\text{const.}$$

When the substitutions (65) and (67) are made and $\dot{u}$ is eliminated by means of (69), this integral takes the form

$$\left.\begin{aligned} &\dot{p}^2+\frac{d^2}{(1+p)^2}+4\mu(1+\mu)(\dot{\psi}+\dot{w})^2= \\ &\qquad(1+\delta)\left[\frac{1}{4(1+p)}+\frac{2\mu}{[(1+p)^2+4(1+\mu)^2(\psi+w)^2]^{\frac{1}{2}}}+\text{const.}\right]. \end{aligned}\right\} \tag{91}$$

Let us make in (91) the usual substitutions

$$p=p(0)+\overline{p}, \qquad \dot{p}=\dot{p}(0)+\dot{\overline{p}} \qquad w=0+\overline{w}, \qquad \dot{w}=\dot{w}(0)+\dot{\overline{w}}, \tag{92}$$

where $\overline{p}$, $\dot{\overline{p}}$, $\overline{w}$, and $\dot{\overline{w}}$ vanish at $\tau=0$, and let us denote the resulting equation by (91a). By putting $\tau=0$, we obtain from (91a) an equation (91b) connecting the terms in (91a) independent of $\overline{p}$, $\dot{\overline{p}}$, $\overline{w}$, and $\dot{\overline{w}}$. When this equation (91b) is substituted in (91a) there results an equation of the form

$$F(\overline{p}, \dot{\overline{p}}, \overline{w}, \dot{\overline{w}})=0, \tag{93}$$

in which there are no terms independent of the arguments indicated. The linear term in $\dot{\overline{w}}$ enters (93) with the coefficient

$$8\mu(1+\mu)[\dot{\psi}+\dot{w}(0)],$$

which, we shall show, is different from zero at $\tau=2\pi$. Since the third body is assumed to be finite, $8\mu(1+\mu)$ is distinct from zero. The coefficient of $\dot{\overline{w}}$ is therefore different from zero at $\tau=2\pi$ unless $\dot{w}(0)=-\dot{\psi}(2\pi)=-a$. Now the third body, when assumed to be infinitesimal, has the initial speed $a/\sqrt{1/8(1+\delta)}$, and $\dot{w}(0)/\sqrt{1/8(1+\delta)}$ is the additional initial speed to be so determined that the orbits shall be periodic in τ with the period 2π when the third body becomes finite. If this additional initial speed is $-a/\sqrt{1/8(1+\delta)}$, then the whole initial speed is zero, and the third body remains at the center of gravity since there is then no force component normal to the $\xi\eta$-plane. In order therefore to obtain solutions in which ζ is not identically zero, we must consider $\dot{w}(0)\neq -a$. Hence the coefficient of the linear term $\dot{\overline{w}}$ in (93) is distinct from zero at $\tau=2\pi$, and therefore (93) can be solved for $\dot{\overline{w}}(2\pi)$ as a power series in $\overline{p}(2\pi)$, $\dot{\overline{p}}(2\pi)$, and $\overline{w}(2\pi)$ which vanishes with $\overline{p}(2\pi)$, $\dot{\overline{p}}(2\pi)$, and $\overline{w}(2\pi)$. This power series is unique if λ, which appears in the coefficients, is assigned. Hence if the first three conditions of (90) are imposed, then $\overline{p}(2\pi)=\dot{\overline{p}}(2\pi)=\overline{w}(2\pi)=0$; and since $\dot{\overline{w}}(2\pi)$ vanishes with $\overline{p}(2\pi)$, $\dot{\overline{p}}(2\pi)$, and $\overline{w}(2\pi)$, it follows that $\dot{\overline{w}}(2\pi)=0$ or $\dot{w}(2\pi)-\dot{w}(0)=0$. Therefore the fourth equation of (90) is a consequence of the first three and may be suppressed.

Equations (70) will be integrated as power series in μ and $\alpha_i (i=1, 2, 3)$, but only in so far as the α_i enter linearly in the solutions. The differential equations from which these linear terms in α_i are obtained are the same as (73). Their solutions satisfying (89), in so far as the linear terms are concerned, are

$$\left.\begin{aligned} p_1&=\alpha_1\cos\sqrt{1/8(1+\delta)}\,\tau+\frac{\alpha_2}{\sqrt{1/8(1+\delta)}}\sin\sqrt{1/8(1+\delta)}\,\tau,\\ w_1&=\alpha_3[\chi+A\tau\varphi]. \end{aligned}\right\} \tag{94}$$

When the first three conditions of (90) are imposed on p and w, we have as a consequence of (94)

$$\left.\begin{aligned}
0&=\alpha_1\left[\cos\sqrt{1/8(1+\delta)}\,2\pi-1\right]+\frac{\alpha_2}{\sqrt{1/8(1+\delta)}}\sin\sqrt{1/8(1+\delta)}\,2\pi\\
&\qquad+\text{terms in }\mu,\ \mu\alpha_3,\text{ and higher degree terms},\\
0&=-\sqrt{1/8(1+\delta)}\,\alpha_1\sin\sqrt{1/8(1+\delta)}\,2\pi+\alpha_2\left[\cos\sqrt{1/8(1+\delta)}\,2\pi-1\right]\\
&\qquad+\text{terms in }\mu,\ \mu\alpha_3,\text{ and higher degree terms},\\
0&=2\alpha_3A\pi+\text{terms in }\alpha_1,\ \alpha_2,\ \mu,\text{ and higher degree terms}.
\end{aligned}\right\}\quad(95)$$

The determinant of the coefficients of the linear terms in α_i in (95) is

$$4A\pi\left[1-\cos\sqrt{1/8(1+\delta)}\,2\pi\right].$$

This determinant does not vanish when a is not zero, and consequently (95) can be solved for α_i as power series in μ, vanishing with μ. These solutions are therefore unique if λ, which enters the coefficients, is assigned. Hence the periodic solutions of (70) for p and w, with the initial conditions (89), are of the same form as those obtained in (88) for the symmetrical orbits. The unrestricted and the symmetrical orbits are unique for a not zero and for any value of λ, and therefore, since the unrestricted orbits include the symmetrical orbits, all the periodic orbits are symmetrical.

176. Direct Construction of the Periodic Solutions of (70).—In order to construct the periodic solutions of (70), we substitute (88) in (70) and equate the coefficients of the various powers of μ. The arbitrary constants of integration are to be so determined that $w(0)=0$ and that each p_i and w_i shall be periodic in τ with the period 2π.

The differential equations for the terms in μ are

$$\left.\begin{aligned}
\ddot{p}_1+\tfrac{1}{8}(1+\delta)p_1&=P_1=2d_0\lambda-\frac{(1+\delta)}{(1+4\psi^2)^{3/2}},\\
\ddot{w}_1+\Big[1+\sum_{j=1}^{\infty}\theta_{2j}\,a^{2j}\Big]w_1&=W_1=-\frac{(1+\delta)\psi}{(1+4\psi^2)^{3/2}}\\
&\quad-(1+\delta)\psi(2p_1+8\psi^2)\sum_{j=1}^{\infty}\binom{-3/2}{j}j(4\psi^2)^{j-1}.
\end{aligned}\right\}\quad(96)$$

The general solution of the p_1-equation is

$$p_1=A_1\cos\sqrt{1/8(1+\delta)}\tau+B_1\sin\sqrt{1/8(1+\delta)}\tau+C_1(\tau),$$

where $C_1(\tau)$ is the same triply even power series as in (74). Since the complementary function does not have the period 2π when $\sqrt{1/8(1+\delta)}$ is not an integer, the arbitrary constants A_1 and B_1 must be zero in order that p_1 shall be periodic with the period 2π. Hence the desired solution of the p_1-equation is

$$p_1=C_1(\tau).\qquad(97)$$

When (97) is substituted in W_1 all the terms of W_1 are known. The general solution of the w_1-equation is the same as (85) except that the particular integral does not carry the factor μ. Hence

$$w_1 = \eta_1^{(1)}\varphi + \eta_2^{(1)}[\chi + A\tau\varphi] + S_1 - aM_1^{(0)}\tau\varphi. \tag{98}$$

Since all the periodic orbits are symmetrical we impose the condition $w(0)=0$, from which it follows that

$$w_i(0) = 0 \qquad (i=1, \ldots \infty). \tag{99}$$

As a consequence of (99), the constant $\eta_1^{(1)}$ is zero. In order that w_1 shall be periodic, the right member of (98) must contain no terms in τ except those in which it occurs under the trigonometric symbols. The non-periodic terms in (98) disappear if the constant $\eta_2^{(1)}$ is so determined that $A\eta_2^{(1)} = aM_1^{(0)}$, from which it follows that

$$\eta_2^{(1)} = \frac{1}{a}P^{(1)}(a^2),$$

where $P^{(1)}(a^2)$ is a power series in a^2 with constant coefficients. When these values of $\eta_1^{(1)}$ and $\eta_2^{(1)}$ are substituted in (98), the solution for w_1 becomes

$$w_1 = \frac{1}{a^2}\sum_{j=0}^{\infty} S_1^{(2j+1)} a^{2j+1}, \tag{100}$$

where each $S_1^{(2j+1)}$ is a sum of sines of odd multiples of τ, the highest multiple being $2j+1$.

The differential equations for the terms in μ^2 are

$$(a)\quad \ddot{p}_2 + \tfrac{1}{8}(1+\delta)p_2 = P_2, \qquad (b)\quad \ddot{w}_2 + \Big[1 + \sum_{j=1}^{\infty}\theta_{2j}a^{2j}\Big]w_2 = W_2. \tag{101}$$

The first equation is independent of the second, and the terms in P_2 are completely known. The complementary function of (101a) does not have the proper period and is excluded from the solution by taking zero values for the constants of integration. Since P_2 has the period 2π the particular integral has the same period and is the solution desired. The function P_2 is a triply even power series multiplied by $1/a^2$, and since the particular integral has the same form as P_2 it is denoted by

$$p_2 = \frac{1}{a^2}\sum_{j=1}^{\infty} C_2^{(2j)} a^{2j}, \tag{102}$$

where each $C_2^{(2j)}$ is a sum of cosines of even multiples of τ, the highest multiple being $2j$.

When p_2 has been obtained, all the terms in W_2 are known. That part of W_2 which is independent of p_2 and w_1 is a triply odd power series. The term p_2 is multiplied by a triply odd power series and yields a triply odd power series multiplied by $1/a^2$. The terms w_1 and w_1^2 are multiplied by triply even and triply odd power series respectively and together yield a triply odd

power series multiplied by $1/a^4$. The lowest power to which a enters $a^4 W_2$ is found from the term $w_1^2 \psi$ to be 3. Hence the form of W_2 is

$$W_2 = \frac{1}{a^4} \sum_{j=1}^{\infty} W_2^{(2j+1)} a^{2j+1},$$

where each $W_2^{(2j+1)}$ is a sum of sines of odd multiples of τ, the highest multiple being $2j+1$.

The complementary function of (101b) is the same as that of (75), viz.,

$$w_2 = n_1^{(2)} \varphi + n_2^{(2)} [\chi + A\tau\varphi]. \tag{103}$$

The general solution of (101b) has the same form as (85) and is denoted by

$$w_2 = \eta_1^{(2)} \varphi + \eta_2^{(2)} [\chi + A\tau\varphi] + S_2 - \frac{1}{a} M_2^{(0)} \tau\varphi, \tag{104}$$

where S_2 has the same form as W_2 and where $M_2^{(0)}$ is a power series in a^2 with constant coefficients. The $\eta_1^{(2)}$ and $\eta_2^{(2)}$ are the constants of integration. The constant $\eta_1^{(2)}$ must be zero to satisfy (99), and in order that w_2 shall be periodic $\eta_2^{(2)}$ must have the value

$$\eta_2^{(2)} = \frac{1}{a} \frac{M_2^{(0)}}{A} = \frac{1}{a^3} P^{(2)}(a^2), \tag{105}$$

where $P^{(2)}(a^2)$ is a power series in a^2 with constant coefficients. With these values of $\eta_1^{(2)}$ and $\eta_2^{(2)}$, the solution (104) becomes

$$w_2 = \frac{1}{a^4} \sum_{j=0}^{\infty} S_2^{(2j+1)} a^{2j+1}, \tag{106}$$

where the $S_2^{(2j+1)}$ have the same form as the $S_1^{(2j+1)}$ in (100).

The form of the w_j is apparent from (100) and (106). The form of the $p_j (j>2)$ is not apparent from (97) and (102), and, before the induction can be made, it is necessary to consider the term in μ^3 in so far as p_3 is concerned. The differential equation for p_3 is

$$\ddot{p}_3 + \frac{1}{8}(1+\delta) p_3 = P_3, \tag{107}$$

where all the terms of P_3 are known. That part of P_3 independent of the w_j is a triply even power series multiplied by $1/a^2$. The w_j enter P_3 multiplied by power series which are triply odd or triply even according as the w_j, considered together, enter to odd or even degrees respectively. These terms form a triply even power series multiplied by $1/a^4$. Since P_3 contains the term $w_2 \psi$, the lowest power of a^2 in $a^4 P_3$ is unity. The complementary function of (107) does not have the period 2π, and the solution desired is the particular integral. This solution has the same form as P_3 and is denoted by

$$p_3 = \frac{1}{a^4} \sum_{j=1}^{\infty} C_3^{(2j)} a^{2j}, \tag{108}$$

where the $C_3^{(2j)}$ have the same form as the $C_2^{(2j)}$ in (102).

Let us suppose that the p_i, w_i, $\eta_j^{(i)}$ $(i=1, \ldots, n-1;\ j=1, 2)$, have been computed and that

$$\left.\begin{aligned} p^i &= \frac{1}{a^{2i-2}}\sum_{j=1}^{\infty} C_i^{(2j)}\, a^{2j}, & w_i &= \frac{1}{a^{2i}}\sum_{j=0}^{\infty} S_i^{(2j+1)}\, a^{2j+1}, \\ \eta_1^{(i)} &= 0, & \eta_2^{(i)} &= \frac{1}{a^{2i-1}} P^{(i)}(a^2), \end{aligned}\right\} \qquad (109)$$

where the $C_i^{(2j)}$, $S_i^{(2j+1)}$, and $P^{(i)}(a^2)$ have the same form as $C_1^{(2j)}$, $S_1^{(2j+1)}$, and $P^{(1)}(a^2)$ respectively. From these assumptions and the differential equations arising from the coefficients of μ^n it will be shown that equations (109) hold when $i=n$. The differential equations for the terms in μ^n are

$$(a)\quad \ddot{p}_n+\tfrac{1}{8}(1+\delta)p_n=P_n, \qquad (b)\quad \ddot{w}_n+\left[1+\sum_{j=1}^{\infty}\theta_{2j}a^{2j}\right]w_n=W_n. \qquad (110)$$

As in the previous steps, the first equation is independent of the second. Since the right member of the first equation in (70) carries μ as a factor, the p_j and w_j which enter P_n have $j<n$. Hence all the terms of P_n are completely known. The general term of P_n has the form

$$p_{\lambda_1}^{\lambda_1'} \cdots p_{\lambda_k}^{\lambda_k'} w_{\mu_1}^{\mu_1'} \cdots w_{\mu_j}^{\mu_j'} \qquad (111)$$

multiplied by a triply odd or by a triply even power series according as $\mu_1'+ \cdots +\mu_j'$ is odd or even respectively. The λ_1, λ_1', $\ldots$, μ_j, μ_j' are positive integers (or zero) such that

$$\lambda_1\lambda_1'+ \cdots +\lambda_k\lambda_k'+\mu_1\mu_1'+ \cdots +\mu_j\mu_j' \leqq n-1. \qquad (112)$$

From the form of the general term it follows that P_n is a triply even power series multiplied by $1/a^i$, where

$$i=(2\lambda_1-2)\lambda_1'+ \cdots +(2\lambda_k-2)\lambda_k'+2(\mu_1\mu_1'+ \cdots +\mu_j\mu_j'). \qquad (113)$$

This expression is even and has its highest value when $\lambda_1'= \cdots =\lambda_k'=0$, i. e., in the terms of P_n in which only the w_j appear. Hence, from (112), the highest value of i is $2n-2$. Since P_n contains the term $w_{n-1}\psi$, the lowest power of a^2 in $a^{2n-2}P_n$ is found to be unity. Therefore the form of P_n is

$$P_n=\frac{1}{a^{2n-2}}\sum_{j=1}^{\infty} P_n^{(2j)}\, a^{2j},$$

where the $P_n^{(2j)}$ have the same form as the $C_i^{(2j)}$. The only solution of (110a) which has the period 2π is the particular integral, and it has the form

$$p_n=\frac{1}{a^{2n-2}}\sum_{j=1}^{\infty} C_n'^{(2j)}\, a^{2j},$$

where the $C_n'^{(2j)}$ are of the same form as the $C_i^{(2j)}$.

When p_n has been determined, all the terms in W_n are known since they arise from $p_j (j \gtreqless n)$ and $w_k (k<n)$. The general term of W_n has the same form as (111), but it is multiplied by a triply odd or triply even power series according as $\mu_j' + \cdots + \mu_j'$ is even or odd respectively. The $\lambda_1, \lambda_1', \ldots, \mu_j, \mu_j'$ are positive integers (or zero) such that

$$\lambda_1 \lambda_1' + \cdots + \mu_j \mu_j' \leqq n. \tag{114}$$

From the form of the general term it follows that W_n is a triply odd power series multiplied by $(1/a)^l$, where

$$l = (2\lambda_1 - 2)\lambda_1' + \cdots + (2\lambda_k - 2)\lambda_k' + 2\mu_1\mu_1' + \cdots + 2\mu_j\mu_j'.$$

This expression is even and has its highest value, $2n$, in the terms of W_n in which only the w_j appear. The lowest power to which a enters $a^{2n}W_n$ is obtained from the term in which ψ enters to the lowest power. This term is $w_1 w_{n-1} \psi$, and therefore the lowest power of a in $a^{2n}W_n$ is found to be 3. Hence the form of W_n is

$$W_n = \frac{1}{a^{2n}} \sum_{j=1}^{\infty} W_n^{(2j+1)} a^{2j+1},$$

where the $W_n^{(2j+1)}$ have the same form as the $S_i^{(2j+1)}$.

The complementary function of the second equation of (110) is

$$w_n = n_1^{(n)} \varphi + n_2^{(n)} [\chi + A\tau\varphi],$$

and by varying the parameters $n_1^{(n)}$ and $n_2^{(n)}$ we have as the complete solution

$$w_n = \eta_1^{(n)} \varphi + \eta_2^{(n)} [\chi + A\tau\varphi] + S_n - \frac{1}{a^{2n-3}} M_n^{(0)} \tau\varphi, \tag{115}$$

where S_n has the same form as W_n, and where $M_n^{(0)}$ is a power series in a^2 with constant coefficients. The $\eta_1^{(n)}$ and $\eta_2^{(n)}$ are the constants of integration. The constant $\eta_1^{(n)}$ must be zero to satisfy (99), and in order that w_n shall be periodic, $\eta_2^{(n)}$ must have the value

$$\eta_2^{(n)} = \frac{M_n^{(0)}}{A a^{2n-3}} = \frac{1}{a^{2n-1}} P^{(n)}(a^2),$$

where $P^{(n)}(a^2)$ is a power series in a^2 with constant coefficients. With these values of the constants of integration, the solution (115) becomes

$$w_n = \frac{1}{a^{2n}} \sum_{j=0}^{\infty} S_n^{(2j+1)} a^{2j+1},$$

where the $S_n^{(2j+1)}$ have the same form as the $S_i^{(2j+1)}$. This completes the induction.

177. The Periodic Solution of Equation (69).—When the solution for p is substituted in (69), u can be obtained by a single integration. Since p is a power series in μ with coefficients which are triply even power series multiplied by $1/a^{2j}$, the right side of (69) will contain terms independent of τ. After the integration, u will therefore contain a term in τ with λ appearing in the coefficient. We shall now show that λ can be so determined that this coefficient shall be zero. When λ has been so determined u will be periodic in τ with the period 2π.

The solution for p_1 obtained in (97), in so far as the term in λ is concerned, is

$$p_1 = \frac{2\lambda}{\sqrt{1/8(1+\delta)}}. \tag{116}$$

When (116) is substituted in (69) and d is replaced by $d_0 + \lambda\mu$, the constant terms appearing on the right side of (69) are

$$-3\lambda\mu + \text{higher degree terms in } \lambda\mu \text{ and } \mu. \tag{117}$$

Since λ carries the factor μ, we may replace $\lambda\mu$ by σ, and then the expression (117) becomes

$$-3\sigma + \text{higher degree terms in } \sigma \text{ and } \mu. \tag{118}$$

If (118) is equated to zero, the resulting equation can be solved uniquely for σ as a power series in μ, vanishing with μ (and converging for $|\mu|$ and $|a|$ sufficiently small). Let this series be denoted by

$$\sigma = \lambda\mu = \sum_{j=1}^{\infty} \sigma_j \mu^j,$$

from which the value of λ is found to be

$$\lambda = \sum_{j=0}^{\infty} \lambda_j \mu^j, \tag{119}$$

the σ_j and λ_j being constants. With this value of λ the u will be periodic, and since λ and p are power series in μ, the u is also a power series in μ and is denoted by

$$u = \sum_{j=1}^{\infty} u_j \mu^j, \tag{120}$$

where each u_j is separately periodic for $|\mu|$ sufficiently small. The solution (120) converges for $|a|$ and $|\mu|$ sufficiently small.

In order to construct the u_j directly, we substitute in (69) the series (119) and (120), and the solution already obtained for p, in which λ is to be replaced by (119). The various λ_j are determined so that the right side of

(69) shall contain no constant terms in the coefficients of the μ^j. The constant term appearing in the coefficient of μ is

$$-3[\lambda_0 - \sum_{j=0}^{\infty} \lambda_0^{(2j)} a^{2j}],$$

the $\lambda_0^{(2j)}$ being known constants. This term is zero if

$$\lambda_0 = \sum_{j=0}^{\infty} \lambda_0^{(2j)} a^{2j}.$$

It is necessary to consider the terms up to μ^3 before the induction to the general term can be made. The constant term appearing in the coefficient of μ^2 has the form

$$-3[\lambda_1 - \sum_{j=0}^{\infty} \lambda_1^{(2j)} a^{2j}],$$

where the $\lambda_1^{(2j)}$ are known constants. This term vanishes if λ_1 has the value

$$\lambda_1 = \sum_{j=0}^{\infty} \lambda_1^{(2j)} a^{2j}.$$

The constant term in the coefficient of μ^3 has a similar form; that is, it can be written

$$-3[\lambda_2 - \frac{1}{a^2} \sum_{j=0}^{\infty} \lambda_2^{(2j)} a^{2j}],$$

the $\lambda_2^{(2j)}$ being known constants, and this term vanishes if λ_2 is so determined that

$$\lambda_2 = \frac{1}{a^2} \sum_{j=0}^{\infty} \lambda_2^{(2j)} a^{2j}.$$

Suppose $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ have been uniquely determined in the same way and that

$$\lambda_0 = \sum_{j=0}^{\infty} \lambda_0^{(2j)} a^{2j}, \qquad \lambda_i = \frac{1}{a^{2i-2}} \sum_{j=0}^{\infty} \lambda_i^{(2j)} a^{2j} \qquad (i=1, \ldots, n-1),$$

the $\lambda_i^{(2j)}$ being known constants. From the form of p_i in (115) it follows that the constant term in the coefficient of μ^{n+1} has the form

$$-3[\lambda_n - \frac{1}{a^{2n-2}} \sum_{j=0}^{\infty} \lambda_n^{(2j)} a^{2j}],$$

where the $\lambda_n^{(2j)}$ are constants derived from the p_j and $\lambda_i (i=1, \ldots, n-1)$, and are therefore known. This constant term is zero if

$$\lambda_n = \frac{1}{a^{2n-2}} \sum_{j=0}^{\infty} \lambda_n^{(2j)} a^{2j}.$$

The same process of determining the λ_j obviously can be indefinitely continued.

With λ thus determined as a power series in μ, the integration of (69) yields periodic terms only, and from the form of the p_j it follows that the u_i have the form

$$u_i = \frac{1}{a^{2i-2}} \sum_{j=1}^{\infty} U_i^{(2j)} a^{2j} \qquad (i=1, \ldots \infty),$$

where the $U_i^{(2j)}$ are sums of sines of even multiples of τ, the highest multiple being $2j$. The periodic solution of (69) is therefore

$$u = \sum_{i=1}^{\infty} \frac{1}{a^{2i-2}} \sum_{j=1}^{\infty} U_i^{(2j)} a^{2j} \mu^i + U,$$

where U is the constant of integration. Since the mass m_1 is started from the point 0, 0, 1/2, at $t=t_0$ or at $\tau=0$, it follows that $v=0$ at $\tau=0$ and therefore, from (67), the value of u at $\tau=0$ is zero. Hence the constant U is zero.

178. The Character of the Periodic Solutions.—When the periodic solutions for p, u, and w have been determined, the solutions for ξ_i, η_i, $\zeta_i (i=1, 2)$ and ζ are obtained by means of the equations (67), (65), (63), and (62). These solutions are all periodic in t with the period $P=2\pi\sqrt{1/8(1+\delta)}$. Three arbitrary constants appear in the solutions, viz., a, μ, and t_0. The expression $a/\sqrt{1/8(1+\delta)}$ represents the initial speed of the third body in I, μ the mass of the third body, and t_0 the epoch. The mass μ is restricted in magnitude but can be increased step by step by making the analytic continuation of the solutions already obtained, provided the series do not pass through any singularities in the intervals. This can be done by the process already developed. The parameter a is restricted in magnitude and so that $\sqrt{1/8(1+\delta)}$ is not an integer. As already stated in §166, it can be shown from equation (2) that the motion of the infinitesimal body will be periodic if the initial conditions are chosen so that the constant C is negative. With the initial conditions chosen as in (4), the constant C has the value $4\{2a^2/(1+\delta)-1\}$. Now if $2a^2/(1+\delta)=1$ or >1, the infinitesimal body recedes to infinity with a velocity which is zero or greater than zero respectively, and therefore the motion will not be periodic. Hence a must be restricted so that $2a^2/(1+\delta)<1$. The epoch t_0 is arbitrary and may be chosen to be zero without loss of geometric generality. Hence for given values of $|\mu|$ and of $|a|$ sufficiently small and such that $\sqrt{1/8(1+\delta)}$ is not an integer, there exists one and only one set of periodic orbits which are geometrically distinct with the period P in t, and which reduce to those obtained in I for $\mu=0$.

In proving the existence of the periodic solutions of (70), if the period were chosen to be $2\nu\pi$ in τ, ν an integer, it could be shown by the same process that periodic solutions would exist under the same restrictions on

a and μ. The constant λ could be determined so that the solution for u would have the period $2\nu\pi$, and hence the periodic solutions of (64) would be unique as soon as ν were chosen. Therefore, for given values of a and of μ sufficiently small, and such that $\sqrt{1/8(1+\delta)}$ is not an integer, there exists one and but one set of orbits which re-enter after ν synodic revolutions. Hence the result is unique for every ν, and since the orbits reëntering after ν revolutions include those reëntering after one revolution, there are no orbits which for $\mu=0$ reduce to those obtained in I, having the period $2\nu\pi$ in τ which do not have the period 2π also.

Since ζ_1, ζ_2, and ζ are odd in a and in t, the orbits are symmetrical with respect to the $\xi\eta$-plane, both geometrically and in t. The two masses m_1 and m_2 move in the same orbit and always remain 180° apart.

CHAPTER XI.

PERIODIC ORBITS OF INFINITESIMAL SATELLITES AND INFERIOR PLANETS.

179. Introduction.—This chapter is devoted to the discussion of certain periodic orbits of the problem of three bodies in which two of the masses are finite, while that of the third is infinitesimal. The finite bodies are assumed to revolve in circles, and the infinitesimal body to move in the plane of their motion, relatively near one or the other of the finite bodies. The periodic orbits which are obtained are those in which the periods of the solutions are equal to the synodic periods of the bodies. The nearer the infinitesimal body is to one of the finite bodies the less its motion is disturbed by the more distant one. The orbits under discussion reduce to circles as the disturbance from the more distant body becomes zero, and they are therefore of the class called by Poincaré "*Solutions de la première sorte.*"*

The results of this chapter are of direct practical application, particularly in the Lunar Theory. They are coextensive in this domain with the *Researches*† of Hill and the first memoir by Brown.‡ When the masses of the two finite bodies have the ratio ten to one the problem reduces to that which Sir George Darwin treated by numerical processes,§ and the results obtained include the orbits called by him "Satellites A" and "Planets A." Darwin's "Satellites B" and "Satellites C" are imaginary for small values of the disturbing forces and belong to values of the parameter for which the series obtained in this chapter do not converge. It appears from the present investigations that there are three families of satellites and of inferior planets whose motion is direct. It follows that Darwin's search for them was exhaustive so far as the satellites are concerned; but he found only one family of planets whose motion is direct. The work of this chapter shows that there is an equal number of retrograde orbits; that is, three families of real or imaginary periodic orbits around each of the finite bodies.

As compared with previous work on this subject, the methods of this chapter are characterized by the fact that the validity of all the processes employed is proved. They have the merit of generality, not only in showing the number of orbits that exist, but also in being applicable to any ratio of masses of the finite bodies. They have the disadvantage that the series do not converge for all values of the parameter. The numerical processes

Les Méthodes Nouvelles de la Mécanique Céleste, vol. I, p. 97.

†*Amer. Jour. Mathematics*, vol. I (1878) pp. 5–26, 129–147, 245–260, and Hill's *Collected Works*, vol. I.

‡*Amer. Jour. of Mathematics*, vol. XIV (1892), pp. 141–150.

§*Acta Mathematica*, vol. XXI (1897), pp. 99–242; *Mathematische Annalen*, vol. LI (1898–9), pp. 523–583.

employed by Darwin are easily applicable to the cases where the series fail. Instead of arranging the solutions as Fourier series, as was done by Hill and as has been customary among astronomers, power series are employed, and the work has the simplicity which is characteristic of processes involving power series. For example, every coefficient is determined by a single step and is modified by no subsequent operations.

180. The Differential Equations.—Let m_1 and m_2 represent the masses and take the origin at m_1 for the determination of the motion of the infinitesimal body. Then the differential equations of motion in polar coördinates are

$$\left.\begin{aligned} &r''-r(v')^2+\frac{k^2m_1}{r^2}=k^2m_2\frac{\partial U}{\partial r}, \qquad rv''+2r'v'=k^2m_2\frac{\partial U}{\partial v}, \\ &U=\frac{1}{[R^2-2rR\cos(v-V)+r^2]^{\frac{1}{2}}}-\frac{r\cos(v-V)}{R^2}, \end{aligned}\right\} \quad (1)$$

where r and v are the polar coördinates of infinitesimal body, and R and V are the polar coördinates of m_2.

In the present discussion those orbits are considered in which r is small relatively to R. In this case U is expansible as a converging power series in r/R, and equations (1) become

$$\left.\begin{aligned} &U=\frac{1}{R}\left\{1+\frac{1}{4}\frac{r^2}{R^2}[1+3\cos 2(v-V)]+\frac{1}{8}\frac{r^3}{R^3}[3\cos(v-V)+5\cos 3(v-V)]+\cdots\right\}, \\ &r''-r(v')^2+\frac{k^2m_1}{r^2}=+\frac{k^2m_2}{2}\frac{r}{R^3}\left\{1+3\cos 2(v-V)+\frac{3r}{4R}[3\cos(v-V)\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left.+5\cos 3(v-V)]+\cdots\right\}, \\ &rv''+2r'v'=-\frac{k^2m_2}{2}\frac{r}{R^3}\left\{3\sin(v-V)+\frac{3r}{4R}[\sin(v-V)+5\sin 3(v-V)]+\cdots\right\}. \end{aligned}\right\} \quad (2)$$

It was assumed in the beginning that the relative motion of m_1 and m_2 is circular. Hence we have, from the two-body problem,

$$R=A=\text{constant}, \qquad V=\frac{k\sqrt{m_1+m_2}}{A^{3/2}}(t-t_0)=N(t-t_0), \quad (3)$$

where N is the angular velocity of the relative motion of the finite bodies.

When the right members of the second and third equations of (2) are put equal to zero, that is, when the infinitesimal body is supposed to revolve about m_1 without being disturbed by m_2, they have the particular solution

$$r=a=\text{constant}, \qquad v=\frac{k\sqrt{m_1}}{a^{3/2}}(t-t_0)=n(t-t_0), \quad (4)$$

where n is the angular velocity of the infinitesimal body with respect to m_1. In the periodic solutions of (2) when the right members are included, the mean angular velocity will be kept equal to n and a will be defined by the equation

$$n^2a^3=k^2m_1. \quad (5)$$

Since m_1 is given, a has three determinations, two of which are complex. In the physical two-body problem there is interest only in the real value of a, and it is immaterial whether a or n is regarded as given. It will be observed, however, from the purely astronomical point of view, that n can be determined by observation much more accurately than a, because the former is an angular variable and any error in its determination causes the theory to deviate secularly from the observations; while the latter is a linear variable and the discrepancies which arise from errors in its determination do not accumulate. But in the three-body problem all three values of a must be included in determining orbits whose period is defined by n, because all the orbits may become real for certain values of the parameters which occur in the right members of the differential equations. In fact, Darwin found three real orbits for certain values of the Jacobian constant.*

In the Lunar Theory, as developed by de Pontécoulant, for example, the solutions were forced into the trigonometric form by various artifices. But it will be noticed that the method of procedure was the opposite of that adopted here, so far as comparison can be made, in that the mean motion was continually modified by de Pontécoulant in order to preserve the trigonometric form; while here the mean motion is regarded as being given arbitrarily in advance by the observations or otherwise, and it is kept fixed.

New quantities ρ, w, τ, and m will be introduced by the equations

$$\left.\begin{aligned}
&r=a(1+\rho), \qquad v=n(t-t_0)+w,\\
&m=\frac{N}{n-N}, \qquad n=\frac{N(1+m)}{m}, \qquad \tau=(n-N)(t-t_0)=\frac{N}{m}(t-t_0),\\
&\frac{a}{A}=\left(\frac{m_1}{m_1+m_2}\right)^{1/3}\left(\frac{m}{1+m}\right)^{2/3}=\left(\frac{m_1}{m_1+m_2}\right)^{1/3}m^{2/3}\left(1-\frac{2}{3}m\cdots\right).
\end{aligned}\right\} \qquad (6)$$

For brevity a/A will be written in place of its expression as a power series. As a result of these transformations, the last two equations of (2) become

$$\left.\begin{aligned}
&\ddot{\rho}-(1+\rho)(1+m+\dot{w})^2+\frac{(1+m)^2}{(1+\rho)^2}=\frac{m^2}{2}\frac{m_2}{m_1+m_2}(1+\rho)\Big\{\big[1+3\cos 2(\tau+w)\big]\\
&\qquad\qquad +\frac{3}{4}\frac{a}{A}(1+\rho)\big[3\cos(\tau+w)+5\cos 3(\tau+w)\big]+\cdots\Big\},\\
&(1+\rho)\ddot{w}+2\dot{\rho}(1+m+\dot{w})=-\frac{m^2}{2}\frac{m_2}{m_1+m_2}(1+\rho)\Big\{3\sin 2(\tau+w)\\
&\qquad\qquad +\frac{3}{4}\frac{a}{A}(1+\rho)\big[\sin(\tau+w)+5\sin 3(\tau+w)\big]+\cdots\Big\},
\end{aligned}\right\} \qquad (7)$$

where the dots over ρ and w indicate derivatives with respect to τ. These equations are valid for the determination of the motion of the infinitesimal body as long as $a(1+\rho)$ is less than A.

**Acta Mathematica*, vol. XXI (1897), pp. 99-242.

It follows from equations (6) that $a\rho$ and w are the deviations from uniform circular motion due to the right members of the differential equations. They are functions of m, and the initial conditions are to be determined so that they shall be periodic in τ with the period 2π. Since the right members of (7) contain m^2 as a factor, the periodic expressions for ρ and w will contain m^2 as a factor.

181. Proof of the Existence of the Periodic Solutions.—For $m=0$ equations (7) admit the periodic solution $\rho=w=0$. It will now be proved that for m distinct from zero, but sufficiently small, equations (7) admit a periodic solution which has the period 2π in τ, and which is expansible as a power series in $m^{1/3}$, vanishing with m. It is the analytic continuation of the solution $\rho=w=m=0$ with respect to m as the parameter. Since τ enters explicitly in the right members of (7) in terms having the period 2π, it follows that the period of the solution must be 2π, or a multiple of 2π.

Equations (2) are not altered if we change the sign of v, V, and t. It easily follows from this that, if $v(t_0)=V(t_0)=\rho'(t_0)=0$, the dependent variables ρ and v are even and odd functions respectively of $t-t_0$. An orbit in which these conditions are satisfied is symmetrical with respect to a line always passing through m_1 and m_2. Such an orbit will be called *symmetrical*. Expressed in the variables of (7), ρ and w are respectively even and odd functions of τ in the case of symmetrical orbits; and $w(0)=\dot{\rho}(0)=0$.

The existence of symmetrical periodic orbits will be established, and then it will be shown, in connection with the construction of the solutions, that the condition that the solutions are periodic implies that they are also symmetrical. It will follow from this that all of the periodic orbits of the type under consideration are symmetrical.

Suppose that the initial conditions are

$$\left.\begin{array}{ll} r=a(1+\rho)=a(1+\alpha)(1-e), & w=0, \\ \dot{r}=a\dot{\rho}=0, & 1+m+\dot{w}=\dfrac{(1+m)\sqrt{1+e}}{(1+\alpha)^{3/2}(1-e)^{3/2}}. \end{array}\right\} \tag{8}$$

If the right members of equations (7) are neglected and the transformation

$$r=a(1+\rho), \qquad \theta=(1+m)\tau+w$$

is made, then equations (7) reduce to the ordinary polar equations of the two-body problem for the motion of the infinitesimal body with respect to m_1. In this two-body problem with the initial conditions (8), it is found that $a\alpha$ is the increment to the semi-axis a, and e is the eccentricity of the orbit of the infinitesimal body. Since the properties of the solution of the two-body problem in terms of the major semi-axis and eccentricity are known, we can at once write down the properties of the solution of (7) in terms of α and e, so far as they are independent of the right members of the equations. It was precisely for this reason that ρ and $\dot{w}$ were given the peculiar initial values defined in (8).

Equations (7) are now integrated as power series in a, e, and $m^{1/3}$. The results have the form

$$\left.\begin{aligned} \rho &= p_1\,(a, e, m^{1/3};\, \tau), & w &= p_3\,(a, e, m^{1/3};\, \tau), \\ \dot\rho &= p_2\,(a, e, m^{1/3};\, \tau), & \dot w &= p_4\,(a, e, m^{1/3};\, \tau). \end{aligned}\right\} \tag{9}$$

The moduli of a, e, and $m^{1/3}$ can be taken so small that the series converge while τ runs through any finite range of values starting from zero. The period must be a multiple of 2π, say $2k\pi$, and consequently it will be supposed that these parameters have such small moduli that (9) converge for $0 \gtreqless \tau \leqq k\pi$.

It follows from the symmetry of the orbits under the initial conditions (8), that if, at $\tau = k\pi$, the three bodies are in a line, and if the infinitesimal body is crossing perpendicularly the rotating line which joins m_1 and m_2, then the orbit necessarily re-enters at $2k\pi$, and the motion is periodic with the period $2k\pi$. Conversely, if, at $\tau = 0$, the orbit crosses perpendicularly the rotating line which joins m_1 and m_2, and if the motion is periodic with the period 2π, then the orbit at $\tau = k\pi$ necessarily crosses perpendicularly the rotating line which joins m_1 and m_2. Therefore, necessary and sufficient conditions for the existence of symmetrical periodic solutions having the period $2k\pi$ are

$$p_2(a, e, m^{1/3};\, k\pi) = 0, \qquad p_3(a, e, m^{1/3};\, k\pi) = 0. \tag{10}$$

Since $p_2 = p_3 = 0$ at $\tau = 0$, it follows that equations (10) are identically satisfied by $a = e = m^{1/3} = 0$. The problem of solving them for a and e as power series in $m^{1/3}$, vanishing for $m = 0$, will now be considered. All the terms of the solutions which do not carry m^2 as a factor are obtained from the solutions of the left members of (7) set equal to zero. Since these terms belong to the two-body problem equations (9) become, when these terms are explicitly exhibited,

$$\left.\begin{aligned} \rho = p_1 &= -e\Big[\cos\nu\tau + \frac{e}{2}(\cos 2\nu\tau - 1) + \cdots\Big] + m^2 q_1(a, e, m^{1/3};\, \tau), \\ \dot\rho = p_2 &= +e\Big[\nu\sin\nu\tau + e\nu\sin 2\nu\tau + \cdots\Big] + m^2 q_2(a, e, m^{1/3};\, \tau), \\ w = p_3 &= \Big[-(1+m)\tau + \nu\tau + 2e\sin\nu\tau + \cdots\Big] + m^2 q_3(a, e, m^{1/3};\, \tau), \\ \dot w = p_4 &= \Big[-(1+m) + \nu + 2\nu e\cos\nu\tau + \cdots\Big] + m^2 q_4(a, e, m^{1/3};\, \tau), \end{aligned}\right\} \tag{11}$$

where

$$\nu = \frac{1+m}{(1+a)^{3/2}}. \tag{12}$$

The series $q_1, \ldots, q_4$ depend upon the right members of the differential equations (7). All terms in the [ ] which are not written contain e^2 as a factor.

The conditions, (10), for the existence of symmetrical periodic solutions become as a consequence of (11)

$$\left.\begin{aligned} p_2(k\pi) &= e\nu\Big[\sin\nu k\pi + e\sin 2\nu k\pi + \cdots\Big] + m^2 q_2(a, e, m^{1/3};\, k\pi) = 0, \\ p_3(k\pi) &= \Big[-(1+m)k\pi + \nu k\pi + 2e\sin 2\nu k\pi + \cdots\Big] + m^2 q_3(a, e, m^{1/3};\, k\pi) = 0, \end{aligned}\right\} \tag{13}$$

where all the unwritten terms in the [ ] carry e^2 as a factor and are linear functions of sines of multiples of $\nu k\pi$. On substituting the value of ν from (12), it is found that

$$\sin j\nu k\pi = \sin\left[1+m-\frac{3}{2}\alpha+\cdots\right]jk\pi = (-1)^{jk}\sin\left[m-\frac{3}{2}\alpha+\cdots\right]jk\pi,$$

where j is an integer. Every term in the [ ] contains either m or α as a factor; therefore every term of the second equation of (13) contains either m or α as a factor. The coefficient of α to the first power is $-3/2\ k\pi$, which is distinct from zero; therefore the second equation of (13) can be solved for α as a power series in m and e, vanishing with $m=0$. The term of lowest degree in m alone is the second, and its coefficient depends upon the right member of the differential equations (7). Therefore the solution of the second equation for α carries m^2 as a factor, and has the form

$$\alpha = m^2 p(m^{1/3}, e). \tag{14}$$

Suppose α is eliminated from the first of (13) by means of (14). After the elimination, a factor m can be divided out. The result then contains a term in e alone to the first degree, and its coefficient is $(-1)^k$. Therefore the resulting equation can be solved for e as a power series in $m^{1/3}$, vanishing with $m=0$, of the form

$$e = mq(m^{1/3}). \tag{15}$$

As a matter of fact, the expression for e contains m^2 as a factor, as can easily be shown. Suppose equations (7) are integrated as power series in $m^{1/3}$, and let the initial conditions be $\rho=\rho=w=\dot{w}=0$, in order to get the terms which are independent of α and e. The series will have the form

$$\rho = \rho_1 m + \rho_2 m^2 + \rho_3 m^{8/3} + \cdots, \qquad w = w_1 m + w_2 m^2 + w_3 m^{8/3} + \cdots.$$

The explicit result of the integration is

$$\begin{aligned}
\rho_1 &= w_1 = 0,\\
\rho_2 &= \frac{m_2}{m_1+m_2}\left[-1+2\cos\tau-\cos 2\tau\right],\\
w_2 &= \frac{m_2}{m_1+m_2}\left[\frac{5}{4}\tau-4\sin\tau+\frac{11}{8}\sin 2\tau\right],\\
p_3 &= \frac{m_2}{m_1+m_2}\left[\frac{1}{2}+\frac{2}{3}\cos\tau-\frac{7}{6}\cos 2\tau-2\tau\sin\tau\right],\\
w_3 &= \frac{m_2}{m_1+m_2}\left[\tau-\frac{4}{3}\sin\tau+\frac{13}{6}\sin 2\tau-4\tau\cos\tau\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

From these equations it follows that $p_2(k\pi)$ has m^2 as a factor, but that it does not have m^3 as a factor. Therefore the expression for e as a power series in $m^{1/3}$ contains m^2 as a factor. Then (15) and (14) together give

$$\alpha = m^2 P_1(m^{1/3}), \qquad e = m^2 P_2(m^{1/3}), \tag{16}$$

where P_1 and P_2 are power series in $m^{1/3}$ which converge for the modulus of m sufficiently small. Therefore the symmetrical periodic orbits exist, and equations (16) and (8) give the initial values of the dependent variables in terms of the parameter $m^{1/3}$ which is defined, except for the cube root of unity, by the data of the problem and the third equation of (6).

182. Properties of the Periodic Solutions.—The periodic orbits whose existence has been proved re-enter after the period $2k\pi$, where k is any integer. Those orbits for which k is greater than unity include those for which k equals unity. Since, according to the discussion which has just been made, the number of periodic orbits is the same for all values of k, it follows that the period of the solutions is 2π. When $m=0$ the infinitesimal body makes a revolution in 2π, and m can be taken so small that the orbit is as near this undisturbed orbit as may be desired. Therefore a synodic revolution is made in 2π for all $|m|$ sufficiently small.

If the expressions for α and e given in (16) are substituted in (9), the result becomes

$$\rho = \sum_{j=6}^{\infty} \rho_j(\tau)\, m^{j/3}, \qquad w = \sum_{j=6}^{\infty} w_j(\tau) m^{j/3}, \tag{17}$$

where the summation starts with $j=6$, because the expressions for α and e have m^2 as a factor, and ρ and w have no terms in m alone of degree less than the second.

The ρ and w are periodic with the period 2π for $|m|$ sufficiently small because the conditions for periodicity have been satisfied. Therefore

$$\sum_{j=6}^{\infty} \rho_j(\tau+2\pi)m^{j/3} \equiv \sum_{j=6}^{\infty} \rho_j(\tau)m^{j/3}, \qquad \sum_{j=6}^{\infty} w_j(\tau+2\pi)m^{j/3} \equiv \sum_{j=6}^{\infty} w_j(\tau)m^{j/3};$$

whence

$$\rho_j(\tau+2\pi) = \rho_j(\tau), \qquad w_j(\tau+2\pi) = w_j(\tau) \qquad (j=6, \ldots \infty). \tag{18}$$

Since $\dot{\rho}(0) = w(0) = 0$, it follows that

$$\sum_{j=6}^{\infty} \dot{\rho}_j(0)\, m^{j/3} = 0, \qquad \sum_{j=6}^{\infty} w_j(0) m^{j/3} = 0;$$

whence

$$\dot{\rho}_j(0) = 0, \qquad w_j(0) = 0 \qquad (j=6, \ldots \infty). \tag{19}$$

If the orbit of the infinitesimal body is retrograde, n is negative and m has the definition

$$m = -N/(n+N)$$

for a given sidereal period. Therefore, for a given numerical value of n, the parameter m is smaller in retrograde motion than it is in direct motion. For a given sidereal period the deviations from circular motion are less in the retrograde orbits than they are in the direct. The physical reason is that the disturbance of the motion of the infinitesimal body by m_2 is greatest when the three bodies are in a line, as can be seen from (7) or by graphically resolving the disturbing acceleration; and in retrograde motion this approximate condition lasts a shorter time than in direct motion.

DIRECT CONSTRUCTION OF THE PERIODIC SOLUTIONS.

183. General Considerations.—It has been proved that equations (7) have solutions of the form (17) which satisfy (18) and (19). The solutions are in $m^{1/3}$ only because a/A is a series in $m^{1/3}$, given explicitly in (6). The expression for a/A can be modified by writing

$$\frac{a}{A}=Mm, \tag{20}$$

where M is to be regarded in the analysis as a constant independent of m. This amounts to a generalization of the m as it appears in certain places in the last equation of (6). The particular transformation (20) is made in order that the right members of (7) shall be in integral powers of m. The proof of the existence of the periodic solutions can be made precisely as before, because the transformation (20) affects only the higher terms which were not explicitly used. While there is nothing essential* in the transformation, it will be made for the sake of convenience, after which equations (7) become

$$\left.\begin{aligned}
&\ddot{\rho}-(1+\rho)(1+m+\dot{w})^2+\frac{(1+m)^2}{(1+\rho)^2}=\frac{m^2}{2}\eta(1+p)\Big\{\big[1+3\cos 2(\tau+w)\big]\\
&\qquad\qquad+\tfrac{3}{4}Mm(1+p)\big[3\cos(\tau+w)+5\cos 3(\tau+w)\big]+\cdots\Big\},\\
&(1+\rho)\,\ddot{w}+2\dot{\rho}\,(1+m+\dot{w})=-\frac{m^2}{2}\eta(1+p)\Big\{3\sin 2(\tau+w)\\
&\qquad\qquad+\tfrac{3}{4}Mm(1+p)\big[\sin(\tau+w)+5\sin 3(\tau+w)\big]+\cdots\Big\},
\end{aligned}\right\} \tag{21}$$

where

$$\eta=\frac{m_2}{m_1+m_2}.$$

In the right member of the first equation of (21) the coefficient of m^j is a sum of cosines of integral multiples of $\tau+w$, the highest multiple being j; the coefficient of m^j in the right member of the second equation is a sum of sines of integral multiples of $(\tau+w)$, the highest multiple being j.

In a closed orbit around m_1 there are two points at which w is zero. The arbitrary t_0 will be so determined that $w(0)=0$. The first condition of (8) will not be imposed in advance, and it will be shown that it is a consequence of the others. It will follow from this that all of the periodic solutions of the type under consideration are symmetrical. Equations (21) will therefore be integrated in the form

$$\rho=\sum_{j=2}^{\infty}\rho_j m^j,\qquad w=\sum_{j=2}^{\infty}w_j m^j \tag{22}$$

subject to the conditions (18) and the second of (19).

*A different transformation was made in *Transactions of the American Mathematical Society*, vol. VII, (1906), p. 542.

184. Coefficients of m².—These terms are defined by the equations

$$\ddot{\rho}_2-3\rho_2-2\dot{w}_2=\frac{1}{2}\eta\,(1+3\cos 2\tau),\qquad \ddot{w}_2+2\dot{\rho}_2=-\frac{3}{2}\eta\sin 2\tau,$$

the general solution of which is

$$\left.\begin{aligned}\rho_2&=\frac{1}{2}\eta+2c_1^{(2)}+c_2^{(2)}\cos\tau+c_3^{(2)}\sin\tau-\eta\cos 2\tau,\\ w_2&=c_4^{(2)}-(\eta+3c_1^{(2)})\tau-2c_2^{(2)}\sin\tau+2c_3^{(2)}\cos\tau+\frac{11}{8}\eta\sin 2\tau,\end{aligned}\right\}\quad(23)$$

where $c_1^{(2)}, \ldots, c_4^{(2)}$ are the constants of integration. By conditions (18) and the second of (19), it follows that

$$c_1^{(2)}=-\frac{1}{3}\eta,\qquad c_4^{(2)}=-2c_3^{(2)}.\qquad(24)$$

Therefore the solution (23) becomes

$$\left.\begin{aligned}\rho_2&=-\frac{1}{6}\eta+c_2^{(2)}\cos\tau+c_3^{(2)}\sin\tau-\eta\cos 2\tau,\\ w_2&=-2c_3^{(2)}-2c_2^{(2)}\sin\tau+2c_3^{(2)}\cos\tau+\frac{11}{8}\eta\sin 2\tau,\end{aligned}\right\}\quad(25)$$

where $c_2^{(2)}$ and $c_3^{(2)}$ are constants which remain so far undetermined.

185. Coefficients of m³.—The differential equations which define the terms of the third degree in m are

$$\left.\begin{aligned}\ddot{\rho}_3-3\rho_3-2\dot{w}_3&=6\rho_2+2\dot{w}_2+\frac{3}{8}M\eta\,[3\cos\tau+5\cos 3\tau]\\ &=-\eta+\left(2c_2^{(2)}+\frac{9}{8}M\eta\right)\cos\tau+2c_3^{(2)}\sin\tau-\frac{1}{2}\eta\cos 2\tau+\frac{15}{8}\eta M\cos 3\tau,\\ \ddot{w}_3+2\dot{\rho}_3&=-2\dot{\rho}_2-\frac{3}{8}M\eta\,[\sin\tau+5\sin 3\tau]\\ &=\left(2c_2^{(2)}-\frac{3}{8}M\eta\right)\sin\tau-2c_3^{(2)}\cos\tau-4\eta\sin 2\tau-\frac{15}{8}M\eta\sin 3\tau.\end{aligned}\right\}\quad(26)$$

From the second of these equations it is found that

$$\dot{w}_3=-2\rho_3+c_1^{(3)}-\left(2c_2^{(2)}-\frac{3}{8}M\eta\right)\cos\tau-2c_3^{(2)}\sin\tau+2\eta\cos 2\tau+\frac{5}{8}M\eta\cos 3\tau,\quad(27)$$

which substituted in the first gives

$$\ddot{\rho}_3+\rho_3=-\eta+2c_1^{(3)}-\left(2c_2^{(2)}-\frac{15}{8}M\eta\right)\cos\tau-2c_3^{(2)}\sin\tau+\frac{7}{2}\eta\cos 2\tau+\frac{25}{8}M\eta\cos 3\tau.\quad(28)$$

In order that the solution of this equation shall be periodic the coefficients of $\cos\tau$ and $\sin\tau$ must be zero; whence

$$c_2^{(2)} = \frac{15}{16} M\eta, \qquad c_3^{(2)} = 0, \tag{29}$$

after which the general solution of (28) becomes

$$\rho_3 = -\eta + 2c_1^{(3)} + c_2^{(3)}\cos\tau + c_3^{(3)}\sin\tau - \frac{7}{6}\eta\cos 2\tau - \frac{25}{64}M\eta\cos 3\tau, \tag{30}$$

where $c_1^{(3)}$, $c_2^{(3)}$, and $c_3^{(3)}$ are undetermined constants. The result of substituting this expression in (27) is

$$\dot{w}_3 = 2\eta - 3c_1^{(3)} - \left(2c_2^{(3)} + \frac{3}{2}M\eta\right)\cos\tau - 2c_3^{(3)}\sin\tau + \frac{13}{3}\eta\cos 2\tau + \frac{45}{32}M\eta\cos 3\tau. \tag{31}$$

In order that the solution of this equation shall be periodic the right member must contain no constant term; whence

$$c_1^{(3)} = \frac{2}{3}\eta. \tag{32}$$

With this value of $c_1^{(3)}$ it is found, upon integrating (31) and imposing the second condition of (19), that, so far as the computation has been made,

$$\left.\begin{aligned}
\rho_2 &= -\frac{1}{6}\eta + \frac{15}{16}M\eta\cos\tau - \eta\cos 2\tau, \qquad w_2 = -\frac{15}{8}M\eta\sin\tau + \frac{11}{8}\eta\sin 2\tau,\\
\rho_3 &= +\frac{1}{3}\eta + c_2^{(3)}\cos\tau + c_3^{(3)}\sin\tau - \frac{7}{6}\eta\cos 2\tau - \frac{25}{64}M\eta\cos 3\tau,\\
w_3 &= -2c_3^{(3)} + 2c_3^{(3)}\cos\tau - \left(2c_2^{(3)} + \frac{3}{2}M\eta\right)\sin\tau + \frac{13}{6}\eta\sin 2\tau + \frac{15}{32}M\eta\sin 3\tau,
\end{aligned}\right\} \tag{33}$$

where $c_2^{(3)}$ and $c_3^{(3)}$ are constants which are as yet undetermined.

186. Coefficients of m⁴.—The integration will be carried one step further and then the induction to the general term of the solution will be made. The differential equations which define the coefficients of m^4 are

$$\left.\begin{aligned}
\ddot{\rho}_4 - 3\rho_4 - 2\dot{w}_4 &= 6\rho_3 - 3\rho_2^2 + 2\rho_2\dot{w}_2 + 2\dot{w}_3 + 3\rho_2 + \dot{w}_2^2 + \frac{1}{2}\eta\rho_2 + \frac{3}{2}\rho_2\cos 2\tau\\
&\quad - 3w_2\sin 2\tau + \frac{1}{16}M^2\eta\,[9 + 20\cos 2\tau + 35\cos 4\tau],\\
\ddot{w}_4 + 2\dot{\rho}_4 &= -\rho_2\ddot{w}_2 - 2\dot{\rho}_3 - 2\dot{\rho}_2\dot{w}_2 - \frac{3}{2}\rho_2\sin 2\tau - 3w_2\cos 2\tau\\
&\quad - \frac{5}{16}M^2\eta\,[2\sin 2\tau + 7\sin 4\tau].
\end{aligned}\right\} \tag{34}$$

Upon developing the explicit values of the right members of (34) by means of (33), it is found that

$$\left.\begin{aligned}
\ddot{\rho}_4-3\rho_4-2\dot{w}_4=&\left[-\tfrac{331}{96}\eta^2+\tfrac{3}{2}\eta-\tfrac{675}{512}M^2\eta^2+\tfrac{9}{16}M^2\eta\right]+2c_3^{(3)}\sin\tau\\
&+\left[2c_2^{(3)}+\tfrac{245}{32}M\eta^2-\tfrac{3}{16}M\eta\right]\cos\tau+\left[-\tfrac{8}{3}\eta^2-\tfrac{4}{3}\eta-\tfrac{675}{512}M^2\eta^2\right.\\
&\left.+\tfrac{5}{4}M^2\eta\right]\cos 2\tau+\tfrac{15}{32}M\eta\cos 3\tau+\left[\tfrac{27}{32}\eta^2+\tfrac{35}{16}M^2\eta\right]\cos 4\tau,\\
\ddot{w}_4+2\dot{\rho}_4=&-2c_3^{(3)}\cos\tau+\left[2c_2^{(3)}-\tfrac{25}{64}M\eta^2\right]\sin\tau\\
&+\left[-\tfrac{2}{3}\eta^2-\tfrac{14}{3}\eta-\tfrac{675}{256}M^2\eta^2-\tfrac{5}{8}M^2\eta\right]\sin 2\tau\\
&+\left[\tfrac{765}{64}M\eta^2-\tfrac{75}{32}M\eta\right]\sin 3\tau+\left[-\tfrac{153}{16}\eta^2-\tfrac{35}{16}M^2\eta\right]\sin 4\tau.
\end{aligned}\right\}\quad(35)$$

The first integral of the second of these equations is

$$\left.\begin{aligned}
\dot{w}_4=&-2\rho_4+c_1^{(4)}-2c_3^{(3)}\sin\tau-\left[2c_2^{(3)}-\tfrac{25}{64}M\eta^2\right]\cos\tau\\
&+\left[\tfrac{1}{3}\eta^2+\tfrac{7}{3}\eta+\tfrac{675}{512}M^2\eta^2+\tfrac{5}{16}M^2\eta\right]\cos 2\tau\\
&-\left[\tfrac{255}{64}M\eta^2-\tfrac{25}{32}M\eta\right]\cos 3\tau+\left[\tfrac{153}{64}\eta^2+\tfrac{35}{64}M^2\eta\right]\cos 4\tau,
\end{aligned}\right\}\quad(36)$$

where $c_1^{(4)}$ is an undetermined constant. Then, on substituting this value of $\dot{w}_4$, the first equation of (35) becomes

$$\left.\begin{aligned}
\ddot{\rho}_4+\rho_4=&\left[2c_1^{(4)}-\tfrac{331}{96}\eta^2+\tfrac{3}{2}\eta-\tfrac{675}{512}M^2\eta^2+\tfrac{9}{16}M^2\eta\right]-2c_3^{(3)}\sin\tau\\
&+\left[-2c_2^{(3)}+\tfrac{135}{16}M\eta^2-\tfrac{3}{16}M\eta\right]\cos\tau+\left[-2\eta^2+\tfrac{10}{3}\eta\right.\\
&\left.+\tfrac{675}{512}M^2\eta^2+\tfrac{15}{8}M^2\eta\right]\cos 2\tau+\left[-\tfrac{255}{32}M\eta^2+\tfrac{65}{32}M\eta\right]\cos 3\tau\\
&+\left[\tfrac{45}{8}\eta^2+\tfrac{105}{32}M^2\eta\right]\cos 4\tau.
\end{aligned}\right\}\quad(37)$$

In order that the solution of (37) shall be periodic, the conditions

$$c_2^{(3)}=\tfrac{135}{32}M\eta^2-\tfrac{3}{32}M\eta,\qquad c_3^{(3)}=0 \quad(38)$$

must be satisfied. Then its general solution becomes

$$\left.\begin{aligned}
\rho_4=&\,2c_1^{(4)}-\tfrac{331}{96}\eta^2+\tfrac{3}{2}\eta-\tfrac{675}{512}M^2\eta^2+\tfrac{9}{16}M^2\eta+c_2^{(4)}\cos\tau+c_3^{(4)}\sin\tau\\
&+\left[+\tfrac{2}{3}\eta^2-\tfrac{10}{9}\eta-\tfrac{225}{512}M^2\eta^2-\tfrac{5}{8}M^2\eta\right]\cos 2\tau\\
&+\left[+\tfrac{255}{256}M\eta^2-\tfrac{65}{256}M\eta\right]\cos 3\tau+\left[-\tfrac{3}{8}\eta^2-\tfrac{7}{32}M^2\eta\right]\cos 4\tau,
\end{aligned}\right\}\quad(39)$$

where $c_1^{(4)}$, $c_2^{(4)}$, and $c_3^{(4)}$ are as yet undetermined constants.

If (39) is substituted in (36), it is found, by using (38), that

$$\left.\begin{aligned}\dot{w}_4 = &-3c_1^{(4)}+\tfrac{331}{48}\eta^2-3\eta+\tfrac{675}{256}M^2\eta^2-\tfrac{9}{8}M^2\eta+\left[-2c_2^{(4)}-\tfrac{515}{64}M\eta^2+\tfrac{3}{16}M\eta\right]\cos\tau\\ &-2c_3^{(4)}\sin\tau+\left[-\eta^2+\tfrac{41}{9}\eta+\tfrac{1125}{512}M^2\eta^2+\tfrac{25}{16}M^2\eta\right]\cos2\tau\\ &+\left[-\tfrac{765}{128}M\eta^2+\tfrac{165}{128}M\eta\right]\cos3\tau+\left[\tfrac{201}{64}\eta^2+\tfrac{63}{64}M^2\eta\right]\cos4\tau.\end{aligned}\right\}\quad(40)$$

The periodicity condition determines $c_1^{(4)}$ by the equation

$$c_1^{(4)}=+\tfrac{331}{144}\eta^2-\eta+\tfrac{225}{256}M^2\eta^2-\tfrac{3}{8}M^2\eta. \quad (41)$$

Then the integral of (40) satisfying the second condition of (19) is

$$\left.\begin{aligned}w_4 = &-2c_3^{(4)}+2c_3^{(4)}\cos\tau-\left[2c_2^{(4)}+\tfrac{515}{64}M\eta^2-\tfrac{3}{16}M\eta\right]\sin\tau\\ &+\left[-\tfrac{1}{2}\eta^2+\tfrac{41}{18}\eta+\tfrac{1125}{1024}M^2\eta^2+\tfrac{25}{32}M^2\eta\right]\sin2\tau\\ &+\left[-\tfrac{255}{128}M\eta^2+\tfrac{55}{128}M\eta\right]\sin3\tau+\left[\tfrac{201}{256}\eta^2+\tfrac{63}{256}M^2\eta\right]\sin4\tau.\end{aligned}\right\}\quad(42)$$

The results so far obtained are

$$\left.\begin{aligned}&\rho_2=-\tfrac{1}{6}\eta+\tfrac{15}{16}M\eta\cos\tau-\eta\cos2\tau,\quad w_2=-\tfrac{15}{8}M\eta\sin\tau+\tfrac{11}{8}\eta\sin2\tau,\\ &\rho_3=+\tfrac{1}{3}\eta+\left[\tfrac{135}{32}M\eta^2-\tfrac{3}{32}M\eta\right]\cos\tau-\tfrac{7}{6}\eta\cos2\tau-\tfrac{25}{64}M\eta\cos3\tau,\\ &w_3=-\left[\tfrac{135}{16}M\eta^2+\tfrac{21}{16}M\eta\right]\sin\tau+\tfrac{13}{6}\eta\sin2\tau+\tfrac{15}{32}M\eta\sin3\tau,\\ &\rho_4=+\tfrac{331}{288}\eta^2-\tfrac{1}{2}\eta+\tfrac{225}{512}M^2\eta^2-\tfrac{3}{16}M^2\eta+c_2^{(4)}\cos\tau+c_3^{(4)}\sin\tau\\ &\qquad+\left[+\tfrac{2}{3}\eta^2-\tfrac{10}{9}\eta-\tfrac{225}{512}M^2\eta^2-\tfrac{5}{8}M^2\eta\right]\cos2\tau\\ &\qquad+\left[+\tfrac{255}{256}M\eta^2-\tfrac{65}{256}M\eta\right]\cos3\tau-\left[\tfrac{3}{8}\eta^2+\tfrac{7}{32}M^2\eta\right]\cos4\tau,\\ &w_4=-2c_3^{(4)}+2c_3^{(4)}\cos\tau-\left[2c_2^{(4)}+\tfrac{515}{64}M\eta^2-\tfrac{3}{16}M\eta\right]\sin\tau\\ &\qquad+\left[-\tfrac{1}{2}\eta^2+\tfrac{41}{18}\eta+\tfrac{1125}{1024}M^2\eta^2+\tfrac{25}{32}M^2\eta\right]\sin2\tau\\ &\qquad+\left[-\tfrac{255}{128}M\eta^2+\tfrac{55}{128}M\eta\right]\sin3\tau+\left[\tfrac{201}{256}\eta^2+\tfrac{63}{256}M^2\eta\right]\sin4\tau,\end{aligned}\right\}\quad(43)$$

where $\eta=\dfrac{m_2}{m_1+m_2}$, and where $c_2^{(4)}$ and $c_3^{(4)}$ are so far undetermined.

It is observed that, so far as the variables are completely determined, the ρ_j and w_j are sums of cosines and sines respectively of integral multiples of τ, the highest multiple being j. At the j^{th} step of the integration one of the four arbitrary constants which arise at that step is determined by the periodicity condition on the w_j, and another by the initial condition on the w_j. The other two constants remain undetermined until the next step, but two which arose at the preceding step are determined by the periodicity condition on ρ_j. It will be shown that these properties are general.

187. Induction to the General Step of the Integration.—Suppose $\rho_2, \ldots, \rho_{n-1}; w_2, \ldots, w_{n-1}$ have been computed and have the properties expressed in the following equations:

$$\left.\begin{aligned}
\rho_j &= a_0^{(j)} + a_1^{(j)}\cos\tau + a_2^{(j)}\cos 2\tau + \cdots + a_j^{(j)}\cos j\tau \qquad (j=2, \ldots, n-2),\\
w_j &= \qquad \beta_1^{(j)}\sin\tau + \beta_2^{(j)}\sin 2\tau + \cdots + \beta_j^{(j)}\sin j\tau \qquad (j=2, \ldots, n-2),\\
\rho_{n-1} &= +c_3^{(n-1)}\sin\tau + a_0^{(n-1)} + c_2^{(n-1)}\cos\tau + a_2^{(n-1)}\cos 2\tau + \cdots + a_{n-1}^{(n-1)}\cos(n-1)\tau,\\
w_{n-1} &= -2c_3^{(n-1)} + 2c_3^{(n-1)}\cos\tau + [-2c_2^{(n-1)} + b_2^{(n-1)}]\sin\tau + \beta_2^{(n-1)}\sin 2\tau\\
&\qquad + \cdots + \beta_{n-1}^{(n-1)}\sin(n-1)\tau,
\end{aligned}\right\} \quad (44)$$

where the $a_k^{(j)}$, $\beta_k^{(j)}$, and $b_2^{(n-1)}$ are known constants, and $c_2^{(n-1)}$ and $c_3^{(n-1)}$ are undetermined constants.

In writing the differential equations which define ρ_n and w_n all unknown quantities will be given explicitly. The terms involving these undetermined coefficients are the same at every step. It is found from equations (7) that the coefficients of m^n are defined by

$$\left.\begin{aligned}
\ddot{\rho}_n - 3\rho_n - 2\dot{w}_n &= +2c_2^{(n-1)}\cos\tau + 2c_3^{(n-1)}\sin\tau + P_n(\rho_j, w_j, \dot{w}_j; \tau),\\
\ddot{w}_n + 2\dot{\rho}_n &= +2c_2^{(n-1)}\sin\tau - 2c_3^{(n-1)}\cos\tau + Q_n(\rho_j, \dot{\rho}_j, w_j, \dot{w}; \tau).
\end{aligned}\right\} \quad (45)$$

where P_n and Q_n are polynomials in ρ_j, $\dot{\rho}_j$, w_j, and $\dot{w}_j$ $(j=2, \ldots, n-2)$ and the known parts of ρ_{n-1}, $\dot{\rho}_{n-1}$, w_{n-1}, and $\dot{w}_{n-1}$, and where τ enters in the coefficients only in sines and cosines.

It follows from (7) that, aside from numerical coefficients, P_n has terms of the types

$$\begin{aligned}
P_n^{(1)} &= \rho_{j_1}\dot{w}_{j_2} && (j_1+j_2=n, \text{ or } n-1),\\
P_n^{(2)} &= \rho_{j_1}\dot{w}_{j_2}\dot{w}_{j_3} && (j_1+j_2+j_3=n),\\
P_n^{(3)} &= \rho_{j_1}^{k_1}\cdots\rho_{j_\nu}^{k_\nu} && (k_1 j_1+\cdots+k_\nu j_\nu=n,\ n-1, \text{ or } n-2),\\
P_n^{(4)} &= M^j\rho_{j_1}^{k_1}\cdots\rho_{j_\nu}^{k_\nu} w_{p_1}^{\lambda_1}\cdots w_{p_\mu}^{\lambda_\mu}{\cos i\tau \atop \sin i\tau} && (j=0, 1, \ldots, n-2;\ k_1+\cdots+k_\nu \leqq j+1;\\
&&& i \leqq j+2;\ j+k_1 j_1+\cdots+k_\nu j_\nu+\lambda_1 p_1+\cdots+\lambda_\mu p_\mu=n-2).
\end{aligned}$$

If $\lambda_1+\cdots+\lambda_\mu$ is even, the term is multiplied by $\cos i\tau$; and if $\lambda_1+\cdots+\lambda_\mu$ is odd, the term is multiplied by $\sin i\tau$. The terms $P_n^{(1)}$, $P_n^{(2)}$, and $P_n^{(3)}$ come from the left member of (7), and $P_n^{(4)}$ comes from the right member.

It follows at once from (44) and the conditions on the j_ι and k_ι that $P_n^{(1)}$, $P_n^{(2)}$, and $P_n^{(3)}$ are sums of cosines of integral multiples τ, the highest multiple being n at most. If $\lambda_1 + \cdots + \lambda_\mu$ is even, the product $w_{p_1}^{\lambda_1} \cdots w_{p_\mu}^{\lambda_\mu}$ is a sum of cosine terms, and it follows therefore that in this case $P_n^{(4)}$ is a sum of cosines of integral multiples of τ. The highest multiple of τ is

$$N = k_1 j_1 + \cdots + k_\nu j_\nu + \lambda_1 p_1 + \cdots + \lambda_\mu p_\mu + i,$$

which becomes, as a consequence of the relations to which the exponents and subscripts are subject,

$$N = n - 2 + i - j \leqq n - 2 + j + 2 - j = n.$$

If $\lambda_1 + \cdots + \lambda_\mu$ is odd, the product $w_{p_1}^{\lambda_1} \cdots w_{p_\mu}^{\lambda_\mu}$ is a sum of sines of integral multiples of τ. Therefore, in this case also, $P_n^{(4)}$ is a sum of cosines of integral multiples of τ; and it is shown, precisely as before, that the highest multiple is n. Hence the general conclusion is that P_n is a sum of cosines of integral multiples of τ, the highest multiple being n.

By a similar discussion it can be proved that Q_n is a sum of sines of integral multiples of τ, the highest being n. Hence equations (45) can be written in the form

$$\left.\begin{aligned} \ddot{\rho}_n - 3\rho_n - 2\dot{w}_n &= +2c_3^{(n-1)} \sin\tau + A_0^{(n)} + [2c_2^{(n-1)} + A_1^{(n)}]\cos\tau \\ &\quad + A_2^{(n)} \cos 2\tau + \cdots + A_n^{(n)} \cos n\tau, \\ \ddot{w}_n + 2\dot{\rho}_n &= -2c_3^{(n-1)} \cos\tau + [2c_2^{(n-1)} + B_1^{(n)}] \sin\tau \\ &\quad + B_2^{(n)} \sin 2\tau + \cdots + B_n^{(n)} \sin n\tau, \end{aligned}\right\} \quad (46)$$

where the $A_j^{(n)}$ and the $B_j^{(n)}$ $(j = 0, \ldots, n)$ are known constants.

The first integral of the second equation of (46) is

$$\left.\begin{aligned} \dot{w}_n = -2\rho_n + c_1^{(n)} - 2c_3^{(n-1)} \sin\tau - [2c_2^{(n-1)} + B_1^{(n)}] \cos\tau - \tfrac{1}{2} B_2^{(n)} \cos 2\tau \\ - \cdots - \frac{1}{n} B_n^{(n)} \cos n\tau, \end{aligned}\right\} \quad (47)$$

where $c_1^{(n)}$ is an undetermined constant.

On substituting equation (47) in the first of (46), it is found that

$$\left.\begin{aligned} \ddot{\rho}_n + \rho_n = -2c_3^{(n-1)} \sin\tau + [2c_1^{(n)} + A_0^{(n)}] + [-2c_2^{(n-1)} + A_1^{(n)} - 2B_1^{(n)}] \cos\tau \\ + \left[A_2^{(n)} - \tfrac{2}{2} B_2^{(n)}\right] \cos 2\tau + \cdots + \left[A_n^{(n)} - \frac{2}{n} B_n^{(n)}\right] \cos n\tau. \end{aligned}\right\} \quad (48)$$

In order that the solution of this equation shall be periodic, the conditions

$$c_3^{(n-1)} = 0, \qquad 2c_2^{(n-1)} = A_1^{(n)} - 2B_1^{(n)} \quad (49)$$

must be imposed. They uniquely determine the constants $c_3^{(n-1)}$ and $c_2^{(n-1)}$, which remained undetermined at the preceding step of the integration.

After equations (49) are fulfilled, the general solution of (48) is of the form

$$\rho_n = c_3^{(n)} \sin\tau + a_0^{(n)} + c_2^{(n)} \cos\tau + a_2^{(n)} \cos 2\tau + \cdots + a_n^{(n)} \cos n\tau, \tag{50}$$

where $c_3^{(n)}$ and $c_2^{(n)}$ are arbitrary constants, and where

$$\left.\begin{aligned} a_0^{(n)} &= 2c_1^{(n)} + A_0^{(n)}, \\ a_j^{(n)} &= -\frac{1}{j^2-1}\left[A_j^{(n)} - \frac{2}{j}B_j^{(n)}\right] = -\frac{[jA_j^{(n)} - 2B_j^{(n)}]}{j(j^2-1)} \qquad (j=2, \ldots, n). \end{aligned}\right\} \tag{51}$$

If equations (49), (50), and (51) are substituted in (47), the result is

$$\left.\begin{aligned} \dot{w}_n = &-\left[3c_1^{(n)} + 2A_0^{(n)}\right] - 2c_3^{(n)}\sin\tau - \left[2c_2^{(n)} + A_1^{(n)} - B_1^{(n)}\right]\cos\tau \\ &- \left[2a_2^{(n)} + \frac{1}{2}B_2^{(n)}\right]\cos 2\tau - \cdots - \left[2a_j^{(n)} + \frac{1}{j}B_j^{(n)}\right]\cos j\tau - \cdots \\ &- \left[2a_n^{(n)} + \frac{1}{n}B_n^{(n)}\right]\cos n\tau. \end{aligned}\right\} \tag{52}$$

In order that the solution of this equation shall be periodic, its right member must contain no constant term; whence

$$c_1^{(n)} = -\frac{2}{3}A_0^{(n)}. \tag{53}$$

Then the integral of (52) satisfying the condition $w_n(0) = 0$ is

$$\left.\begin{aligned} w_n = &-2c_3^{(n)} + 2c_3^{(n)}\cos\tau - \left[2c_2^{(n)} + A_1^{(n)} - B_1^{(n)}\right]\sin\tau \\ &- \frac{1}{2}\left[2a_2^{(n)} + \frac{1}{2}B_2^{(n)}\right]\sin 2\tau - \cdots - \frac{1}{j}\left[2a_j^{(n)} + \frac{1}{j}B_j^{(n)}\right]\sin j\tau \\ &- \cdots - \frac{1}{n}\left[2a_n^{(n)} + \frac{1}{n}B_n^{(n)}\right]\cos n\tau. \end{aligned}\right\} \tag{54}$$

The results obtained at this step are

$$\left.\begin{aligned} &\rho_n = c_3^{(n)}\sin\tau + a_0^{(n)} + c_2^{(n)}\cos\tau + a_2^{(n)}\cos 2\tau + \cdots + a_j^{(n)}\cos j\tau \\ &\qquad + \cdots + a_n^{(n)}\cos n\tau, \\ &w_n = -2c_3^{(n)} + 2c_3^{(n)}\cos\tau - [2c_2^{(n)} + A_1^{(n)} - B_1^{(n)}]\sin\tau + \beta_2^{(n)}\sin 2\tau \\ &\qquad + \cdots + \beta_n^{(n)}\sin n\tau, \\ &a_0^{(n)} = -\frac{1}{3}A_0^{(n)}, \qquad a_j^{(n)} = -\frac{[jA_j^{(n)} - 2B_j^{(n)}]}{j(j^2-1)} \qquad (j=2, \ldots, n), \\ &c_2^{(n-1)} = \frac{A_1^{(n)} - 2B_1^{(n)}}{2}, \qquad \beta_j^{(n)} = +\frac{[2jA_j^{(n)} - (j^2+3)B_j^{(n)}]}{j^2(j^2-1)} \qquad (j=2, \ldots, n), \\ &c_3^{(n-1)} = 0, \end{aligned}\right\} \tag{55}$$

where $c_2^{(n)}$ and $c_3^{(n)}$ are as yet undetermined constants.

Since the results expressed in the first two equations of this set are identical in properties with the equations (44), with which the discussion of the general step was started, and since the properties of (44) were fulfilled for the subscripts 2, 3, and 4, it follows that the induction is complete. The process of integration can be carried as far as may be desired.

The hypotheses under which the discussion has been made are that the solutions are periodic and that $w(0)=0$. Solutions satisfying these properties and $\dot{\rho}(0)=0$ were known to exist from the existence discussion, and therefore they could certainly be found because the assumed properties are included in those of the symmetrical orbits. It appears in the construction that the hypotheses adopted imply also that $\dot{\rho}(0)=0$. Therefore *all periodic orbits of the type under discussion which are expansible as power series in m are symmetrical orbits.* It can be shown by direct consideration of the series that there are no others expansible in any fractional powers of m.

188. Application of Jacobi's Integral.—The differential equations admit Jacobi's integral, of which no use has yet been made, and it is the only integral not involving the independent variable. Upon transforming the integral to the variables used in this chapter, it is found without difficulty that its explicit form is

$$\left.\begin{aligned}&\dot{\rho}^2+(1+\rho)^2(1+\dot{w})^2-m^2(1+\rho)^2-\frac{2(1+m)^2}{1+\rho}\\&-2m^2\left(\frac{A}{a}\right)^2\frac{m_2}{m_1+m_2}\left\{\left[1-2\frac{a}{A}(1+\rho)\cos(\tau+w)+\left(\frac{a}{A}\right)^2(1+\rho)^2\right]^{-\frac{1}{2}}\right.\\&\qquad\qquad\left.-\frac{a}{A}(1+\rho)\cos(\tau+w)\right\}=-C,\end{aligned}\right\}\qquad(56)$$

where C is the constant of integration. It will be shown that this integral can be used as a searching test on the accuracy of the computations of the solutions, or to replace the second differential equation of (7).

Since the periodic solutions are developable as power series in m, the integral can be expanded as a power series in m and written in the form

$$F_0+F_1(\rho_j,\dot{\rho}_j,w_j,\dot{w}_j;\tau)m+\cdots+F_n(\rho_j,\dot{\rho}_j,w_j,\dot{w}_j;\tau)m^n+\cdots=C.\qquad(57)$$

In the F_n the highest value of j is n. Since the integral converges for all $|m|$ sufficiently small, each F_n separately is constant, and therefore

$$F_n(\rho_j,\dot{\rho}_j,w_j,\dot{w}_j;\tau)=C_n.\qquad(58)$$

It follows from the form of (55) and (56) that F_n is a sum of cosines of integral multiples of τ. In F_n the sum of the products of the exponents and subscripts of the factors of any term not involving $\cos j\tau$ or $\sin j\tau$ can not exceed n; and in any term involving $\cos j\tau$ or $\sin j\tau$ the sum can not exceed $n-i$. Therefore the highest multiple of τ in F_n is n, and (58) can be written in the form

$$F_n=\gamma_0^{(n)}+\gamma_1^{(n)}\cos\tau+\cdots+\gamma_j^{(n)}\cos j\tau+\cdots+\gamma_n^{(n)}\cos n\tau=C_n.\qquad(59)$$

Since this relation is an identity in τ, it follows that

$$\gamma_0^{(n)}=C_n, \qquad \gamma_j^{(n)}=0 \qquad (j=1, \ldots, n). \tag{60}$$

These relations are functions of $\alpha_k^{(2)}, \ldots, \alpha_k^{(n)}$; $\beta_k^{(2)}, \ldots, \beta_k^{(n)}$, and their fulfillment for $j=1, \ldots, n$ serves as a thorough check on the expansion of U and on all of the computations.

It will now be shown how the relations (60) can be used in place of the second equation of (7). If (56) is expanded as a power series in m, it is found that

$$F_n=4\rho_n+2\dot{w}_n+4\rho_{n-1}+G_n(\rho_j, \dot{\rho}_j, w_j, \dot{w}_j; \tau), \tag{61}$$

where G_n is a polynomial in ρ_j, $\dot{\rho}_j$, w_j, and $\dot{w}_j$ and involves τ only in sines and cosines. Moreover, the greatest value of j in G_n is $n-2$. Suppose that $\rho_2, \ldots, \rho_{n-2}$; $w_2, \ldots, w_{n-2}$ are entirely known, and that ρ_{n-1} and w_{n-1} are known except for the undetermined coefficients $c_2^{(n-1)}$; it will be shown that equations (60) and the third and fourth equations of (55) define the $\alpha_j^{(n)}$ and $\beta_j^{(n)}$ uniquely.

It follows from the properties of F_n and equation (61) that this function can be written in the form

$$\begin{aligned}F_n=&[4\alpha_0^{(n)}+C_0^{(n)}]+[4c_2^{(n-2)}-2A_1^{(n)}+2B_1^{(n)}+C_1^{(n)}]\cos\tau\\&+[4\alpha_2^{(n)}+4\beta_2^{(n)}+C_2^{(n)}\cos 2\tau+\cdots+[4\alpha_j^{(n)}+2j\beta_j^{(n)}+C_j^{(n)}]\cos j\tau\\&+\cdots+[4\alpha_n^{(n)}+2n\beta_n^{(n)}+C_n^{(n)}]\cos n\tau.\end{aligned}$$

Consequently equations (60) become

$$\left.\begin{aligned}&\gamma_0^{(n)}=4\alpha_0^{(n)}+C_0^{(n)}=C_n,\\&\gamma_1^{(n)}=4c_2^{(n-1)}-2A_1^{(n)}+2B_1^{(n)}+C_1^{(n)}=0,\\&\gamma_j^{(n)}=4\alpha_j^{(n)}+2j\beta_j^{(n)}+C_j^{(n)}=0 \qquad (j=2, \ldots, n).\end{aligned}\right\} \tag{62}$$

It follows from equations (55) that

$$\left.\begin{aligned}&4c_2^{(n-1)}-2A_j^{(n)}+4B_1^{(n)}=0,\\&4\alpha_j^{(n)}+2j\beta_j^{(n)}+\frac{2}{j}B_j^{(n)}=0 \qquad (j=2, \ldots, n).\end{aligned}\right\} \tag{63}$$

Upon comparing equations (62) and (63), it is found that

$$2B_j^{(n)}=jC_j^{(n)} \qquad (n=2, \ldots \infty;\ j=1, \ldots, n). \tag{64}$$

Therefore the third to the seventh equations of (55) can be written

$$\left.\begin{aligned}&\alpha_0^{(n)}=-\tfrac{1}{3}A_0^{(n)}, \qquad c_2^{(n-1)}=\frac{A_1^{(n)}-C_1^{(n)}}{2}, \qquad c_3^{(n-1)}=0 \qquad (j=2, \ldots, n),\\&\alpha_j^{(n)}=-\frac{[A_j^{(n)}-C_j^{(n)}]}{(j^2-1)}, \qquad \beta_j^{(n)}=\frac{4A_j^{(n)}-(j^2+3)C_j^{(n)}}{2j(j^2-1)} \qquad (j=2, \ldots, n),\end{aligned}\right\} \tag{65}$$

These equations express the coefficients, which are determined at this step of the integration, uniquely in terms of constants which depend only upon the first equation of (7) and upon the integral. In practical computation it is more convenient to make the determination of the coefficients depend upon the $A_j^{(n)}$ and $C_j^{(n)}$ than upon the $A_j^{(n)}$ and $B_j^{(n)}$, for the former have many terms in common, except for numerical multipliers, and both are coefficients of cosine series, which are easier to check than are the sine series on which the $B_j^{(n)}$ depend. But the chances of error in lengthy computations are so great that if the developments are to be made to high powers of m, the only safe method is to use both the second equation of (7) and the integral, or, what is the same thing, to secure the fulfillment of equations (64).

In order to illustrate the process the expression for F_4 will be developed. It is found from equation (56) that

$$\left.\begin{aligned} F_4 = {} & 4\rho_4 + 2\dot{w}_4 + 4\rho_3 + \dot{\rho}_2^2 - \rho_2^2 + \dot{w}_2^2 + 4\rho_2\dot{w}_2 - \rho_2\eta - 3\rho_2\eta\cos 2\tau \\ & + 3w_2\eta\sin 2\tau + \tfrac{1}{32}M^2\eta\left[9 + 20\cos 2\tau + 35\cos 4\tau\right]. \end{aligned}\right\} \quad (66)$$

Upon developing the right member explicitly by means of the first four equations of (43), it is found that

$$\begin{aligned} F_4 = {} & C_0^{(4)} + \left[4\,c_2^{(3)} - \tfrac{25}{32}\,M\eta^2\right]\cos\tau + \left[\,4a_2^{(4)} + 4\beta_2^{(4)} - \tfrac{2}{3}\eta^2 - \tfrac{14}{3}\eta \right. \\ & \left. - \tfrac{675}{256}M^2\eta^2 - \tfrac{5}{8}\,M^2\eta\right]\cos 2\tau + \left[\,4a_3^{(4)} + 6\beta_3^{(4)} + \tfrac{255}{32}M\eta^2 \right. \\ & \left. - \tfrac{25}{16}M\eta\right]\cos 3\tau + \left[4\,a_4^{(4)} + 8\,\beta_4^{(4)} - \tfrac{153}{32}\eta^2 - \tfrac{35}{32}M^2\eta\right]\cos 4\tau. \end{aligned}$$

It is found in the notation of (62), and by comparing with the right member of the second equation of (35), that

$$\left.\begin{aligned} C_1^{(4)} &= +4c_2^{(3)} - \tfrac{25}{32}M\eta^2 = 2\,B_1^{(4)}, \\ 2C_2^{(4)} &= -\tfrac{4}{3}\eta^2 - \tfrac{28}{3}\eta - \tfrac{675}{128}M^2\eta^2 - \tfrac{5}{4}M^2\eta = 2B_2^{(4)}, \\ 3C_3^{(4)} &= +\tfrac{765}{32}M\eta^2 - \tfrac{75}{16}M\eta = 2B_3^{(4)}, \\ 4C_4^{(4)} &= -\tfrac{953}{8}\eta^2 - \tfrac{35}{8}M^2\eta = 2B_4^{(4)}, \end{aligned}\right\} \quad (67)$$

exactly fulfilling equations (64).

189. The Solutions as Functions of the Jacobian Constant.—It follows from equation (57) that when the periodic solution is given, the constant C is uniquely defined. The relation between C and the constant of the Jacobian integral, when it is expressed in terms of the variables in more ordinary use, will be found. If the origin is taken at the center of gravity of the system, the differential equations of motion in rectangular coördinates are

$$\left.\begin{aligned} &x'' = \frac{\partial U}{\partial x}, \qquad y'' = \frac{\partial U}{\partial y}, \qquad U = \frac{k^2 m_1}{r_1} + \frac{k^2 m_2}{r_2}, \\ &r_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2}, \qquad r_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2}. \end{aligned}\right\} \tag{68}$$

These equations admit the integral

$$x'^2 + y'^2 - 2N(xy' - yx') = 2U - C_0, \tag{69}$$

where C_0 is the constant of the Jacobian integral and N is defined in (3). The relation between C_0 and C of equation (56) is required.

The variables x and y are expressed in terms of the polar coördinates, r and v of (1), by the equations

$$x = r\cos v - \frac{m_2}{m_1+m_2} A\cos Nt, \qquad y = r\sin v - \frac{m_2}{m_1+m_2} A\sin Nt;$$

from which it follows that

$$\left.\begin{aligned} x'^2 + y'^2 - 2N(xy' - yx') &= r'^2 + r^2 v'^2 - 2Nr^2 v' \\ &+ \frac{2m^2}{m_1+m_2} AN^2 r\cos(v - Nt) - m_2^2 A^2 N^2. \end{aligned}\right\} \tag{70}$$

Upon making the transformations (6) and referring to (3) and (5), it is easily found that

$$\left.\begin{aligned} &k^2 m_1 = n^2 a^3 = \frac{N^2}{m^2}(1+m)^2 a^3, \qquad k^2 m_2 = N^2 A^3 - n^2 a^3 = N^2 A^3 \frac{m_2}{m_1+m_2}, \\ &r'^2 = a^2 \frac{N^2}{m^2}\dot{\rho}^2, \qquad r^2 v'^2 - 2Nr^2 v' = \frac{a^2 N^2}{m^2}(1+\rho)^2(1+\dot{w})^2 - a^2 N^2 (1+\rho)^2, \\ &2m_2 A N^2 r\cos(v - Nt) = \frac{2m_2}{m_1+m_2} aAN^2(1+\rho)\cos(\tau + w), \\ &\frac{k^2 m_1}{r_1} + \frac{k^2 m_2}{r_2} = \frac{N^2(1+m)^2 a^2}{m^2(1+\rho)} \\ &\qquad + N^2 A^2 \frac{m_2}{m_1+m_2}\left[1 - 2\frac{a}{A}(1+\rho)\cos(\tau+w) + \left(\frac{a}{A}\right)^2 (1+\rho)^2\right]^{-\frac{1}{2}}. \end{aligned}\right\} \tag{71}$$

Upon substituting (71) in (70) and (69) and comparing with (56), the relation between C_0 and C is found to be

$$C_0 - m_2^2 A^2 N^2 = \frac{a^2 N^2}{m^2} C = \left(\frac{m_1}{m_1+m_2}\right)^{2/3} \frac{A^2 N^2}{(1+m)^{4/3} m^{1/3}} C. \tag{72}$$

It is seen from (56) that when C is expanded as a power series in m, or in $m^{1/3}$ if a/A is eliminated by the last equation of (6), it starts off with a term which is independent of m. Therefore C_0, the Jacobian constant for the integral in the ordinary form, is expansible as a power series in $m^{1/3}$ and is infinite for $m = 0$. The three periodic orbits, of which two are complex for $|m|$ sufficiently small, corresponding to the three determinations of a/A in (6), coincide and branch at $m=0$ or $C_0 = \infty$. Since the coördinates in the periodic orbits are analytic functions of $m^{1/3}$, and $m^{1/3}$ is an analytic function of C_0 through the inversion of (72), it follows that the coördinates in the periodic orbits are analytic functions of C_0. One branch-point is at $C_0 = \infty$. In the special problem treated by Darwin,* in which the ratio of the finite masses is 10 to 1, he found by computation in the case of the orbits around the smaller finite mass that there is another branch-point for a certain value of C_0, at which the complex orbits first become real and coincident, and then real and distinct.

190. Applications to the Lunar Theory.—In the development of the Lunar Theory the differential equations have been so treated that the resulting expression for the distance, or its reciprocal, is a sum of terms which involve the time only under the cosine and sine functions. The longitude involves terms of the same type and the time multiplied by a constant factor. Considering the problem in the plane of the ecliptic, there are terms whose period is equal to one-half the synodic period of the moon. They are known as the *variational* terms. Now the period of the periodic orbits which have been found above is the synodic period of the revolution of the infinitesimal body, or twice that of the variational terms. The terms of the solutions which are of even degree in μ have the period of half the synodical period of revolution. The variational terms in fact belong to the class of periodic orbits treated here. The detailed comparison, up to m^9, with the work of Delaunay was made in the *Transactions of the American Mathematical Society*, vol. VII (1906), p. 562, and perfect agreement was found except in the coefficients of the higher powers of m, where errors are almost unavoidable in Delaunay's complicated method.

Hill wrote a remarkable series of papers on the Lunar Theory in the *American Journal of Mathematics*, vol. I (1878), in which he proposed to start from the variational orbit, instead of from an ellipse, as an intermediate orbit for the determination of the motion of the moon. The elliptic orbit as an intermediate orbit came down from Newton and his successors, and the inertia of the human mind is such that it was retained for over a century in spite of the fact that it has little to recommend it. Hill has the great honor of initiating a new movement which, it seems certain, will be of the highest importance.

*_Acta Mathematica_, vol. XXI (1897), pp. 99–242.

The results obtained by Hill are coextensive with those given here if we put $M=0$ and $\eta=1$ in the latter series. The method employed by Hill was entirely different from that of this chapter. It was convenient in practice, but its validity can not easily be established. The same method was extended by Brown to include terms which contain M as a factor* to the first, second, and third degrees. A comparison of the results obtained by the methods of this paper with those of several writers on the Lunar Theory, especially in the coefficient of a/A which converges most slowly, will be found in the *Transactions of the American Mathematical Society*, vol. VII (1906), p. 569.

191. Applications to Darwin's Periodic Orbits.—In Darwin's computations,† the ratio of the masses of the finite bodies was ten to one. It is found from the definition of η and the last equation of (6) that for the motion around the smaller of the finite bodies

$$\eta=\frac{m_2}{m_1+m_2}=\frac{10}{11}, \qquad \frac{a}{A}=\left(\frac{m_1}{m_1+m_2}\right)^{1/3}\left(\frac{N}{n}\right)^{2/3}=\left(\frac{1}{11}\right)^{1/3}\left(\frac{m}{1+m}\right)^{2/3}. \tag{73}$$

Darwin defined his orbits by the value of the Jacobian constant, and their periods were found from the detailed computations. In comparing with his work it is simpler to take the periods which he obtained and to find the orbits from equation (43). The comparison will be made first with his "Satellite A" for the Jacobian constant in his notation equal to 40.5, *loc. cit.*, p. 199. The synodic period was found to be $61°\ 23'=61.383°$, where the period of the finite bodies is $360°$. Therefore

$$m=\frac{61.383}{360}=0.17051, \qquad \frac{a}{A}=\left(\frac{1}{11}\right)^{1/3}\left(\frac{m}{m+1}\right)^{2/3}=0.12449. \tag{74}$$

The m for this orbit is more than twice that occurring in the Lunar Theory. With these values of the constants substituted in the series of §186, it is found that

$$\left.\begin{aligned} r&=0.12427+0.00652\cos\tau-0.00420\cos 2\tau+0.00004\cos 3\tau\\ &\qquad\qquad\qquad\qquad-0.00006\cos 4\tau+\cdots,\\ w&=-0.12062\sin\tau+0.05079\sin 2\tau-0.00184\sin 3\tau\\ &\qquad\qquad\qquad\qquad+0.00095\sin 4\tau+\cdots \end{aligned}\right\} \tag{75}$$

The infinitesimal body is in a line with the finite bodies and between them when $\tau=0$. The value of r at this time is found from (75) to be $r(0)=0.12657$. The corresponding value given by Darwin is 0.1265. The infinitesimal body is in opposition at $\tau=\pi$, and it is found from (75) that $r(\pi)=0.11345$. Darwin's value is 0.1135. These agreements show that the "Satellites A" are of the class treated in this chapter.

**American Journal of Mathematics*, vol. XIV (1891), pp. 140–160.

†*Acta Mathematica*, vol. XXI (1887), pp. 99–242.

In a retrograde orbit having the same sidereal period, the expression for m is

$$m=\frac{-N}{n+N}=\frac{-N/n}{1+N/n}. \tag{76}$$

In this case

$$\frac{N}{n}=\frac{613.83}{61.383+360};$$

therefore

$$m=-0.12715.$$

The value of a/A is the same as before, and the series for r gives

$$\left.\begin{aligned} r&=0.12412-0.00057\cos\tau-0.00158\cos 2\tau-0.00000\cos 3\tau\\ &\qquad\qquad -0.00001\cos 4\tau+\cdots,\\ w&=0.00820\sin\tau+0.01654\sin 2\tau+0.00047\sin 3\tau\\ &\qquad\qquad +0.00010\sin 4\tau+\cdots,\end{aligned}\right\} \tag{77}$$

which are seen to converge somewhat more rapidly than the series of (95). No retrograde orbits were computed by Darwin in his memoir in the *Acta Mathematica*.

Comparison will also be made with one of Darwin's "Planets A." In this case

$$m_1=10,\qquad m_2=1,\qquad A=1,\qquad \eta=\frac{m_2}{m_1+m_2}=\frac{1}{11}.$$

The orbit will be taken for which the Jacobian constant is 40.0. The period given by Darwin (*loc. cit.* p. 225) is 154° 13′. Therefore

$$m=\frac{154.216}{360}=0.42838,\qquad \frac{a}{A}=0.43404. \tag{78}$$

With these values of the parameters, the series for r gives

$$r=0.43373+0.00776\cos\tau-0.01286\cos 2\tau-0.00104\cos 3\tau+\cdots. \tag{79}$$

From this series it is found that

$$r(0)=0.42759,\qquad r(\pi)=0.41415.$$

Darwin's results in the respective cases were $r(0)=0.423$ and $r(\pi)=0.4140$.

The agreement of these results shows the identity of his "Planets A" and the orbits covered by the analysis of this chapter.

CHAPTER XII.

PERIODIC ORBITS OF SUPERIOR PLANETS.

192. Introduction.—The preceding chapter was devoted to the consideration of the motion of an infinitesimal body subject to the attraction of two finite bodies which revolve in circles. The periodic orbits whose existence was there proved inclose only one of the finite bodies, and they are more nearly circular the smaller their dimensions and the shorter their periods.

The present chapter also will be devoted to the consideration of the motion of an infinitesimal body subject to the attraction of two finite bodies which revolve in circles; but the periodic orbits now under discussion inclose both of the finite bodies and are more nearly circular the larger their dimensions and the longer their periods. There are three families of orbits of this class in which the motion is direct, and three in which it is retrograde. For small values of the parameter in terms of which the solutions are developed, only one family each of the direct and of the retrograde orbits is real.

The mode of treatment of the problem of this chapter is similar to that of the preceding. A certain parameter μ naturally enters the problem. When μ is zero, the problem reduces to that of two bodies, which admits a circular orbit as a periodic solution. The existence of the analytic continuation of this orbit with respect to the parameter μ is proved, and direct methods of constructing the solutions are developed. It is shown also how the integral can be used as a check on the computations, or as a substitute for one of the differential equations in the construction of the solutions.

The results of the preceding chapter were directly applicable to the Lunar Theory; those of this chapter have no direct bearing on the practical problems of the solar system, at least as they are at present treated. Their chief value at present is that they cover a part of the field of the problem of three bodies in which one is infinitesimal and in which the finite bodies revolve in circles.

193. The Differential Equations.—Let the origin of coördinates be at the center of gravity of the finite bodies m_1 and m_2, and take the xy-plane as the plane of their motion. Suppose the infinitesimal body moves in the xy-plane. Let the coördinates of m_1, m_2, and the infinitesimal body be (x_1, y_1), (x_2, y_2), and (x, y) respectively. Then the differential equations of motion for the infinitesimal body are

$$\left.\begin{aligned} &\frac{d^2x}{dt^2}=\frac{\partial U}{\partial x}, \qquad \frac{d^2y}{dt^2}=\frac{\partial U}{\partial y}, \qquad U=\frac{k^2m_1}{r_1}+\frac{k^2m_2}{r_2}, \\ &r_1=\sqrt{(x-x_1)^2+(y-y_1)^2}, \qquad r_2=\sqrt{(x-x_2)^2+(y-y_2)^2}. \end{aligned}\right\} \tag{1}$$

Let

$$\left.\begin{aligned} r&=\sqrt{x^2+y^2}, & R_1&=\sqrt{x_1^2+y_1^2}=\frac{m_2}{m_1+m_2}R, \\ R&=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}, & R_2&=\sqrt{x_2^2+y_2^2}=\frac{m_1}{m_1+m_2}R. \end{aligned}\right\} \tag{2}$$

Then, in polar coördinates, equations (1) become

$$\frac{d^2r}{dt^2}-r\left(\frac{dv}{dt}\right)^2=\frac{\partial U}{\partial r}, \qquad r\frac{d^2v}{dt^2}+2\frac{dr}{dt}\frac{dv}{dt}=\frac{1}{r}\frac{\partial U}{\partial v}. \tag{3}$$

The potential function U will now be developed. From (1) and (2) it is found that

$$\left.\begin{aligned} U&=\frac{k^2m_1}{r}\left[1-\frac{2R_1}{r}\cos(v-v_1)+\left(\frac{R_1}{r}\right)^2\right]^{-\frac{1}{2}}+\frac{k^2m_2}{r}\left[1+\frac{2R_2}{r}\cos(v-v_1)+\left(\frac{R_2}{r}\right)^2\right]^{-\frac{1}{2}} \\ &=\frac{k^2(m_1+m_2)}{r}+\frac{k^2m_1m_2R^2}{m_1+m_2\,r^3}\left\{\frac{1}{4}\left[1+3\cos 2(v-v_1)\right]\right. \\ &\qquad\left.+\frac{R_1-R_2}{8r}\left[3\cos(v-v_1)+5\cos 3(v-v_1)\right]+\cdots\right\}. \end{aligned}\right\} \tag{4}$$

Then equations (3) become

$$\left.\begin{aligned} \frac{d^2r}{dt^2}-r\left(\frac{dv}{dt}\right)^2+\frac{k^2(m_1+m_2)}{r^2}&=-\frac{k^2m_1m_2}{m_1+m_2}\frac{R^2}{r^4}\left\{\left[\frac{3}{4}1+3\cos 2(v-v_1)\right]\right. \\ &\qquad\left.+\frac{R_1-R_2}{2r}\left[3\cos(v-v_1)+5\cos 3(v-v_1)\right]+\cdots\right\}, \\ r\frac{d^2v}{dt^2}+2\frac{dr}{dt}\frac{dv}{dt}&=-\frac{k^2m_1m_2}{m_1+m_2}\frac{R^2}{r^4}\left\{\frac{3}{2}\sin 2(v-v_1)\right. \\ &\qquad\left.+\frac{3(R_1-R_2)}{8r}\left[\sin(v-v_1)+5\sin 3(v-v_1)\right]+\cdots\right\}. \end{aligned}\right\} \tag{5}$$

If the orbits of m_1 and m_2 are circles, which is assumed to be the case, equations (1) admit the Jacobian integral

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2-2n_1\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=2U-C, \qquad n_1=\frac{k\sqrt{m_1+m_2}}{R^{3/2}}. \tag{6}$$

It follows that n_1 is the mean motion of the finite bodies and that $v_1=n_1(t-t_0)$. In polar coördinates the integral becomes

$$r'^2+r^2v'^2-2n_1r^2v'=2U-C, \tag{7}$$

where the primes indicate derivatives with respect to t.

When the right members of (5) are put equal to zero, the equations admit the particular solution

$$r=a, \qquad v=\frac{k\sqrt{m_1+m_2}}{a^{3/2}}(t-t_0)=n(t-t_0), \tag{8}$$

where n is the angular velocity of the infinitesimal body in its orbit and t_0 is an arbitrary constant. It will be supposed that n is given by the observations, or that its value is assumed, and that a is determined by the second equation of (8). The constant a has three values, only one of which is real.

New variables, ρ, θ, and τ, and new constants, μ and M, will be introduced by the equations

$$r=a(1+\rho), \quad v=n(t-t_0)+\theta, \quad (n_1-n)(t-t_0)=\tau, \quad \frac{n}{n_1}=\mu, \quad \frac{m_1 m_2}{(m_1+m_2)^2}=M. \quad (9)$$

It follows from (6), (8), and (9) that

$$\frac{R}{a}=\mu^{2/3}; \tag{10}$$

and equations (5) become

$$\left.\begin{aligned}
&\ddot{\rho}-(1+\rho)\left[\frac{\mu}{1-\mu}+\dot{\theta}\right]^2+\frac{\mu^2}{(1-\mu)^2}\frac{1}{(1+\rho)^2}= \\
&\quad -\frac{M}{(1-\mu)^2}\frac{\mu^{10/3}}{(1+\rho)^4}\left\{\frac{3}{4}\left[1+3\cos 2(\tau+\theta)\right]\right. \\
&\quad \left.+\frac{m_2-m_1}{m_1+m_2}\frac{\mu^{2/3}}{2(1+\rho)}\left[3\cos(\tau+\theta)+5\cos 3(\tau+\theta)\right]+\cdots\right\}, \\
&(1+\rho)\ddot{\theta}+2\dot{\rho}\left[\frac{\mu}{1-\mu}+\dot{\theta}\right]=-\frac{M}{(1-\mu)^2}\frac{\mu^{10/3}}{(1+\rho)^4}\left\{\frac{3}{2}\sin 2(\tau+\theta)\right. \\
&\quad \left.+\frac{m_2-m_1}{m_1+m_2}\frac{3\mu^{2/3}}{8(1+\rho)}\left[\sin(\tau+\theta)+5\sin 3(\tau+\theta)\right]+\cdots\right\},
\end{aligned}\right\} \tag{11}$$

where the dots over the letters indicate derivatives with respect to τ. These equations are valid for the determination of the motion of the infinitesimal body provided $|\mu|<1$. The right members of equations (11) involve only cosines and sines respectively of integral multiples of $\tau+\theta$. The parts in the brackets proceed according to powers of $\mu^{2/3}$, the coefficients of even powers of $\mu^{2/3}$ in the first and second equations being cosines and sines respectively of even multiples of τ, and the coefficients of odd powers of $\mu^{2/3}$ being cosines and sines respectively of odd multiples of τ.

194. Proof of the Existence of Periodic Solutions.—Suppose $\rho=\beta$, $\dot{\rho}=0$, $\theta=0$, $\dot{\theta}=\gamma$ at $\tau=0$, and let the solution of (11) be written in the form

$$\rho=f(\beta, \gamma; \tau), \qquad \theta=\varphi(\beta, \gamma; \tau). \tag{12}$$

Now make the transformation

$$\rho=\rho_1, \qquad \theta=-\theta_1, \qquad \tau=-\tau_1. \tag{13}$$

The resulting equations have precisely the form (11). Consequently their solutions with the initial conditions $\rho_1=\beta$, $\dot{\rho}_1=0$, $\theta=0$, $\dot{\theta}_1=\gamma$ are

$$\rho_1=f(\beta, \gamma; \tau_1)=f(\beta, \gamma; -\tau)=\rho, \qquad \theta_1=\varphi(\beta, \gamma; \tau_1)=\varphi(\beta, \gamma; -\tau)=-\theta. \tag{14}$$

Therefore, with these initial conditions, ρ is an even function of τ, and θ is an odd function of τ. The orbit is symmetrical with respect to the ρ-axis both geometrically and in τ. Such an orbit will be called *symmetrical,* whether it is periodic or not.

Now consider the conditions for a closed symmetrical orbit. Since the right members of (11) involve only sines and cosines of integral multiples of τ, sufficient conditions that in symmetrical orbits ρ and θ shall be periodic with the period $2j\pi$ are

$$\dot{\rho}=\dot{f}(\beta, \gamma; j\pi)=0, \qquad \theta=\varphi(\beta, \gamma; j\pi)=0; \tag{15}$$

and these conditions are necessary, provided they are distinct.

In order to examine the solutions of (15), it is convenient to use parameters other than β and γ. Suppose that, at $\tau=0$,

$$\left.\begin{aligned} r&=a(1+\rho)=a(1+\alpha)(1-e), \qquad \dot{r}=a\dot{\rho}=0, \\ \frac{\mu}{1-\mu}+\dot{\theta}&=\frac{\mu}{1-\mu}\,\frac{\sqrt{1+e}}{(1+\alpha)^{3/2}(1-e)^{3/2}}. \end{aligned}\right\} \tag{16}$$

It follows that $a(1+\alpha)$ and e are the major semi-axis and eccentricity of the elliptic orbit which would be obtained if the right members of equations (11) were zero. Because of the well-known properties of the solutions of the two-body problem in terms of these elements, the properties of the general solutions, so far as they do not depend upon the right members of (11), are known. These properties will be important in solving the conditions for periodic solutions.

Equations (11) are regular in the vicinity of $\mu=0$, $\rho=0$, $\dot{\rho}=0$, $\theta=0$, $\dot{\theta}=0$ for all values of τ. It follows that the moduli of α, e, and $\mu^{1/3}$ can be taken so small that the solutions will be regular while τ runs through any finite preassigned range of values. We shall choose as the interval for τ the range $0 \gtreqless \tau \leqq 2j\pi$ and integrate (11) as power series in α, e, and $\mu^{1/3}$, vanishing with $\alpha=e=\mu^{1/3}=0$. That is, the results will be the analytic continuation with respect to these parameters of the particular solution $r=a$, $v=nt$, which exists when $\mu=0$. The results may be written in the form

$$\left.\begin{aligned} \rho&=p_1(\alpha, e, \mu^{1/3}; \tau), \qquad \theta=p_3(\alpha, e, \mu^{1/3}; \tau), \\ \dot{\rho}&=p_2(\alpha, e, \mu^{1/3}; \tau), \qquad \dot{\theta}=p_4(\alpha, e, \mu^{1/3}; \tau), \end{aligned}\right\} \tag{17}$$

where $p_1, \ldots, p_4$ are power series in α, e, and $\mu^{1/3}$, with τ in the coefficients.

The conditions for a periodic solution, (15), become

$$p_2(\alpha, e, \mu^{1/3}; j\pi)=0, \qquad p_3(\alpha, e, \mu^{1/3}; j\pi)=0. \tag{18}$$

It will be shown that these equations can be solved for α and e as power series in $\mu^{1/3}$, vanishing with $\mu^{1/3}=0$, which converge if the modulus of $\mu^{1/3}$ is sufficiently small.

Since the right members of (11) carry $\mu^{10/3}$ as a factor, the part of the solution depending on the right members will be divisible by $\mu^{10/3}$. If the right members of (11) were zero and if the solution were formed with the initial conditions (16), the mean angular motion of the infinitesimal body in its orbit would be

$$\nu=\frac{\mu}{(1-\mu)(1+\alpha)^{3/2}}. \tag{19}$$

Consequently, from the solution of the two-body problem, it follows that equations (18) have the form

$$\left.\begin{aligned} p_2(j\pi) &= e\nu\left\{\sin \nu j\pi + e\sin 2\nu j\pi + \cdots\right\} + \mu^{10/3}q_2(\alpha, e, \mu^{1/3}; j\pi) = 0, \\ p_3(j\pi) &= -\left\{\frac{\mu}{1-\mu}j\pi - \nu j\pi - 2e\sin \nu j\pi - \cdots\right\} + \mu^{10/3}q_3(\alpha, e, \mu^{1/3}; j\pi) = 0, \end{aligned}\right\} \quad (20)$$

where the unwritten parts in the brackets are sines of multiples of $\nu j\pi$, and carry e^2 as a factor.

Upon referring to (19), it is observed that the first equation of (20) is divisible by μ^2, and the second by μ. After dividing by these factors the equations are still satisfied by $\alpha = e = \mu = 0$; moreover, the determinant of their linear terms in α and e is

$$\Delta = \begin{vmatrix} 0, & j\pi \\ -\frac{3}{2}j\pi, & 2j\pi \end{vmatrix} = +\frac{3}{2}j^2\pi^2 \neq 0. \quad (21)$$

Therefore, besides the solution $\mu = 0$, equations (20) have a unique solution for α and e as power series in $\mu^{1/3}$, vanishing with $\mu^{1/3} = 0$, which converge for the modulus of $\mu^{1/3}$ sufficiently small. These power series carry $\mu^{4/3}$ as a factor, and can be written in the form

$$\alpha = \mu^{4/3}P_1(\mu^{1/3}), \qquad e = \mu^{4/3}P_2(\mu^{1/3}). \quad (22)$$

Upon substituting these series in the right members of (17), which vanish with $\alpha = e = \mu^{1/3} = 0$, the result is

$$\rho = \mu^{4/3}Q_1(\mu^{1/3};\ \tau), \qquad \theta = \mu^{4/3}Q_2(\mu^{1/3};\ \tau). \quad (23)$$

The series Q_1 and Q_2 are periodic in τ with the period $2j\pi$ because the conditions that the solutions shall have this period have been satisfied. Since (17) converge for all $0 \gtreqless \tau \leqq 2j\pi$ if the moduli of α, e, and $\mu^{1/3}$ are sufficiently small, and since the expressions for α and e given in (22) vanish for $\mu = 0$, it follows that the modulus of $\mu^{1/3}$ can be taken so small that the series (23) converge for all τ in the interval; and since they are periodic with the period $2j\pi$, the convergence holds for all finite values of τ.

The integer j has so far been undetermined. When j is unity, the periodic solutions exist uniquely and their period is 2π. When j is greater than unity the periodic solutions also exist uniquely. Since the periodic orbits for j greater than unity include those for j equal to unity, and since in both cases there is precisely one periodic orbit for a given value of $\mu^{1/3}$, *it follows that all the symmetrical periodic orbits of the class under consideration have the period 2π in the independent variable τ.*

It follows from (6) and (9) that $\tau+\theta=v-v_1$. Since in the periodic solution θ is periodic with the period 2π, the period of the solution is the synodic period of the three bodies. Hence, if the motion of the infinitesimal body is referred to a set of axes having their origin at the center of gravity of the system and rotating in the direction of motion of the finite bodies at the angular rate at which they move, and if the x-axis passes through the finite bodies, then the periodic orbit of the infinitesimal body, which has been proved to exist, will be symmetrical with respect to the x-axis. Since, by hypothesis, $a>R$, it follows from (6) and (8) that $n_1>n$. Therefore, even if the motion of the infinitesimal body is forward with respect to fixed axes, it is retrograde with respect to the rotating axes.

It is supposed that the period of the finite bodies, and therefore n_1, is given in advance and remains fixed. The variation of the parameter $\mu^{1/3}$ corresponds to a variation of the period of the infinitesimal body defined by n. If the motion with respect to fixed axes is forward, n has the same sign as n_1, and $\mu^{1/3}$ has three values, one of which is real and positive while the other two are complex. If the motion is retrograde, $\mu^{1/3}$ has three different values, one of which is real and negative while the other two are complex. Therefore, for a given period, there are six symmetrical orbits, three direct and three retrograde; and for small μ one direct orbit is real and one retrograde orbit is real, while in the others the coördinates are complex. This means, of course, that the corresponding solutions do not exist in the physical problem. The coördinates of the complex orbits are conjugate in pairs. For a certain value of $\mu^{1/3}$ they may become equal, and therefore real, and, for larger values of $\mu^{1/3}$, real and distinct.

Upon transforming the integral (7) by (9), it is found that

$$\left.\begin{aligned}
&\dot{\rho}^2+(1+\rho)^2\left[\frac{\mu}{1-\mu}+\dot{\theta}\right]^2-2\frac{(1+\rho)^2}{1-\mu}\left[\frac{\mu}{1-\mu}+\dot{\theta}\right]=\frac{\mu^2}{(1-\mu)^2(1+\rho)}\\
&+\frac{2M\mu^{10/3}}{(1-\mu)^2(1+\rho)^3}\left\{\frac{1}{4}\left[1+3\cos 2(\tau+\theta)\right]+\frac{m_2-m_1}{m_1+m_2}\frac{\mu^{2/3}}{8(1+\rho)}\left[3\cos(\tau+\theta)\right.\right.\\
&\qquad\qquad\left.\left.+5\cos 3(\tau+\theta)\right]+\cdots\right\}-C_1,
\end{aligned}\right\} \quad (24)$$

where $C=n_1^2(1-\mu)^2a^2C_1$. It follows from this equation and (23) that C_1 can be expanded as a power series in $\mu^{1/3}$, vanishing with $\mu^{1/3}$. The term of lowest degree in $\mu^{1/3}$, after substituting (23), is $\mu^{3/3}$. Therefore, $\mu^{1/3}$ can be expanded as a power series in $C_1^{1/3}$. For $C_1=0$, the three branches of the function are the same. Since $a=R/\mu^{2/3}$, the relation between C and C_1 is

$$C=\frac{n_1^2R^2(1-\mu)^2C_1}{\mu^{4/3}}=\frac{n_1^2R^2}{\mu^{1/3}}[1+\text{power series in }\mu]. \quad (25)$$

From this it follows that $C=\infty$ for $\mu^{1/3}=0$. Therefore the periodic orbits branch at $C=\infty$, and there are two cycles of three each.

195. Practical Construction of the Periodic Solutions.—It has been proved that the symmetrical periodic solutions under discussion are expressible in the form

$$\rho=\sum_{i=4}^{\infty}\rho_i\mu^{i/3},\qquad \theta=\sum_{i=4}^{\infty}\theta_i\mu^{i/3},\tag{26}$$

where the ρ_i and θ_i are functions of τ. Since these series are periodic and converge for all $|\mu^{1/3}|$ sufficiently small, it follows that each ρ_i and θ_i separately is periodic; that is,

$$\rho_i(\tau+2\pi)\equiv\rho_i(\tau),\qquad \theta_i(\tau+2\pi)\equiv\theta_i(\tau).\tag{27}$$

In every closed orbit there are points at which $d\rho/d\tau=0$. Suppose t_0 of (9) is so determined that this condition is satisfied at $\tau=0$; it will follow from this and the convergence of (26) for all $|\mu^{1/3}|$ sufficiently small that

$$\dot{\rho}_i=0\qquad (i=4,\ \ldots\ \infty).\tag{28}$$

In the *symmetrical* periodic orbits the value of θ is zero at $\tau=0$. But this condition will not be imposed, because the general periodic orbits, whose initial conditions are not specialized, include those which are symmetrical; and in the construction it will appear that the conditions for symmetry are a consequence of those for periodicity. Hence all the periodic orbits of the class under consideration are symmetrical.

Equations (26) are to be substituted in (11), arranged as power series in $\mu^{1/3}$, and the coefficients of the several powers of $\mu^{1/3}$ set equal to zero. The coefficients of $\mu^{4/3}$ set equal to zero give the equations

$$\ddot{\rho}_4=0,\qquad \ddot{\theta}_4=0.\tag{29}$$

The solutions of these equations satisfying (27) and (28) are

$$\rho_4=a_4,\qquad \theta_4=b_4,\tag{30}$$

where a_4 and b_4 are so far undetermined constants.

The coefficients of $\mu^{5/3},\ \ldots\ ,\mu^{9/3}$ are the same as (29) except for their subscripts, and their solutions satisfying (27) and (28) are similarly

$$\rho_j=a_j,\qquad \theta_j=b_j\qquad (j=5,\ \ldots\ ,9),\tag{31}$$

where all the a_j and b_j are so far undetermined constants.

The coefficients of $\mu^{10/3}$ give the equations

$$\ddot{\rho}_{10}=3a_4-\frac{3}{4}M[1+3\cos 2\tau],\qquad \ddot{\theta}_{10}=-\frac{3}{2}M\sin 2\tau.\tag{32}$$

In order that the solution of the first of these equations shall be periodic the condition

$$a_4=\frac{1}{4}M\tag{33}$$

must be imposed, which uniquely determines the constant a_4. Then the solution of (32) satisfying (27) and (28) is

$$\rho_{10}=a_{10}+\frac{9}{16}M\cos 2\tau,\qquad \theta_{10}=b_{10}+\frac{3}{8}M\sin 2\tau,\tag{34}$$

where a_{10} and b_{10} are as yet undetermined.

The coefficients of $\mu^{11/3}$ are defined by the equations

$$\ddot{\rho}_{11} = 3a_5, \qquad \ddot{\theta}_{11} = 0,$$

from which it follows that

$$a_5 = 0, \qquad \rho_{11} = a_{11}, \qquad \theta_{11} = b_{11}, \tag{35}$$

where a_{11} and b_{11} are as yet undetermined.

The coefficients of $\mu^{12/3}$ in the solutions satisfy the equations

$$\ddot{\rho}_{12} = 3a_6 + \frac{M(m_1 - m_2)}{2(m_1 + m_2)}[3\cos\tau + 5\cos 3\tau], \quad \ddot{\theta}_{12} = \frac{3M(m_1 - m_2)}{8(m_1 + m_2)}[\sin\tau + 5\sin 3\tau].$$

Upon imposing conditions (27) and integrating, it is found that $a_6 = 0$, and

$$\left.\begin{aligned} \rho_{12} &= a_{12} - \frac{M(m_1 - m_2)}{2(m_1 + m_2)}\left[3\cos\tau + \tfrac{5}{9}\cos 3\tau\right], \\ \theta_{12} &= b_{12} - \tfrac{3}{8}\frac{M(m_1 - m_2)}{m_1 + m_2}\left[\sin\tau + \tfrac{5}{9}\sin 3\tau\right], \end{aligned}\right\} \tag{36}$$

where a_{12} and b_{12} remain so far undetermined.

In a similar way it is found from the coefficients of $\mu^{13/3}$ that

$$a_7 = 0, \qquad \rho_{13} = a_{13} + \tfrac{3}{4}M\cos 2\tau, \qquad \theta_{13} = b_{13} + \tfrac{3}{16}M\sin 2\tau, \tag{37}$$

where a_{13} and b_{13} remain undetermined at this step.

So far all the b_j have remained arbitrary, and it is necessary to carry the integration one step further in order to see how they are determined. The coefficients of $\mu^{14/3}$ are defined by

$$\left.\begin{aligned} \ddot{\rho}_{14} &= 3a_8 - 3a_4^2 + \tfrac{3}{2}M[3b_4\sin 2\tau + 2a_4 + 6a_4\cos 2\tau], \\ \ddot{\theta}_{14} &= -a_4\ddot{\theta}_{10} - 3M[b_4\cos 2\tau - 2a_4\sin 2\tau]. \end{aligned}\right\} \tag{38}$$

Upon substituting the values of a_4 and θ_{10} from (33) and (34), imposing the conditions (27) and integrating, it is found that $a_8 = -\frac{3}{16}M^2$, and

$$\left.\begin{aligned} \rho_{14} &= a_{14} - \tfrac{9}{8}b_4 M\sin 2\tau - \tfrac{9}{16}M^2\cos 2\tau, \\ \theta_{14} &= b_{14} + \tfrac{3}{4}b_4 M\cos 2\tau - \tfrac{15}{32}M^2\sin 2\tau. \end{aligned}\right\} \tag{39}$$

The condition (28) for $j = 14$ gives the equations

$$b_4 = 0, \qquad \rho_{14} = a_{14} - \tfrac{9}{16}M^2\cos 2\tau, \qquad \theta_{14} = b_{14} - \tfrac{15}{32}M^2\sin 2\tau. \tag{40}$$

It is found in a similar way from the coefficients of $\mu^{15/3}$ that $b_5 = 0$, and

$$\left.\begin{aligned} \rho_{15} &= a_{15} - \tfrac{9}{4}M\frac{m_1 - m_2}{m_1 + m_2}\left[\cos\tau + \tfrac{5}{27}\cos 3\tau\right], \\ \theta_{15} &= b_{15} + \tfrac{1}{4}M\frac{m_1 - m_2}{m_1 + m_2}\left[9\sin\tau + \tfrac{25}{27}\sin 3\tau\right], \end{aligned}\right\} \tag{41}$$

where a_{15} and b_{15} remain so far arbitrary.

It will be observed that, so far as the computation has been carried, the coefficients of the ρ_j are cosines of integral multiples of τ, and that the coefficients of the θ_j, except for the undetermined additive constants, are sines of integral multiples of τ. In the computation of ρ_j the periodicity conditions have uniquely determined a_{j-6}, and the condition $\dot{\rho}_j=0$ at $\tau=0$ has required that $b_{j-10}=0$. It will now be shown that these properties are general. Suppose $\rho_4, \ldots, \rho_n$; $\theta_4, \ldots, \theta_n$ have been computed and that the coefficients are all known except $a_{n-5}, \ldots, a_n$, which enter additively in $\rho_{n-4}, \ldots, \rho_n$ respectively, and $b_{n-9}, \ldots, b_n$, which enter additively in $\theta_{n-9}, \ldots, \theta_n$ respectively. The differential equations for the determination of ρ_{n+1} and θ_{n+1} are

$$\ddot{\rho}_{n+1}=3a_{n-5}+\tfrac{9}{2}Mb_{n-9}\sin 2\tau+F_{n+1}(\tau), \qquad \ddot{\theta}_{n+1}=G_{n+1}(\tau), \tag{42}$$

where $F_{n+1}(\tau)$ and $G_{n+1}(\tau)$ are entirely known functions of τ. It follows from the assumptions respecting $\rho_4, \ldots, \rho_n$; $\theta_4, \ldots, \theta_n$ and the properties of equations (11) that $F_{n+1}(\tau)$ is a sum of *cosines* of integral multiples of τ, and that $G_{n+1}(\tau)$ is a sum of *sines* of integral multiples of τ. Hence they may be written in the form

$$F_{n+1}(\tau)=\Sigma\, A_j^{(n+1)}\cos j\tau, \qquad G_{n+1}(\tau)=\Sigma\, B_j^{(n+1)}\sin j\tau.$$

In order that the solution of the first equation of (42) shall be periodic the condition

$$3a_{n-5}+A_0^{(n+1)}=0 \tag{43}$$

must be imposed, and this condition uniquely determines a_{n-5}.

After equation (43) has been satisfied, the solution of the first equation of (42) is

$$\rho_{n+1}=a_{n+1}-\tfrac{9}{8}Mb_{n-9}\sin 2\tau+\Sigma\, \alpha_j^{(n+1)}\cos j\tau, \qquad \alpha_j^{(n+1)}=-\frac{1}{j^2}A_j^{(n+1)}. \tag{44}$$

The condition $\rho=0$ at $\tau=0$ makes it necessary to take

$$b_{n-9}=0. \tag{45}$$

Then ρ_{n+1} is completely determined except for the additive constant a_{n+1}, and it is a sum of cosines of integral multiples of τ.

The solution of the second equation of (42) is

$$\theta_{n+1}=b_{n+1}+\Sigma\,\beta_j^{(n+1)}\sin j\tau, \qquad \beta_j^{(n+1)}=-\frac{1}{j^2}B_j^{(n+1)}. \tag{46}$$

Hence θ_{n+1} is a sum of sines of integral multiples of τ, except for the undetermined constant b_{n+1}, which must be put equal to zero in order to satisfy the condition on ρ_{n+11}. These results lead, by induction, to the conclusion that the ρ_j and θ_j ($j=4, \ldots \infty$) are sums of cosines and sines respectively of integral multiples of τ whose coefficients are uniquely determined.

From the properties of the solutions which have just been established, it follows that not only is $\dot{\rho}=0$ at $\tau=0$, but also $\theta(0)=0$. Therefore these periodic orbits are the symmetrical orbits whose existence was established in §194. In the construction it was not assumed that the orbits were symmetrical, and since this property is a necessary consequence of the periodicity conditions, it follows that all periodic solutions which are expansible as power series in $\mu^{1/3}$ are symmetrical. It is easily shown, by direct consideration of the construction of periodic solutions, that they can not be expanded as power series in $\mu^{1/3}$ except when j is a multiple of 3, and that then they reduce to those found above.

196. Application of the Integral.—The differential equations admit the integral (24), which, for brevity, can be written in the form

$$F(\rho,\, \dot{\rho},\, \theta,\, \dot{\theta},\, \tau,\, \mu^{1/3})=0.$$

It follows from the form of (24) and the expansions (26) that the left member of this equation can be developed as a power series in $\mu^{1/3}$, giving

$$F=F_0+F_1\mu^{1/3}+F_2\mu^{2/3}+\cdots+F_n\mu^{n/3}+\cdots=0. \tag{47}$$

Since the ρ_j and θ_j are sums of cosines and sines respectively of integral multiples of τ, and since $\dot{\rho}$ enters in (24) only in the second degree and θ only in even degrees, it follows that the F_j are sums of cosines of integral multiples of τ. Equation (47) is an identity in $\mu^{1/3}$, whence

$$F_n=\Sigma\, C_j^{(n)}\cos j\tau=0 \qquad (n=0,\ \ldots\ \infty).$$

Since these equations hold for all values of τ, it follows that

$$C_j^{(n)}=0 \qquad (n=0,\ \ldots\ \infty;\ j=0,\ \ldots\ \infty). \tag{48}$$

The $C_j^{(n)}$ are functions of the $\alpha_j^{(0)},\ \ldots\ ,\ \alpha_j^{(n)}$ and $\beta_j^{(0)},\ \ldots\ ,\ \beta_j^{(n)}$. Hence equations (48) can be used as check formulas on the computation of the coefficients of the solutions.

Equations (48) can be used in place of the second equation of (11) for the determination of the $\beta_j^{(n)}$, the coefficients of the trigonometric terms in the expression for θ_n. Suppose $\rho_4,\ \ldots\ ,\ \rho_{n-1}$ and $\theta_4,\ \ldots\ ,\ \theta_{n-1}$ have been determined except for additive constants in $\rho_{n-6},\ \ldots\ ,\ \rho_{n-1}$. It follows from (24) that F_n is

$$F_n=-2\dot{\theta}_n+P_n(\rho_j,\, \dot{\rho}_j,\, \theta_j,\, \dot{\theta}_j) \qquad (j=4,\ \ldots\ ,\ n-1),$$

where P_n is a polynomial in the arguments indicated. Consequently equations (48) are of the form

$$C_j^{(n)}=-2j\beta_j^{(n)}+D_j^{(n)}(\alpha_k^{(\nu)},\, \beta_k^{(\nu)})=0 \quad (\nu=4,\ \ldots\ ,\ n-1),$$

which uniquely determine the $\beta_j^{(n)}$.

CHAPTER XIII.

A CLASS OF PERIODIC ORBITS OF A PARTICLE SUBJECT TO THE ATTRACTION OF n SPHERES HAVING PRESCRIBED MOTION.

By WILLIAM RAYMOND LONGLEY.

197. Introduction.—The restricted problem of three bodies furnishes naturally the starting-point* for the consideration of the periodic orbits of an infinitesimal body, or particle, which is subject to the Newtonian attraction of certain finite spheres whose motion is supposed to be known. The two finite bodies are supposed to revolve in circles about their common center of mass, and the motion of the particle is restricted to the plane in which the finite bodies move. One class of orbits occurring in this problem is that in which the particle revolves about one of the finite bodies, and for the consideration of these orbits it is convenient to refer the motion of the particle to a plane rotating with the angular velocity of the finite bodies. All of the known periodic orbits of this type possess one and only one line of symmetry, namely, the line joining the finite bodies, and this property of symmetry plays an important part in the proof of their existence and the construction of series to represent them.

The purpose of this chapter is to generalize the restricted problem by introducing into the plane of motion more than two finite bodies. The coördinates of the finite bodies (spheres) are supposed to be known functions of the time, that is, the motion of the spheres is prescribed. For the analysis which follows the nature of the forces producing this motion is unimportant. The spheres are supposed to attract the particle according to the Newtonian law. Besides involving additional terms in the disturbing function, this generalization modifies the original problem by introducing cases where the periodic orbits have no line of symmetry, and cases where there are more lines of symmetry than one. This modification necessitates some changes in the details of the analysis which must be worked out. In order to avoid cumbersome notation, the analysis will be developed for simple particular cases of the motion of spheres under their Newtonian attraction; with slight changes it is applicable to more general types of prescribed motion of the finite bodies, which are indicated in §207.

*See papers by Hill, *American Journal of Mathematics*, vol. 1 (1878), p. 245; Darwin, *Acta Mathematica*, vol. 21 (1897), p. 99; and Moulton, *Transactions of the American Mathematical Society*, vol. 7 (1906), p. 537.

198. Existence of Periodic Orbits Having no Line of Symmetry.—It was shown by Lagrange* that an equilateral triangle is a possible configuration for three spheres revolving in circles about their common center of mass. This motion of three finite bodies will serve to illustrate the case when the periodic orbits of the particle about one of the bodies possess no line of symmetry. Let the masses of the three finite bodies moving according to the equilateral-triangle solution be denoted by M, M_1, M_2, and suppose the particle P revolves about the mass M. Suppose also that the masses M_1 and M_2 are unequal.†

With reference to M as origin and an axis having a fixed direction in space, let the polar coördinates of M_1, M_2, and P be respectively (R_1, V_1), (R_2, V_2), and (r, v). The coördinates of the bodies are expressed in terms of the time, t, as follows:

$$R_1 = R_2 = A, \qquad V_1 = V_2 - \frac{\pi}{3} = Nt, \tag{1}$$

where

$$N^2 A^3 = k^2(M + M_1 + M_2).$$

Here N denotes the angular velocity, A the length of a side of the triangle, and k is a constant depending upon the units employed.

The differential equations of motion of P are

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2M}{r^2} = \frac{\partial\Omega}{\partial r}, \qquad r\frac{d^2v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} = \frac{1}{r}\frac{\partial\Omega}{\partial v}, \tag{2}$$

where

$$\left.\begin{aligned} &\Omega = k^2\left[\frac{M_1}{r_1} + \frac{M_2}{r_2} - \frac{M_1}{A^2} r\cos(v - V_1) - \frac{M_2}{A^2} r\cos(v - V_2)\right], \\ &r_1 = \sqrt{r^2 + A^2 - 2rA\cos(v - V_1)}, \qquad r_2 = \sqrt{r^2 + A^2 - 2rA\cos(v - V_2)}. \end{aligned}\right\} \tag{3}$$

Let us define m and a by the relations

$$m\nu = N, \qquad \nu^2 a^3 = k^2 M, \tag{4}$$

where ν is a quantity to be assigned later.

By the substitution $v = w + V_1 = w + Nt$ the motion is referred to an axis rotating with the angular velocity of the finite bodies and passing always through M_1; and factors depending upon the units employed are eliminated by the relations $r = a\rho$, $\nu t = \tau$. On making these substitutions in equations (2) and dividing by $\nu^2 a$, the differential equations of relative motion become

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + m\right)^2 + \frac{1}{\rho^2} = \frac{1}{\nu^2 a}\frac{\partial\Omega}{\partial(a\rho)}, \qquad \rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + m\right) = \frac{1}{\nu^2 a^2 \rho}\frac{\partial\Omega}{\partial w}. \tag{5}$$

*Prize memoir, *Essai sur le Problème des Trois Corps, 1772; Coll. Works*, vol. 6, p. 229.

†If $M_1 = M_2$ the periodic orbit of P about M has a line of symmetry, namely, the median of the triangle from the vertex M, which is the line joining M to the center of mass of the system. For the treatment of this special case it is convenient to make use of the property of symmetry and to employ analysis similar to that developed in §§202 and 203.

We can expand Ω as a power series in $a\rho/A$ which is convergent for all values of w provided the distance $MP=a\rho$ is less than A; and in all that follows this condition is supposed to be satisfied. The expansion has the form

$$\begin{aligned}\Omega=\frac{k^2M_1}{A}&\left[1+\tfrac{1}{4}\left(\frac{a}{A}\rho\right)^2\left\{1+3\cos 2w\right\}+\tfrac{1}{8}\left(\frac{a}{A}\rho\right)^3\left\{3\cos w+5\cos 3w\right\}+\cdots\right]\\&+\frac{k^2M_2}{A}\left[1+\tfrac{1}{4}\left(\frac{a}{A}\rho\right)^2\left\{1+3\cos 2\left(w-\frac{\pi}{3}\right)\right\}\right.\\&\left.+\tfrac{1}{8}\left(\frac{a}{A}\rho\right)^3\left\{3\cos\left(w-\frac{\pi}{3}\right)+5\cos 3\left(w-\frac{\pi}{3}\right)\right\}+\cdots\right].\end{aligned}$$

Let λ_1 and λ_2 be defined by the relations

$$M_1=\lambda_1(M_1+M_2),\qquad M_2=\lambda_2(M_1+M_2). \tag{6}$$

From equations (1) we have

$$\frac{k^2(M_1+M_2)}{A^3}=\frac{M_1+M_2}{M+M_1+M_2}N^2.$$

Then, on setting

$$\frac{M_1+M_2}{M+M_1+M_2}=K,$$

it follows that the second members of equations (5) have the form

$$\left.\begin{aligned}\frac{1}{\nu^2a}\frac{\partial\Omega}{\partial(a\rho)}&=Km^2\rho\left[\tfrac{1}{2}\lambda_1\left\{1+3\cos 2w\right\}+\tfrac{3}{8}\lambda_1\left(\frac{a}{A}\rho\right)\left\{3\cos w+5\cos 3w\right\}+\cdots\right.\\&\left.+\tfrac{1}{2}\lambda_2\left\{1+3\cos 2\left(w-\frac{\pi}{3}\right)\right\}+\tfrac{3}{8}\lambda_2\left(\frac{a}{A}\rho\right)\left\{3\cos\left(w-\frac{\pi}{3}\right)+5\cos 3\left(w-\frac{\pi}{3}\right)\right\}+\cdots\right],\\\frac{1}{\nu^2a^2\rho}\frac{\partial\Omega}{\partial w}&=-Km^2\rho\left[\tfrac{3}{2}\lambda_1\sin 2w+\tfrac{3}{8}\lambda_1\left(\frac{a}{A}\rho\right)\left\{\sin w+5\sin 3w)\right\}+\cdots\right.\\&\left.+\tfrac{3}{2}\lambda_2\sin 2\left(w-\frac{\pi}{3}\right)+\tfrac{3}{8}\lambda_2\left(\frac{a}{A}\rho\right)\left\{\sin\left(w-\frac{\pi}{3}\right)+5\sin 3w\left(w-\frac{\pi}{3}\right)\right\}+\cdots\right].\end{aligned}\right\} \tag{7}$$

It is convenient to introduce a parameter μ into the differential equations (5) by the relations

$$m=\mu,\qquad \lambda_2=\lambda\mu,\qquad \frac{a}{A}=\eta\mu \tag{8}$$

wherever the degree of a/A is higher than the *first.* The quantities λ and η are numerical constants. By relating λ_2 and μ the existence proof is made to depend only upon general properties and certain terms of the differential equations which involve λ_1; that is, upon terms in the disturbing function which are due to the body M_1. We shall consider the solution of equations (5) as power series in the parameter μ. The differential equations, and consequently also the solution, do not represent the physical problem under consideration for any value of the parameter except the one satisfying the relations (8). But if the solution is valid when this particular numerical value of μ is substituted, then it is a solution of the differential equations

representing the physical problem and therefore has a physical interpretation. The generalization of the parameter a/A is merely for convenience in having finite expressions in the equations which determine the coefficients at the various steps in the solution.

On introducing the parameter μ as indicated, equations (5) become

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = \mu^2 f, \qquad \rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = \mu^2 g, \tag{9}$$

where

$$f = K\rho\left[\lambda_1\left\{\frac{1}{2}(1+3\cos 2w) + \frac{3}{8}\frac{a}{A}\rho(3\cos w + 5\cos 3w)\right\} + \mu F(\rho, \mu, \cos jw, \sin jw)\right],$$

$$g = -K\rho\left[\lambda_1\left\{\frac{3}{2}\sin 2w + \frac{3}{8}\frac{a}{A}\rho(\sin w + 5\sin 3w)\right\} + \mu G(\rho, \mu, \sin jw, \cos jw)\right],$$

where F and G are functions of the indicated arguments.

Equations (9) are periodic in w with the period 2π and do not involve τ explicitly. Suppose that

$$\rho = \psi_1(\tau), \qquad w = \psi_2(\tau)$$

is a solution. Sufficient conditions that the solution shall be periodic with the period $2p\pi$ (where p is an integer) are

$$\left.\begin{aligned} &\psi_1(2p\pi) = \psi_1(0), && \psi_1'(2p\pi) = \psi_1'(0), \\ &\psi_2(2p\pi) - 2p\pi = \psi_2(0), && \psi_2'(2p\pi) = \psi_2'(0), \end{aligned}\right\} \tag{10}$$

where ψ_1' and ψ_2' denote derivatives of ψ_1 and ψ_2 with respect to τ.

When $\mu = 0$ a periodic solution, which will be called the *undisturbed orbit*, is known, namely,

$$\rho = 1, \qquad w = \tau, \tag{11}$$

and the initial conditions are

$$\rho = 1, \qquad \rho' = 0, \qquad w = 0, \qquad w' = 1. \tag{12}$$

It will be shown by the process of analytic continuation that, for values of μ different from zero, but sufficiently small, there exists a periodic solution which, for $\mu = 0$, reduces to equations (11). For this purpose we consider the solution of equations (9) subject to the initial conditions

$$\rho = 1 + \beta_1, \qquad \rho' = \beta_2, \qquad w = \beta_3, \qquad w' = 1 + \beta_4, \tag{13}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$, are to be determined as functions of μ, vanishing with μ, so that the conditions of periodicity (10) shall hold. It follows from the differential equations that the solution is expressible as power series in β_1, β_2, β_3, β_4, and μ and that, for sufficiently small values of the parameters, the series are convergent for all values of τ from 0 to $2p\pi$. We suppose that this condition on the moduli of the parameters is satisfied.

For the determination of those terms in the series which involve the initial conditions but not μ^2, it is possible to use the known solution of the two-body problem, since for $\mu = 0$ equations (9) reduce to the equations of

motion of a particle P when subject to the attraction of M alone. Hence, instead of the additive increments $\beta_1, \beta_2, \beta_3, \beta_4$, it is convenient to introduce new parameters a, e, θ, φ defined by the relations

$$\left.\begin{aligned} &\rho = 1+\beta_1 = (1+a)(1-e\cos\theta), \qquad \rho' = \beta_2 = \frac{e\sin\theta}{\sqrt{1+a}\,(1-e\cos\theta)}, \\ &w = \beta_3 = \operatorname{arc}\cos\left[\frac{\cos\theta - e}{1-e\cos\theta}\right] - \operatorname{arc}\cos\left[\frac{\cos(\theta-\varphi)-e}{1-e\cos(\theta-\varphi)}\right], \\ &w' = 1+\beta_4 = \frac{\sqrt{1-e^2}}{(1+a)^{3/2}(1-e\cos\theta)^2} - \mu. \end{aligned}\right\} \quad (14)$$

By introducing μ in the fourth of equations (14) it is possible to use the two-body problem for determining all terms of the solution which are independent of μ^2. In terms of the parameters a, e, θ, φ the properties of the solution

P_0 is the position of the particle at $\tau=0$.

W_0 is the longitude of the particle at $\tau=0$.

$W_1 - W_0$ is the longitude of perihelion.

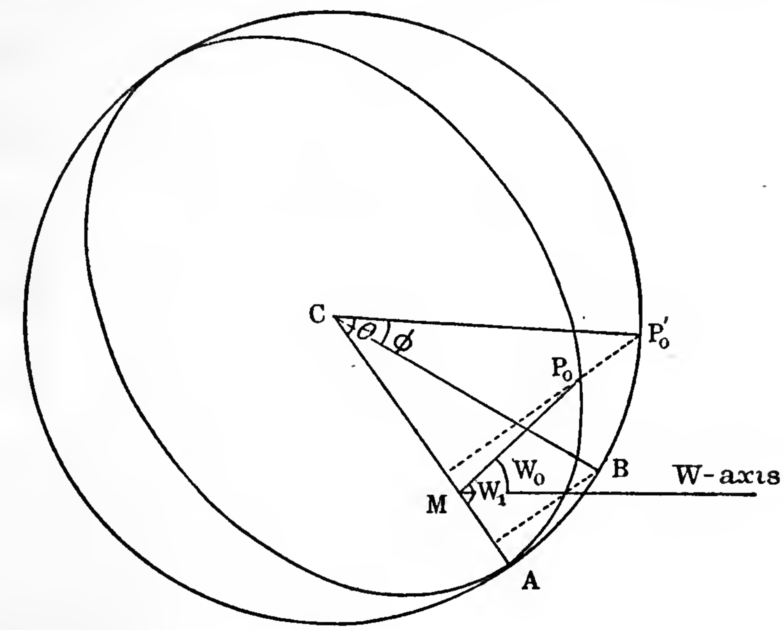

Fig. 8.

are well known, and the conditions of periodicity can be easily discussed. The geometric meaning of the angles θ, φ, W_0, and W_1, which occur below, is shown in Fig. 8.

On making the substitution $w = u - \mu\tau$ and then setting $\mu = 0$ in equations (9), we obtain the differential equations of the two-body problem, of which the solution is

$$\left.\begin{aligned} r = (1+a)(1-e\cos E), \qquad &\cos(u+W_1-W_0) = \frac{\cos E - e}{1-e\cos E}, \\ &\sin(u+W_1-W_0) = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}, \end{aligned}\right\} \quad (15)$$

where E is defined by the relation

$$\frac{\tau}{(1+a)^{3/2}} + \theta - e\sin\theta = E - e\sin E.$$

All other terms in the solution involve μ^2. To find the terms in μ^2 and $\mu^2\varphi$ we will write

$$\rho=1+\rho_2\mu^2+\bar{\rho}\mu^2\varphi+\cdots, \qquad w=\tau-\tau\mu+\varphi+w_2\mu^2+\overline{w}\mu^2\varphi+\cdots.$$

On substituting these expressions in the differential equations, there results for the determination of ρ_2 and w_2 the following set of equations:

$$\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2=\frac{K\lambda_1}{2}(1+3\cos 2\tau)+\frac{3K\lambda_1}{8}\frac{a}{A}(3\cos\tau+5\cos 3\tau),$$

$$\frac{d^2w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}=-\frac{3K\lambda_1}{2}\sin 2\tau-\frac{3K\lambda_1}{8}\frac{a}{A}(\sin\tau+5\sin 3\tau).$$

These equations are integrable, and the result of integration is

$$\rho_2=K\lambda_1\left[-1-2\frac{a}{A}+\left(2+\frac{153}{64}\frac{a}{A}\right)\cos\tau+\frac{15}{16}\frac{a}{A}\tau\sin\tau-\cos 2\tau-\frac{25}{64}\frac{a}{A}\cos 3\tau\right],$$

$$w_2=K\lambda_1\left[\left(\frac{5}{4}+3\frac{a}{A}\right)\tau-\left(4+\frac{201}{32}\frac{a}{A}\right)\sin\tau+\frac{15}{8}\frac{a}{A}\tau\cos\tau+\frac{11}{8}\sin 2\tau+\frac{135}{32}\frac{a}{A}\sin 3\tau\right].$$

By a similar computation it is shown that

$$\bar{\rho}=K\lambda_1\left[-\left(4+\frac{285}{64}\frac{a}{A}\right)\sin\tau+\frac{15}{16}\frac{a}{A}\tau\cos\tau+2\sin 2\tau+\frac{75}{64}\frac{a}{A}\sin 3\tau\right],$$

$$\overline{w}=K\lambda_1\left[\frac{21}{4}+\frac{288}{32}\frac{a}{A}-\left(8+\frac{333}{32}\frac{a}{A}\right)\cos\tau-\frac{15}{8}\frac{a}{A}\tau\sin\tau+\frac{11}{4}\cos 2\tau+\frac{45}{32}\frac{a}{A}\cos 3\tau\right].$$

The terms independent of μ^2 are obtained from equations (15) by Taylor's expansion and the relation $w=u-\mu\tau$; and the solution becomes

$$\left.\begin{aligned}
\rho&=1+a-e\cos\tau-ae\left(\cos\tau-\tfrac{3}{2}\tau\sin\tau\right)+e\theta\sin\tau\\
&\qquad+ae\theta\left(\sin\tau-\tfrac{3}{2}\tau\cos\tau\right)+\rho_2\mu^2+\bar{\rho}\mu^2\varphi+\cdots,\\
w&=\tau-\tfrac{3}{2}\tau a+2e\sin\tau+\varphi-\tau\mu-3ae\tau\cos\tau-e\theta(1-2\cos\tau)\\
&\qquad+e\varphi+ae\theta(5\cos\tau+3\tau\sin\tau)+w_2\mu^2+\overline{w}\mu^2\varphi+\cdots.
\end{aligned}\right\}\quad(16)$$

Applying the conditions (10) that the solution shall be periodic, we have

$$\left.\begin{aligned}
&(a)\qquad 0=-3p\pi ae\theta+\frac{15}{8}\frac{a}{A}p\pi\mu^2\varphi+\cdots,\\
&(b)\qquad 0=3p\pi ae+\frac{15}{8}\frac{a}{A}p\pi\mu^2+\cdots,\\
&(c)\qquad 0=-3p\pi a-2p\pi\mu-6p\pi ae+\cdots,\\
&(d)\qquad 0=6p\pi ae\theta-\frac{15}{4}\frac{a}{A}p\pi\mu^2\varphi+\cdots.
\end{aligned}\right\}\quad(17)$$

The conditions (17) involve the four quantities α, e, θ, φ, and, if independent, would determine them in terms of μ. But the differential equations (9) do not involve τ explicitly and hence admit the integral of Jacobi. This furnishes a relation of the type

$$F(\alpha, e, \theta, \varphi, \mu) = \text{constant},$$

and equations (17) are not independent.* It follows that if (*a*), (*b*), and (*c*) are solved for the three quantities α, e, and θ in terms of μ and φ and the results substituted in (*d*), the equation is satisfied identically in φ. In this problem the dynamical interpretation is simple. Since the finite bodies move in circles the origin of time is arbitrary.† The most convenient choice is $\tau=0$ when $w=0$, which is equivalent to choosing $\varphi=0$.

Consider the solution of equations (*a*), (*b*), and (*c*) for α, e, and θ. The equations have the following properties:

(I) There are no terms independent of α and μ. This follows from the fact that, in the two-body problem, the period does not depend upon e and θ.

(II) There are no terms involving μ to the first degree except the one term $-2p\pi\mu$, which occurs in (*c*).

(III) There are no terms in θ independent of e, since θ does not enter the initial conditions independently of e. It follows from these properties and the particular form of the first terms of the equations that α, e, and θ are determined uniquely as power series in μ by the following steps:

(1) From (*c*) we obtain

$$\alpha=\mu\Big[-\tfrac{2}{3}+\ \cdot\ \cdot\ \cdot\ +\text{function}\,(\mu, e, \theta)\Big].$$

(2) This value of α when substituted in (*b*) permits a factor μ to be divided out. We can then solve the result for e as a power series in μ and θ which contains μ as a factor, and obtain

$$e=\mu\Big[\frac{15}{16}\frac{a}{A}+\ \cdot\ \cdot\ \cdot\ +\text{function}\,(\mu, \theta)\Big].$$

(3) When the values of α and e are substituted in (*a*) a factor μ^2 can be divided out and θ obtained as a power series in μ alone, vanishing with μ.

(4) By the substitution of the value of θ thus found in the expressions for e and α, we obtain finally

$$\alpha=\mu p_1(\mu), \qquad e=\mu p_2(\mu), \qquad \theta=\mu p_3(\mu).$$

The preceding operations are known to be convergent for all values of α, e, θ, and μ which are sufficiently small. Hence, for a given value of μ sufficiently small, it is possible to determine the initial conditions (14) as power series in μ such that the solution of the differential equations (9) shall be periodic in τ with the period $2p\pi$.

*See Poincaré, *loc. cit.*, p. 87.

†When the finite bodies do not form a fixed configuration in the rotating plane the integral of Jacobi does not exist and the origin of time is not arbitrary. In this case it is necessary to determine the four parameters from the conditions of periodicity. The case of the triangular solution when the finite bodies move in ellipses has been treated by Longley in a paper in the *Transactions of the American Mathematical Society*, vol. 8 (1907), pp. 159–188.

When the values of a, e, θ in terms of μ are substituted in equations (16) the periodic solution is obtained. The period of the solution is $2p\pi$ in τ, where p is an integer, and from the conditions of periodicity (10) it is apparent that the particle makes p revolutions in the rotating plane during a period. The process by which the periodic solution was obtained yields a unique result; therefore, for an assigned value of μ, there exists one, and only one, orbit having the period $2p\pi$. Since the orbits having the period $2p\pi$, $p>1$, include those having the period 2π, it follows that *all* the *orbits of this analytic type are closed after one synodic revolution.**

Since $\tau=\nu t$ the period of the solution in t is $2\pi/\nu$, and the quantity ν, which is so far arbitrary, can be determined by assigning the period of the solution. The parameter μ is then determined by the relation $\mu=m=N/\nu$; that is, the numerical value of μ is the ratio of the mean motions of the finite bodies and of the particle. If the direction of revolution of the particle is the same as that of the finite bodies—that is, if the orbit is direct—ν and N have the same sign and μ is positive; if the orbit is retrograde; ν and N have opposite signs and μ is negative. Since for an assigned value of μ there exists one, and only one, periodic orbit, and since values of μ which are numerically equal, but opposite in sign, give orbits having the same period in τ, it follows that for a given period there exist two, and only two, real orbits of the type under consideration. In one the motion is direct, and in the other it is retrograde.

We may now state the result as follows: *The period $2\pi/\nu$ of the solution may be assigned arbitrarily in advance, subject only to the condition that the ratio N/ν is sufficiently small, where $2\pi/N$ is the period of the motion of the finite bodies. Then there exist two, and only two, real periodic orbits of the particle having the required period. In one the motion is direct, and in the other it is retrograde. All the orbits of this type are closed after one synodic revolution.*

In deriving this conclusion no use was made of the explicit values of those terms in the disturbing function which are due to the body M_2. The proof depends entirely upon the form of certain terms of the solution which involve λ_1. Hence the analysis and conclusions are applicable without change to the case where n finite bodies revolve in circles in such a way as to form in the rotating plane a fixed configuration.

199. Construction of Periodic Orbits Having no Line of Symmetry.—It is possible to construct the periodic solutions of the differential equations (9) by the method indicated in the existence proof, but the process is laborious. A method will now be given by which the solution to any desired number of terms can be conveniently constructed. It is not necessary to determine the initial conditions explicitly in advance, and the computation involves only algebraic processes.

*Since no new orbits are obtained by taking $p>1$ we will assume hereafter that $p=1$.

It has been proved that the periodic solutions are expressible in the form

$$\left.\begin{aligned} \rho-1&=\rho_1\mu+\rho_2\mu^2+\rho_3\mu^3+\cdots+\rho_i\mu^i+\cdots, \\ w-\tau&=w_1\mu+w_2\mu^2+w_3\mu^3+\cdots+w_i\mu^i+\cdots. \end{aligned}\right\}\qquad(18)$$

The series (18) satisfy the differential equations (9) uniformly over a finite interval in μ, and hence, when the series are substituted in the differential equations, the coefficient of each power of μ must vanish. Furthermore, the series are periodic with period 2π in τ; and, because the periodicity holds for a continuous range of values of μ, each coefficient ρ_i and w_i separately is periodic with the period 2π in τ. It has been shown also that we can choose $w=0$ when $\tau=0$, and because this holds identically in μ, it follows that $w_i(0)=0$ for every i.

Let the solution (18) be substituted in the differential equations (9) and arrange the results as power series in μ. The terms of the first members have the following forms, where the accents indicate derivatives with respect to τ:

$$\left.\begin{aligned}
\frac{d^2\rho}{d\tau^2}&=\rho_1''\mu+\rho_2''\mu^2+\rho_3''\mu^3+\cdots+\rho_i''\mu^i+\cdots, \\
\rho\left(\frac{dw}{d\tau}+\mu\right)^2&=1+\left[\rho_1+2(w_1'+1)\right]\mu \\
&\quad+\left[\rho_2+2w_2'+2\rho_1(w_1'+1)+(w_1'+1)^2\right]\mu^2+\left[\rho_3+2w_3'\right. \\
&\quad\left.+2(w_1'+1)w_2'+2\rho_1w_2'+2\rho_2(w_1'+1)+\rho_1(w_1'+1)^2\right]\mu^3 \\
&\quad+\cdots+\left[\rho_i+2w_i'+2(w_1'+1)w_{i-1}'+2\rho_{i-1}(w_1'+1)\right. \\
&\quad\left.+2\rho_1w_{i-1}'+\cdots\right]\mu^i+\cdots, \\
\frac{1}{\rho^2}&=1-2\rho_1\mu-(2\rho_2-3\rho_1^2)\mu^2-(2\rho_3-6\rho_1\rho_2-4\rho_1^3)\mu^3 \\
&\quad+\cdots+(2\rho_i-6\rho_{i-1}\rho_1+\cdots)\mu^i+\cdots, \\
\rho\frac{d^2w}{d\tau^2}&=w_1''\mu+(w_2''+\rho_1w_1'')\mu^2+(w_3''+\rho_1w_2''+\rho_2w_1'')\mu^3 \\
&\quad+\cdots+(w_i''+\rho_1w_{i-1}''+\cdots+\rho_{i-1}w_1'')\mu^i+\cdots, \\
\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right)&=\rho_1'\mu+\left[\rho_2'+\rho_1'(w_1'+1)\right]\mu^2 \\
&\quad+\left[\rho_3'+\rho_2'(w_1'+1)+\rho_1'w_2'\right]\mu^3+\cdots \\
&\quad+\left[\rho_i'+\rho_{i-1}'(w_1'+1)+\cdots+\rho_1'w_{i-1}'\right]\mu^i+\cdots.
\end{aligned}\right\}\qquad(19)$$

The second members have no terms independent of μ^2. Therefore, on equating to zero the coefficients of the first power of μ, we have for the determination of ρ_1 and w_1

$$\frac{d^2\rho_1}{d\tau^2}-2\frac{dw_1}{d\tau}-3\rho_1=2, \qquad \frac{d^2w_1}{d\tau^2}+2\frac{d\rho_1}{d\tau}=0. \tag{20}$$

It follows from these equations that

$$\left.\begin{aligned}\rho_1&=2(1+c_1^{(1)})+c_2^{(1)}\cos\tau+c_3^{(1)}\sin\tau,\\ w_1&=c_4^{(1)}-(4+3c_1^{(1)})\tau-2c_2^{(1)}\sin\tau+2c_3^{(1)}\cos\tau,\end{aligned}\right\} \tag{21}$$

where $c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, $c_4^{(1)}$ are constants of integration. Since ρ_1 and w_1 are periodic, the coefficient of τ in w_1 must vanish. This condition determines the constant $c_1^{(1)}$, namely, $c_1^{(1)}=-4/3$. Since $w_1=0$ when $\tau=0$, $c_4^{(1)}=-2c_3^{(1)}$. The constants $c_2^{(1)}$ and $c_3^{(1)}$ are so far undetermined.

On equating to zero the coefficients of the second power of μ, the following set of equations is obtained:

$$\left.\begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=(w_1'+1)^2+2\rho_1(w_1+1)-3\rho_1^2+f_0,\\ \frac{d^2w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=-\rho_1w_1''-2\rho_1'(w_1'+1)+g_0,\end{aligned}\right\} \tag{22}$$

where f_0 and g_0 are obtained from f and g respectively by writing $\mu=0$, $w=\tau$, $\rho=1$. The second members are known functions of τ and the equations are explicitly

$$\left.\begin{aligned}\frac{d^2\rho_2}{d\tau^2}-2\frac{dw_2}{d\tau}-3\rho_2&=A_0^{(2)}+\left(\tfrac{14}{3}c_2^{(1)}+A_1^{(2)}\right)\cos\tau+\tfrac{14}{3}c_3^{(1)}\sin\tau\\ &\qquad+A_2^{(2)}\cos2\tau+A_3^{(2)}\cos3\tau,\\ \frac{d^2w_2}{d\tau^2}+2\frac{d\rho_2}{d\tau}&=\left(\tfrac{10}{3}c_2^{(1)}+D_1^{(2)}\right)\sin\tau-\tfrac{10}{3}c_3^{(1)}\cos\tau+D_2^{(2)}\sin2\tau+D_3^{(2)}\sin3\tau.\end{aligned}\right\} \tag{23}$$

On integrating the second equation, we have

$$\frac{dw_2}{d\tau}+2\rho_2=c_1^{(2)}-\left(\tfrac{10}{3}c_2^{(1)}+D_1^{(2)}\right)\cos\tau-\tfrac{10}{3}c_3^{(1)}\sin\tau-\frac{D_2^{(2)}}{2}\cos2\tau-\frac{D_3^{(2)}}{3}\cos3\tau. \tag{24}$$

On eliminating $\frac{dw_2}{d\tau}$ from the first of equations (23) by means of equation (24), there results

$$\left.\begin{aligned}\frac{d^2\rho_2}{d\tau^2}+\rho_2&=A_0^{(2)}+2c_1^{(2)}+\left(-2c_2^{(1)}+A_1^{(2)}-2D_1^{(2)}\right)\cos\tau-2c_3^{(1)}\sin\tau\\ &\qquad+\left(A_2^{(2)}-\tfrac{2}{2}D_2^{(2)}\right)\cos2\tau+\left(A_3^{(2)}-\tfrac{2}{3}D_3^{(2)}\right)\cos3\tau.\end{aligned}\right\} \tag{25}$$

In order that the solution of equations (25) shall contain no non-periodic term, the coefficients of $\cos\tau$ and $\sin\tau$ must vanish; hence $c_2^{(1)}$ and $c_3^{(1)}$ are determined by the conditions

$$2c_2^{(1)}=A_1^{(2)}-2D_1^{(2)}, \qquad c_3^{(1)}=0.$$

With these values of $c_2^{(1)}$ and $c_3^{(1)}$ the solution becomes

$$\rho_2=A_0^{(2)}+2c_1^{(2)}+c_2^{(2)}\cos\tau+c_3^{(2)}\sin\tau+a_2^{(2)}\cos 2\tau+a_3^{(2)}\cos 3\tau,$$

where

$$a_j^{(2)}=\frac{1}{1-j^2}\left(A_j^{(2)}-\frac{2}{j}D_j^{(2)}\right) \qquad (j=2,\ 3).$$

On substituting this value of ρ_2 in equation (24) and integrating, we obtain for w_2 a solution of the form

$$w_2=c_4^{(2)}-(2A_0^{(2)}+3c_1^{(2)})\tau-2c_2^{(2)}\sin\tau+2c_3^{(2)}\cos\tau+\delta_2^{(2)}\sin 2\tau+\delta_3^{(2)}\sin 3\tau,$$

where

$$\delta_j^{(2)}=-\frac{1}{j^2}\left(D_j^{(2)}+2ja_j^{(2)}\right) \qquad (j=2,\ 3).$$

Since w_2 is periodic, $c_1^{(2)}$ is determined by the condition

$$2A_0^{(2)}+3c_1^{(2)}=0.$$

Since $w_2=0$ when $\tau=0$, $c_4^{(2)}$ is expressible in terms of $c_3^{(2)}$, namely,

$$c_4^{(2)}+2c_3^{(2)}=0.$$

Of the eight constants of integration which have been introduced in the first two steps, five ($c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$, $c_4^{(1)}$, $c_1^{(2)}$) have been determined uniquely; $c_4^{(2)}$ has been expressed uniquely in terms of $c_3^{(2)}$; while the remaining two ($c_2^{(2)}$, $c_3^{(2)}$) are still arbitrary.

By equating to zero the coefficients of the third power of μ the following set of equations is obtained:

$$\left.\begin{aligned}
&\frac{d^2\rho_3}{d\tau^2}-2\frac{dw_3}{d\tau}-3\rho_3=2(\rho_1+w_1'+1)\,w_2'+2\rho_2(w_1'+1)\\
&\qquad\qquad\qquad +\rho_1(w_1'+1)^2-6\rho_1\rho_2+4\rho_1^3+f_1,\\
&\frac{d^2w_3}{d\tau^2}+2\frac{d\rho_3}{d\tau}=-\rho_1w_2''-\rho_2w_1''-2\rho_2'(w_1'+1)-2\rho_1'w_2'+g_1,
\end{aligned}\right\} \quad (26)$$

where f_1 and g_1 denote the coefficients of μ in f and g respectively. The second members are known functions of τ, and the equations have the form

$$\left.\begin{aligned}
&\frac{d^2\rho_3}{d\tau^2}-2\frac{dw_3}{d\tau}-3\rho_3=A_0^{(3)}+\left(\tfrac{14}{3}c_2^{(2)}+A_1^{(3)}\right)\cos\tau+\left(\tfrac{14}{3}c_3^{(2)}+B_1^{(3)}\right)\sin\tau\\
&\qquad\qquad\qquad +A_j^{(3)}\cos j\tau+B_j^{(3)}\sin j\tau \qquad (j=2,\ 3,\ 4).\\
&\frac{d^2w_3}{d\tau^2}+2\frac{d\rho_3}{d\tau}=\left(\tfrac{10}{3}c_2^{(2)}+D_1^{(3)}\right)\sin\tau+\left(-\tfrac{10}{3}c_3^{(2)}+C_1^{(3)}\right)\cos\tau\\
&\qquad\qquad\qquad +D_j^{(3)}\sin j\tau+C_j^{(3)}\cos j\tau.
\end{aligned}\right\} \quad (27)$$

The treatment of equations (27) proceeds by steps similar to those employed in the solution of equations (23). Four new constants of integration are introduced, namely, $c_1^{(3)}$, $c_2^{(3)}$, $c_3^{(3)}$, $c_4^{(3)}$, while $c_2^{(2)}$, $c_3^{(2)}$, and $c_1^{(3)}$ are uniquely determined by the conditions

$$2c_2^{(2)} = A_1^{(3)} - 2D_1^{(3)}, \qquad 2c_3^{(2)} = B_1^{(3)} + 2C_1^{(3)}, \qquad 3c_1^{(3)} = -2A_0^{(3)}.$$

The solution has the form

$$\rho_3 = \alpha_0^{(3)} + c_2^{(3)}\cos\tau + c_3^{(3)}\sin\tau + \sum_{j=2}^{4}\left(\alpha_j^{(3)}\cos j\tau + \beta_j^{(3)}\sin j\tau,\right),$$

$$w_3 = c_4^{(3)} + \left(-2c_2^{(3)} - \tfrac{5}{3}A_1^{(3)} + \tfrac{7}{3}D_1^{(3)}\right)\sin\tau + \left(2c_3^{(3)} + \tfrac{5}{3}B_1^{(3)} + \tfrac{7}{3}C_1^{(3)}\right)\cos\tau$$
$$+ \sum_{j=2}^{4}\left(\delta_j^{(3)}\sin j\tau + \gamma_j^{(3)}\cos j\tau\right).$$

From the condition that $w_3 = 0$ when $\tau = 0$, it follows that $c_4^{(3)}$ is expressible uniquely in terms of $c_3^{(3)}$ by the relation

$$c_4^{(3)} + 2c_3^{(3)} + \sum_{j=2}^{4}\gamma_j^{(3)} = 0.$$

The two constants $c_2^{(3)}$ and $c_3^{(3)}$ are determined in the next step.

It can be established by complete induction that the preceding process can be carried as far as is desired. Suppose $\rho_1, w_1; \rho_1, w_2; \ldots; \rho_{i-1}, w_{i-1}$ have been determined by this process. The expressions have the following form:

$$\rho_l = \alpha_0^{(l)} + \sum_{j=1}^{l+1}\left(\alpha_j^{(l)}\cos j\tau + \beta_j^{(l)}\sin j\tau\right),$$

$$w_l = \gamma_0^{(l)} + \sum_{j=1}^{l+1}\left(\delta_j^{(l)}\sin j\tau + \gamma_j^{(l)}\cos j\tau\right) \qquad (l = 1, 2, \ldots, i-2),$$

$$\rho_{i-1} = \alpha_0^{(i-1)} + c_2^{(i-1)}\cos\tau + c_3^{(i-1)}\sin\tau + \sum_{j=2}^{i}\left(\alpha_j^{(i-1)}\cos j\tau + \beta_j^{(i-1)}\sin j\tau\right),$$

$$w_{i-1} = c_4^{(i-1)} + \left(-2c_2^{(i-1)} - \tfrac{5}{3}A_1^{(i-1)} + \tfrac{7}{3}D_1^{(i-1)}\right)\sin\tau + \left(2c_3^{(i-1)} + \tfrac{5}{3}B_1^{(i-1)} + \tfrac{7}{3}C_1^{(i-1)}\cos\tau\right)$$
$$+ \sum_{j=2}^{i}\left(\delta_j^{(i-1)}\sin j\tau + \gamma_j^{(i-1)}\cos j\tau\right).$$

The constants of integration have been uniquely determined except $c_2^{(i-1)}$, $c_3^{(i-1)}$, and $c_4^{(i-1)}$. The first two are so far arbitrary, while $c_4^{(i-1)}$ is expressible in terms of $c_3^{(i-1)}$ by the relation

$$c_4^{(i-1)} + 2c_3^{(i-1)} + \sum_{j=1}^{i}\gamma_j^{(i-1)} = 0.$$

The equations for the determination of ρ_i and w_i have the form

$$\left.\begin{aligned}\frac{d^2\rho_i}{d\tau^2}-2\frac{dw_i}{d\tau}-3\rho_i&=A_0^{(i)}+\left(\tfrac{14}{3}c_2^{(i-1)}+A_1^{(i)}\right)\cos\tau+\left(\tfrac{14}{3}c_3^{(i-1)}+B_1^{(i)}\right)\sin\tau\\&\quad+\sum_{j=2}^{i+1}\left(A_j^{(i)}\cos j\tau+B_j^{(i)}\sin j\tau\right),\\\frac{d^2w_i}{d\tau^2}+2\frac{d\rho_i}{d\tau}&=\left(\tfrac{10}{3}c_2^{(i-1)}+D_1^{(i)}\right)\sin\tau+\left(-\tfrac{10}{3}c_3^{(i-1)}+C_1^{(i)}\right)\cos\tau\\&\quad+\sum_{j=2}^{i+1}\left(D_j^{(i)}\sin j\tau+C_j^{(i)}\cos j\tau\right).\end{aligned}\right\}\quad(28)$$

The coefficients A, B, C, D are known constants and equations (28) are solved by the steps employed in the solution of equations (23). During the process four constants of integration are introduced, namely $c_1^{(i)}$, $c_2^{(i)}$, $c_3^{(i)}$, $c_4^{(i)}$, and four are uniquely determined by the conditions

$$\left.\begin{aligned}2c_2^{(i-1)}&=A_1^{(i)}-2D_1^{(i)}, &\qquad 2c_3^{(i-1)}&=B_1^{(i)}+2C_1^{(i)},\\c_4^{(i-1)}&=-2c_3^{(i-1)}-\sum_{j=1}^{i}\gamma_j^{(i-1)}, &\qquad 3c_1^{(i)}&=-2A_0^{(i)}.\end{aligned}\right\}\quad(29)$$

The solution of equations (28) is

$$\left.\begin{aligned}\rho_i&=a_0^{(i)}+c_2^{(i)}\cos\tau+c_3^{(i)}\sin\tau+\sum_{j=2}^{i+1}\left(a_j^{(i)}\cos j\tau+\beta_j^{(i)}\sin j\tau\right),\\w_i&=c_4^{(i)}+\delta_1^{(i)}\sin\tau+\gamma_1^{(i)}\cos\tau+\sum_{j=2}^{i+1}\left(\delta_j^{(i)}\sin j\tau+\gamma_j^{(i)}\cos j\tau\right),\end{aligned}\right\}\quad(30)$$

where the coefficients are given by the formulas

$$\left.\begin{aligned}a_0^{(i)}&=-\tfrac{1}{3}A_0^{(i)}, &\qquad \delta_1^{(i)}&=-2c_2^{(i)}-\tfrac{5}{3}A_1^{(i)}+\tfrac{7}{3}D_1^{(i)},\\a_j^{(i)}&=\frac{1}{1-j^2}\left(A_j^{(i)}-\frac{2}{j}D_j^{(i)}\right), &\qquad \delta_j^{(i)}&=-\frac{1}{j^2}\left(D_j^{(i)}+2ja_j^{(i)}\right),\\\beta_j^{(i)}&=\frac{1}{1-j^2}\left(B_j^{(i)}+\frac{2}{j}C_j^{(i)}\right), &\qquad \gamma_1^{(i)}&=2c_3^{(i)}+\tfrac{5}{3}B_1^{(i)}+\tfrac{7}{3}C_1^{(i)},\\&&\gamma_j^{(i)}&=-\frac{1}{j^2}\left(C_j^{(i)}-2j\beta_j^{(i)}\right).\end{aligned}\right\}\quad(31)$$

The formulas (31) together with the conditions (29) are sufficient to construct the periodic solution of equations (9) to any desired degree of accuracy; the computation is entirely algebraic. In order to determine the constants of integration entering in the last (i^{th}) step, it is necessary to compute the coefficients $A_1^{(i+1)}$, $B_1^{(i+1)}$, $C_1^{(i+1)}$, $D_1^{(i+1)}$ of the next following step.

200. Numerical Example 1.—For the purpose of illustration, we assign numbers to the constants involved in the preceding analysis and construct an orbit. In this and the other numerical examples which occur later, it has not been shown that the processes are valid for the numerical values which are employed and which have been selected for convenience in graphical representation. It is probable that the series are convergent, although it has not been found possible to determine the true radii of convergence.

The differential equations of motion are equations (5). On putting $m=\mu$ and writing the second members explicitly as far as terms of the second degree in a/A, we have

$$\left.\begin{aligned}
&\frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\mu\right)^2+\frac{1}{\rho^2}=\frac{k^2M_1\rho}{\nu^2A^3}\left[\frac{1}{2}\left\{1+3\cos 2w\right\}+\frac{3}{8}\frac{a}{A}\rho\left\{3\cos w+5\cos 3w\right\}\right.\\
&\qquad\left.+\frac{1}{16}\left(\frac{a}{A}\right)^2\rho^2\left\{9+20\cos 2w+35\cos 4w\right\}+\cdots\right]\\
&\qquad+\frac{k^2M_2\rho}{\nu^2A^3}\left[\frac{1}{2}\left\{1+3\cos 2\left(w-\frac{\pi}{3}\right)\right\}+\frac{3}{8}\frac{a}{A}\rho\left\{3\cos\left(w-\frac{\pi}{3}\right)+5\cos 3\left(w-\frac{\pi}{3}\right)\right\}\right.\\
&\qquad\left.+\frac{1}{16}\left(\frac{a}{A}\right)^2\rho^2\left\{9+20\cos 2\left(w-\frac{\pi}{3}\right)+35\cos 4\left(w-\frac{\pi}{3}\right)\right\}+\cdots\right],\\
&\rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right)=-\frac{k^2M_1\rho}{\nu^2A^3}\left[\frac{3}{2}\sin 2w\right.\\
&\qquad\left.+\frac{3}{8}\frac{a}{A}\rho\left\{\sin w+5\sin 3w\right\}+\frac{5}{16}\left(\frac{a}{A}\right)^2\rho^2\left\{2\sin 2w+7\sin 4w\right\}+\cdots\right]\\
&\qquad-\frac{k^2M_2\rho}{\nu^2A^3}\left[\frac{3}{2}\sin 2\left(w-\frac{\pi}{3}\right)+\frac{3}{8}\frac{a}{A}\rho\left\{\sin\left(w-\frac{\pi}{3}\right)+5\sin 3\left(w-\frac{\pi}{3}\right)\right\}\right.\\
&\qquad\left.+\frac{5}{16}\left(\frac{a}{A}\right)^2\rho^2\left\{2\sin 2\left(w-\frac{\pi}{3}\right)+7\sin 4\left(w-\frac{\pi}{3}\right)\right\}+\cdots\right].
\end{aligned}\right\}\quad(32)$$

We select M for the unit of mass and suppose $M_1=10$, $M_2=5$. For the unit of distance we take the distance between the finite bodies, that is, $A=1$; and the unit of time is selected so that $N=1$. The period of the solution is assigned so that $\nu=5$, whence

$$\mu=m=\frac{N}{\nu}=0.2.$$

The constant k^2 is determined from the relation

$$N^2A^3 = k^2(M + M_1 + M_2),$$

whence

$$k^2 = 0.06250, \qquad k^2M_1 = 0.62500, \qquad k^2M_2 = 1.56250\mu.$$

The constant a was defined by $\nu^2 a^3 = k^2 M$, whence

$$\frac{a}{A} = a = 0.67860\mu.$$

With these numerical values equations (32) become

$$\left.\begin{aligned}
\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} &= (0.31250 + 0.93750\cos 2w)\rho\mu^2 \\
&+ (0.78125 - 1.17188\cos 2w + 2.02977\sin 2w)\rho\mu^3 \\
&+ (0.47714\cos w + 0.79523\cos 3w)\rho^2\mu^3 \\
&+ (0.59643\cos w + 1.03308\sin w - 1.98808\cos 3w)\rho^2\mu^4 \\
&+ (0.16188 + 0.35974\cos 2w + 0.62954\cos 4w)\rho^3\mu^4 + \cdots, \\
\rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) &= -(0.93750\sin 2w)\rho\mu^2 \\
&+ (1.17188\sin 2w + 2.02977\cos 2w)\rho\mu^3 \\
&- (0.15905\sin w + 0.79523\sin 3w)\rho^2\mu^3 \\
&+ (-0.19881\sin w + 0.34436\cos w + 1.98808\sin 3w)\rho^2\mu^4 \\
&- (0.17987\sin 2w + 0.62954\sin 4w)\rho^3\mu^4 + \cdots.
\end{aligned}\right\} \quad (33)$$

The periodic solution of equations (33) is

$$\left.\begin{aligned}
\rho = 1 - \tfrac{2}{3}\mu &+ (0.45139 + 0.39762\cos\tau - 0.62500\cos 2\tau)\mu^2 \\
&+ (-0.47647 + 1.04542\cos\tau + 0.86090\sin\tau + 0.46875\cos 2\tau \\
&- 1.35318\sin 2\tau - 0.16567\cos 3\tau)\mu^3 + \cdots, \\
w = \tau &+ (-0.79524\sin\tau + 0.85938\sin 2\tau)\mu^2 \\
&+ (0.13882 - 3.25720\sin\tau + 1.72180\cos\tau + 0.27995\sin 2\tau \\
&- 1.86062\cos 2\tau + 0.19881\sin 3\tau)\mu^3 + \cdots.
\end{aligned}\right\} \quad (34)$$

Substituting the numerical value $\mu = 0.2$, equations (34), if convergent, are the equations of motion of the particle P. The orbit is shown in Fig. 9. In this and the figures of the following numerical examples the comparison circles are not the circular orbits which have been called the *undisturbed* orbits. The undisturbed orbits are referred to fixed axes while the drawings are made with reference to rotating axes. The comparison circles represent orbits in which the particle would make a complete revolution

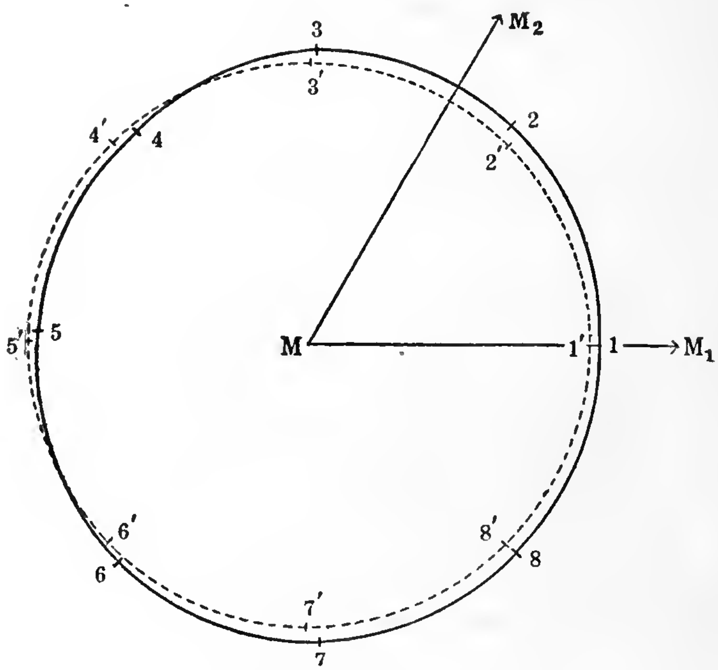

FIG. 9.

with respect to the rotating axes during the period. The points which are numbered 1, 2, . . . , 8 represent positions of the particle in the periodic orbit at intervals of $\tau = \pi/4$. The corresponding positions in the comparison circle are indicated by the numbers $1'$, $2'$, . . . , $8'$.

201. Some Particular Solutions of the Problem of n Bodies.—The existence of symmetrical periodic orbits of the particle depends upon the masses and motion of the finite bodies. So far as the analysis is concerned, this motion may be arbitrarily periodic, without reference to the nature of the forces producing it. It is required only that the motion of the finite bodies shall be known and that they shall attract the particle according to the Newtonian law. It will be interesting, however, in developing the analysis, to prescribe motion for the finite bodies, which is possible under the law of the inverse square of the distance. For three finite bodies the two solutions of Lagrange are well known. In the case of the equilateral-triangle solution the periodic orbits of the particle about one of the bodies

have a line of symmetry if the other two masses are equal. In the case of the straight-line solution the periodic orbits of the particle about any one of the bodies are symmetrical with respect to the line joining the finite bodies. The particular solutions which Lagrange has given for three bodies have been extended to some cases of more than three bodies,* and we shall consider two examples: (1) in which there are five bodies, and (2) in which there are nine bodies.

Let the masses of n finite bodies be represented by $M_1, M_2, \ldots, M_n$. Suppose that the bodies lie always in the same plane, and that their coördinates with respect to their common center of mass as origin and a system of rectangular axes which rotate with the uniform angular velocity N are, respectively, (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n). Supposing that the bodies attract each other according to the Newtonian law, the differential equations of motion are

$$\left.\begin{aligned} &\frac{d^2x_i}{dt^2}-2N\frac{dy_i}{dt}-N^2x_i=-k^2\sum_{j=1}^{n}\frac{M_j(x_i-x_j)}{r_{i,j}^3},\\ &\frac{d^2y_i}{dt^2}+2N\frac{dx_i}{dt}-N^2y_i=-k^2\sum_{j=1}^{n}\frac{M_j(y_i-y_j)}{r_{i,j}^3},\\ &r_{i,j}=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}\qquad (i=1, 2, \ldots, n;\ j\neq i). \end{aligned}\right\} \tag{35}$$

If we assume that each body is revolving in a circle about the common center of mass of the system with the uniform angular velocity N, its coördinates with respect to the rotating axes are constants and the derivatives of the coördinates with respect to the time are zero. Equations (35) therefore reduce to the following system of algebraic equations

$$\left.\begin{aligned} &-N^2x_i+k^2\sum_{j=1}^{n}\frac{M_j(x_i-x_j)}{r_{i,j}^3}=0 \qquad (i=1, 2, \ldots n;\ j\neq i).\\ &-N^2y_i+k^2\sum_{j=1}^{n}\frac{M_j(y_i-y_j)}{r_{i,j}^3}=0. \end{aligned}\right\} \tag{36}$$

It follows from these equations that

$$M_1x_1+M_2x_2+\cdots+M_nx_n=0,\qquad M_1y_1+M_2y_2+\cdots+M_ny_n=0, \tag{37}$$

which express the fact that the origin of coördinates is at the center of mass. These equations may be used instead of two of (36).

*See Hoppe, "Erweiterung der bekannten Special lösungen des Dreikörperproblems;" *Archiv. der Math. und Phys.*, vol. 64, pp. 218–223. Andoyer, "Sur l'équilibre relatif de n corps;" *Bull. astron.*, vol. 23 (1906), pp. 50–59. Longley, "Some particular solutions in the problem of n bodies;" *Bull. Amer. Math. Soc.*, vol. 13 (1907), pp. 324–335.

This system of $2n$ simultaneous algebraic equations involves the square of the angular velocity, N, the $n-1$ ratios of the masses, and the $2n-1$ ratios of the distances x_i and y_i. Accordingly $n-1$ of these quantities may be chosen arbitrarily and, if the resulting equations are independent, the remaining $2n$ quantities are determined by the relations (36). In order to admit physical interpretation, the quantity N^2 and the masses must be real and positive, while the coördinates must be real. With these restrictions it is not easy to discuss the general solutions of equations (36), but some interesting results can be obtained by a study of special cases. If, in the problem of three bodies, the assumption is made that the triangle formed by the three bodies is isosceles, it can be shown that equations (36) can be satisfied only if the triangle is also equilateral.

On supposing the number of bodies to be five, the system of equations to be satisfied is

$$
\left.
\begin{array}{ll}
(a) & M_1x_1+M_2x_2+M_3x_3+M_4x_4+M_5x_5=0,\\[2mm]
(b) & -\dfrac{N^2}{k^2}x_1+\dfrac{M_2(x_1-x_2)}{r_{1,2}^3}+\dfrac{M_3(x_1-x_3)}{r_{1,3}^3}+\dfrac{M_4(x_1-x_4)}{r_{1,4}^3}+\dfrac{M_5(x_1-x_5)}{r_{1,5}^3}=0,\\[2mm]
(c) & -\dfrac{N^2}{k^2}x_2+\dfrac{M_1(x_2-x_1)}{r_{2,1}^3}+\dfrac{M_3(x_2-x_3)}{r_{2,3}^3}+\dfrac{M_4(x_2-x_4)}{r_{2,4}^3}+\dfrac{M_5(x_2-x_5)}{r_{2,5}^3}=0,\\[2mm]
(d) & -\dfrac{N^2}{k^2}x_3+\dfrac{M_1(x_3-x_1)}{r_{3,1}^3}+\dfrac{M_2(x_3-x_2)}{r_{3,2}^3}+\dfrac{M_4(x_3-x_4)}{r_{3,4}^3}+\dfrac{M_5(x_3-x_5)}{r_{3,5}^3}=0,\\[2mm]
(e) & -\dfrac{N^2}{k^2}x_4+\dfrac{M_1(x_4-x_1)}{r_{4,1}^3}+\dfrac{M_2(x_4-x_2)}{r_{4,2}^3}+\dfrac{M_3(x_4-x_3)}{r_{4,3}^3}+\dfrac{M_5(x_4-x_5)}{r_{4,5}^3}=0,\\[2mm]
(f) & M_1y_1+M_2y_2+M_3y_3+M_4y_4+M_5y_5=0,\\[2mm]
(g) & -\dfrac{N^2}{k^2}y_1+\dfrac{M_2(y_1-y_2)}{r_{1,2}^3}+\dfrac{M_3(y_1-y_3)}{r_{1,3}^3}+\dfrac{M_4(y_1-y_4)}{r_{1,4}^3}+\dfrac{M_5(y_1-y_5)}{r_{1,5}^3}=0,\\[2mm]
(h) & -\dfrac{N^2}{k^2}y_2+\dfrac{M_1(y_2-y_1)}{r_{2,1}^3}+\dfrac{M_3(y_2-y_3)}{r_{2,3}^3}+\dfrac{M_4(y_2-y_4)}{r_{2,4}^3}+\dfrac{M_5(y_2-y_5)}{r_{2,5}^3}=0,\\[2mm]
(i) & -\dfrac{N^2}{k^2}y_3+\dfrac{M_1(y_3-y_1)}{r_{3,1}^3}+\dfrac{M_2(y_3-y_2)}{r_{3,2}^3}+\dfrac{M_4(y_3-y_4)}{r_{3,4}^3}+\dfrac{M_5(y_3-y_5)}{r_{3,5}^3}=0,\\[2mm]
(j) & -\dfrac{N^2}{k^2}y_4+\dfrac{M_1(y_4-y_1)}{r_{4,1}^3}+\dfrac{M_2(y_4-y_2)}{r_{4,2}^3}+\dfrac{M_3(y_4-y_3)}{r_{4,3}^3}+\dfrac{M_5(y_4-y_5)}{r_{4,5}^3}=0.
\end{array}
\right\}
\qquad (38)
$$

Let us suppose that M_5 lies at the origin of coördinates and that the other four masses, which are equal in pairs, lie on the coördinate axes at

the vertices of a rhombus with the equal masses opposite each other (see Fig. 10). This is equivalent to the relations

$$\left.\begin{array}{llllll} x_1=A, & x_2=0, & x_3=-A, & x_4=0, & x_5=0, & M_1=M_3=M', \\ y_1=0, & y_2=KA, & y_3=0, & y_4=-KA, & y_5=0, & M_2=M_4=M''. \end{array}\right\} \quad (39)$$

On substituting the assumed values (39) in equations (38), we find that

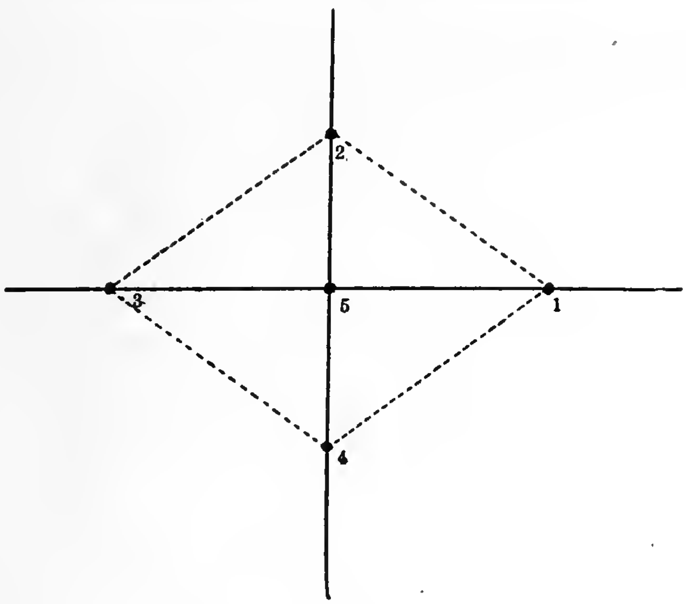

FIG. 10.

(a), (c), (e), (f), (g), and (i) are satisfied. Equations (b) and (d) become identical, yielding

$$\frac{N^2}{k^2}=\frac{2M'}{(2A)^3}+\frac{2M''}{A^3(\sqrt{1+K^2})^3}+\frac{M_5}{A^3}, \quad (40)$$

and equations (h) and (j) become identical, yielding

$$\frac{N^2}{k^2}=\frac{2M'}{A^3(\sqrt{1+K^2})^3}+\frac{2M''}{(2KA)^3}+\frac{M_5}{(KA)^3}. \quad (41)$$

When M', M'', M_5, K, and A have been chosen or determined, these equations insure a positive value for N^2.

On eliminating N^2/k^2 between equations (40) and (41), we obtain for the relation between the masses,

$$M''=\left\{\frac{8K^3-K^3(\sqrt{1+K^2})^3}{8K^3-(\sqrt{1+K^2})^3}\right\}M'+\left\{\frac{4(1-K^3)(\sqrt{1+K^2})^3}{8K^3-(\sqrt{1+K^2})^3}\right\}M_5. \quad (42)$$

The choice of the constant K, which is the ratio of the diagonals of the rhombus, is limited by the condition that the resulting ratio of the masses must be positive. To investigate this condition we set $M_5=1$. Then, regarding K as a parameter, equation (42) represents a straight line in the

$M'M''$ plane. Only those pairs of values (M', M'') which represent a point in the first quadrant are admissible. This condition will certainly be satisfied if the slope is positive, that is, if the coefficient of M' is positive. This condition is easily found to be

$$\frac{1}{\sqrt{3}} < K < \sqrt{3}. \tag{43}$$

If the slope of the line is negative, it may still lie partly in the first quadrant if the intercept on the M''-axis is positive—that is, if the coefficient of M_5 in equation (42) is positive. It is easily verified, however, that values of

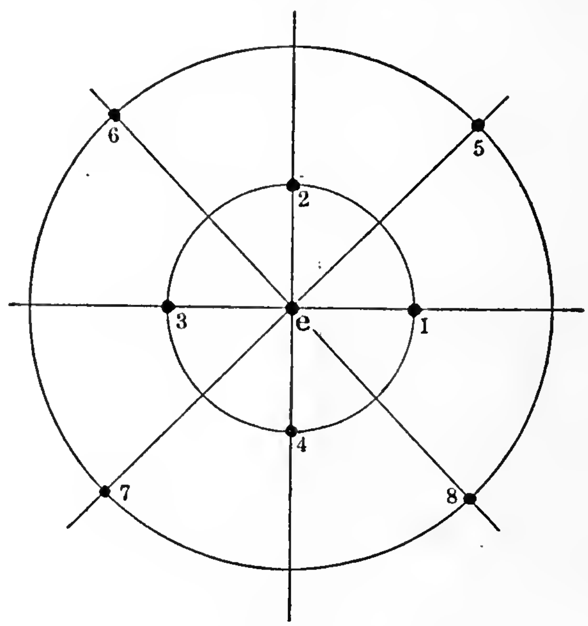

FIG. 11.

K which make the slope negative, also make the intercept on the M''-axis negative. Hence the choice of the ratio of the diagonals is limited by the condition (43); otherwise it is arbitrary.

The conditions for the rhombus configuration with a fifth body at the center may be summarized as follows:

Suppose a value of K satisfying condition (43) *is assigned; then two of the masses M', M'', and M_5 can be chosen arbitrarily* and the third is determined by equation* (42). *The length, A, of one semi-diagonal can be selected at pleasure and the angular velocity is then determined by equation* (40) *or equation* (41).

In the following discussions this configuration will be referred to as configuration (A).

The second configuration, (B), will consist of nine bodies, arranged as shown in Fig. 11. Four bodies M_1, M_2, M_3, M_4 lie on the coördinate axes and on the circumference of a circle of radius A; four others, M_5, M_6,

*Except when $K=1$ and the rhombus becomes a square. Then M' and M'' must be equal.

M_7, M_8 lie on the bisectors of the angles between the coördinate axes and on a circle of radius KA, while the ninth body is at the center. Supposing that all the masses on the same circle are equal, we have the following conditions:

$$\left.\begin{array}{ll}
x_1=A, & y_1=0, \\
x_2=0, & y_2=A, \\
x_3=-A, & y_3=0, \\
x_4=0, & y_4=-A, \\
x_5=\frac{1}{2}\sqrt{2}KA, & y_5=\frac{1}{2}\sqrt{2}KA, \\
x_6=-\frac{1}{2}\sqrt{2}KA, & y_6=\frac{1}{2}\sqrt{2}KA, \\
x_7=-\frac{1}{2}\sqrt{2}KA, & y_7=-\frac{1}{2}\sqrt{2}KA, \\
x_8=\frac{1}{2}\sqrt{2}KA, & y_8=-\frac{1}{2}\sqrt{2}KA, \\
x_9=0, & y_9=0, \\
M_1=M_2=M_3=M_4=M', & M_5=M_6=M_7=M_8=M''.
\end{array}\right\} \qquad (44)$$

Corresponding to equations (38) of the five-body problem, there is a set of equations, which, upon the assumptions (44), reduce to

$$\left.\begin{aligned}
\frac{N^2A^3}{k^2}&=M'\left(\frac{1}{\sqrt{2}}+\frac{1}{4}\right)+2M''\left\{\frac{1-\frac{K}{\sqrt{2}}}{(\sqrt{1-\sqrt{2}K+K^2})^3}+\frac{1+\frac{K}{\sqrt{2}}}{(\sqrt{1+\sqrt{2}K+K^2})^3}\right\}+M_9, \\
\frac{N^2A^3}{k^2}&=M'\left\{\frac{1-\frac{1}{\sqrt{2}K}}{(\sqrt{1-\sqrt{2}K+K^2})^3}+\frac{1+\frac{1}{\sqrt{2}K}}{(\sqrt{1+\sqrt{2}K+K^2})^3}\right\}+\frac{M''}{K^3}\left(\frac{1}{\sqrt{2}}+\frac{1}{4}\right)+\frac{M_9}{K^3}.
\end{aligned}\right\} \qquad (45)$$

Eliminating N^2A^3/k^2 from equations (45), there results the following condition on the masses M', M'', M_9, and the ratio, K, of the radii of the circles,

$$\left.\begin{aligned}
&M''\left\{\left(\frac{1}{\sqrt{2}}+\frac{1}{4}\right)\frac{1}{K^3}+\frac{K\sqrt{2}-2}{(\sqrt{1-\sqrt{2}K+K^2})^3}+\frac{K\sqrt{2}+2}{(\sqrt{1+\sqrt{2}K+K^2})^3}\right\}= \\
&M'\left\{\frac{1}{\sqrt{2}}+\frac{1}{4}-\frac{2K-\sqrt{2}}{(\sqrt{1-\sqrt{2}K+K^2})^3}-\frac{2K+\sqrt{2}}{(\sqrt{1+\sqrt{2}K+K^2})^3}\right\}+M_9\left(1-\frac{1}{K^3}\right).
\end{aligned}\right\} \qquad (46)$$

The choice of the ratio K and two of the masses is limited by the conditions that the third mass, as determined by equation (46) and the square of the angular velocity, N^2, from (45), shall be positive. The limits within which the choice can be made have not been determined, but for the purpose of application in numerical example 3, the following set of values satisfying the conditions has been computed:

$$K=2, \quad M'=1, \quad A=1, \quad M_9=1, \quad M''=8.2526, \quad N^2=1.6399k^2.$$

202. Existence of Symmetrical Periodic Orbits.—For the development of the type of analysis applicable to symmetrical orbits we shall use configuration (A) of the preceding section, the notation being unchanged except that the mass at the center will now be denoted by M instead of M_5. With reference to M as origin and an axis having a fixed direction in space, let the polar coördinates of M_1, M_2, M_3, M_4, and P be, respectively, (R_1, V_1), (R_2, V_2), (R_3, V_3), (R_4, V_4), and (r, v). The coördinates of the bodies are expressed in terms of the time as follows:

$$R_1 = R_3 = A, \qquad R_2 = R_4 = KA, \qquad V_1 = V_2 - \frac{\pi}{2} = V_3 - \pi = V_4 - \frac{3\pi}{2} = Nt,$$

where N is given by equation (40) or equation (41).

The differential equations of motion of P are

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2M}{r^2} = \frac{\partial\Omega}{\partial r}, \qquad r\frac{d^2v}{dt^2} + 2\frac{dr}{dt}\frac{dv}{dt} = \frac{1}{r}\frac{\partial\Omega}{\partial r}, \tag{47}$$

where

$$\Omega = k^2\left[\frac{M_1}{r_1} + \frac{M_2}{r_2} + \frac{M_3}{r_3} + \frac{M_4}{r_4} - \frac{M_1}{A^2}r\cos(v - V_1) \right.$$
$$\left. - \frac{M_2}{K^2A^2}r\cos(v - V_2) - \frac{M_3}{A^2}r\cos(v - V_3) - \frac{M_4}{K^2A^2}r\cos(v - V_4)\right],$$

$$r_1 = \sqrt{r^2 + A^2 - 2rA\cos(v - V_1)}, \qquad r_3 = \sqrt{r^2 + A^2 - 2rA\cos(v - V_3)},$$
$$r_2 = \sqrt{r^2 + A^2 - 2rKA\cos(v - V_2)}, \qquad r_4 = \sqrt{r^2 + K^2A^2 - 2rKA\cos(v - V_4)}.$$

We now define m and a by the relations

$$m\nu = N, \qquad \nu^2a^3 = k^2M,$$

where, as in the preceding case, ν denotes the mean angular velocity of P. The motion is referred to an axis rotating with the angular velocity N and passing always through M_1 by the substitution $v = w + V_1 = w + Nt$, and factors depending on the units employed are eliminated by the substitution $r = a\rho$, $\nu t = \tau$. We obtain then the differential equations of relative motion

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + m\right)^2 + \frac{1}{\rho^2} = \frac{1}{\nu^2a}\frac{\partial\Omega}{\partial(a\rho)}, \qquad \rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + m\right) = \frac{1}{\nu^2a^2\rho}\frac{\partial\Omega}{\partial w}. \tag{48}$$

These equations have the same form as the set (9) and the analysis and results of that problem are applicable in this case. We know, then, that there exists one, and only one, orbit in which the particle P moves with direct motion and with a preassigned period. We shall see that there exists one, and only one, such orbit which is symmetrical to the line joining M and M_1 (also to the line joining M and M_2); hence it will follow that there are no unsymmetrical orbits of this type. Furthermore, *all* the periodic orbits which are given by this analysis are closed after one revolution in the rotating plane; hence in case symmetrical orbits exist, they are also closed after one revolution.

We can expand Ω as a power series in $a\rho/A$ which is convergent for all values of w so long as the distance, $a\rho$, of the particle from M remains less than the distance from M to the nearest finite body; and in all that follows this condition is supposed to be satisfied. The expansion has the form

$$\begin{aligned}
\Omega=\frac{k^2M_1}{A}&\Big[1+\tfrac{1}{4}\Big(\frac{a}{A}\rho\Big)^2\Big\{1+3\cos 2w\Big\}\\
&+\tfrac{1}{8}\Big(\frac{a}{A}\rho\Big)^3\Big\{3\cos w+5\cos 3w\Big\}+\cdots\Big]\\
&+\frac{k^2M_2}{KA}\Big[1+\frac{1}{4K^2}\Big(\frac{a}{A}\rho\Big)^2\Big\{1+3\cos 2\Big(w-\frac{\pi}{2}\Big)\Big\}\\
&+\frac{1}{8K^3}\Big(\frac{a}{A}\rho\Big)^3\Big\{3\cos\Big(w-\frac{\pi}{2}\Big)+5\cos 3\Big(w-\frac{\pi}{2}\Big)\Big\}+\cdots\Big]\\
&+\frac{k^2M_3}{A}\Big[1+\tfrac{1}{4}\Big(\frac{a}{A}\rho\Big)^2\Big\{1+3\cos 2(w-\pi)\Big\}\\
&+\tfrac{1}{8}\Big(\frac{a}{A}\rho\Big)^3\Big\{3\cos(w-\pi)+5\cos 3(w-\pi)\Big\}+\cdots\Big]\\
&+\frac{k^2M_4}{KA}\Big[1+\frac{1}{4K^2}\Big(\frac{a}{A}\rho\Big)^2\Big\{1+3\cos 2\Big(w-\frac{3\pi}{2}\Big)\Big\}\\
&+\frac{1}{8K^3}\Big(\frac{a}{A}\rho\Big)^3\Big\{3\cos\Big(w-\frac{3\pi}{2}\Big)+5\cos 3\Big(w-\frac{3\pi}{2}\Big)\Big\}+\cdots\Big].
\end{aligned}$$

From the conditions of the configuration (A) we have

$$M_1=M_3=M', \qquad M_2=M_0=M''.$$

Let λ_1 and λ_2 be defined by the relations

$$M'=\lambda_1\Big(\frac{M'}{4}+\frac{2M''}{(\sqrt{1+K^2})^3}\Big), \qquad \frac{M''}{K^3}=\lambda_2\Big(\frac{M'}{4}+\frac{2M''}{(\sqrt{1+K^2})^3}\Big).$$

From equations (40) we have

$$\frac{k^2\Big(\dfrac{M'}{4}+\dfrac{2M''}{(\sqrt{1+K^2})^3}\Big)}{A^3}=\frac{[M'\sqrt{(1+K^2)^3}+8M'']N^2}{(M'+4M)\sqrt{(1+K^2)^3}+8M''}.$$

On setting the coefficient of N^2 in this equation equal to κ, it follows that the second members of equations (48) have the form

$$\left.\begin{aligned}
\frac{1}{\nu^2a}\frac{\partial\Omega}{\partial(a\rho)}&=\kappa m^2\rho[\lambda_1(1+3\cos 2w)+\lambda_2(1-3\cos 2w)\\
&\qquad+\text{terms involving only cosines of even multiples of } w],\\
\frac{1}{\nu^2a^2\rho}\frac{\partial\Omega}{\partial w}&=-\kappa m^2\rho[3\lambda_1\sin 2w-3\lambda_2\sin 2w\\
&\qquad+\text{terms involving only sines of even multiples of } w].
\end{aligned}\right\}\quad(49)$$

On introducing the parameter of integration μ by the relations $m=\mu$ and $a/A=\eta\mu$, where η is a numerical constant, equations (48) become

$$\frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\mu\right)^2+\frac{1}{\rho^2}=\mu^2 f, \qquad \rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right)=\mu^2 g, \tag{50}$$

where

$$f=\kappa\rho[\lambda_1(1+3\cos 2w)+\lambda_2(1-3\cos 2w)+\cdots],$$
$$g=-\kappa\rho[3(\lambda_1-\lambda_2)\sin 2w+\cdots].$$

Suppose

$$\rho=\psi_1(\tau), \qquad w=\psi_2(\tau) \tag{51}$$

is a solution of equations (50) such that $\rho'=w=0$ at $\tau=0$. Then it follows from the form of the differential equations that ψ_1 is an even function, and that ψ_2 is an odd function of τ. Hence, if the particle crosses the w-axis orthogonally, the orbit is symmetrical with respect to the line $w=0$, and with respect to the time of crossing this line.* Suppose that when $\tau=\pi$ the particle crosses this line (or, what is the same thing, the line $w=\pi$) again orthogonally; the orbit will be symmetrical with respect to this line and the time of crossing, and the particle will have again its initial position and relative components of velocity at the end of the period $\tau=2\pi$. Hence sufficient conditions that the solution (51) shall be periodic are

$$\rho'(\pi)=0, \qquad w(\pi)-\pi=0. \tag{52}$$

For $\mu=0$ the equations have the form which occurs in the problem of two bodies, and a symmetrical solution having the required period is known, namely,

$$\rho=1, \qquad w=\tau.$$

The initial conditions for this solution are

$$\rho=1, \qquad \rho'=0, \qquad w=0, \qquad w'=1.$$

Consider the solution for values of μ different from zero but sufficiently small, and let the initial conditions for $\tau=0$ be

$$\left.\begin{aligned} &\rho=1+\beta_1=(1+a)(1-e), \qquad \rho'=0, \qquad w=0,\\ &w'=1+\beta_4=\frac{\sqrt{1-e^2}}{(1+a)^{3/2}(1-e)^2}-\mu. \end{aligned}\right\} \tag{53}$$

*It may be remarked also, in this particular example, that if the solution (51) is subject to the initial conditions, $\rho'=0$, $w=\pi/2$, $(\tau=\tau_1)$, then ψ_1 is an even function, and $\psi_2-\pi/2$ is an odd function of $\tau-\tau_1$ and hence, if the particle crosses the line $w=\pi/2$ orthogonally, the orbit is symmetrical to this line and the epoch $\tau=\tau_1$. The sufficient conditions of periodicity (52) might be replaced by the conditions

$$\rho'\left(\frac{\pi}{2}\right)=0, \qquad w\left(\frac{\pi}{2}\right)-\frac{\pi}{2}=0,$$

since the orbit would then have two lines of symmetry. For the purpose of covering more general cases it is better to base the existence proof on only one line of symmetry.

The solution is symmetrical with respect to the epoch $\tau=0$, and can be expressed as power series in α, e, and μ, which are convergent for an interval in τ including the interval 0 to π, if the parameters are sufficiently small. If α and e can be determined in terms of μ, vanishing with μ, so that the conditions (52) are satisfied, then the solution will be periodic with the period 2π. All terms of the solution which are independent of μ^2 can be obtained from the two-body problem by making the substitution $w=u-\mu\tau$. These terms are given in finite form by the expressions

$$\rho=(1+\alpha)(1-e\cos E), \qquad u=\arccos\left(\frac{\cos E-e}{1-e\cos E}\right)=\arcsin\left(\frac{\sqrt{1-e^2}\sin E}{1-e\cos E}\right),$$

where E is defined by the relation

$$\frac{\tau}{(1+\alpha)^{3/2}}=E-e\sin E.$$

On returning to the variable w, writing the terms in α and e as power series by Taylor's expansion, and applying the conditions (52), we obtain the equations

$$0=-\frac{3}{4}\pi\alpha e+\cdots, \qquad 0=-\frac{3}{2}\pi\alpha-\pi\mu+\cdots. \tag{54}$$

It follows from the known properties of the series that there are no terms in e alone, and there are no terms involving μ to the first degree except the term $-\pi\mu$. The equations are satisfied by $\alpha=e=\mu=0$, and in the second the coefficient of the first power of α is not zero; hence the second equation can be solved uniquely for α as a power series in e and μ, which contains μ as a factor. The result has the form

$$\alpha=\mu\left(-\frac{2}{3}+\cdots\right). \tag{55}$$

When this value of α is substituted in the first of equations (54), a factor μ can be divided out, leaving $0=\frac{1}{2}\pi e+\cdots$. This equation is satisfied by $e=\mu=0$, and since the coefficient of the first power of e is not zero, it furnishes a unique determination of e in terms of μ, vanishing with μ. When this value of e is substituted in equation (55), we have α expressed uniquely in terms of μ, vanishing with μ.

Hence for a given value of μ sufficiently small it is possible to determine the initial conditions (53) as power series in μ such that the solution in μ is symmetrical with respect to the line $w=\pi$ and the epoch $\tau=\pi$. Since it is symmetrical also with respect to the line $w=0$ and the epoch $\tau=0$, it is periodic in τ with period 2π.

The orbit is symmetrical with respect to the line joining the bodies M and M_1. If we take for the initial line the line joining M and M_2, it follows from the same analysis that the orbit is symmetrical with respect to

this line and the time of crossing it. Since for a given value of μ there is only one periodic orbit of this type, it follows that it is symmetrical with respect to both the lines joining M with M_1 and with M_2. For the configuration (A) of the finite bodies the periodic orbits of the particle have two lines of symmetry; there are four apses and the apsidal angle is $\pi/2$. For the configuration (B) (see numerical example 3) the periodic orbits have four lines of symmetry, there are eight apses, and the apsidal angle is $\pi/4$

In establishing the uniqueness of the periodic solution of equations (54) it is to be noted that no use was made of the explicit values of the terms from the second members of equations (50). Hence, if the second members have forms which permit symmetrical solutions, the preceding analysis is applicable without change to show the existence of symmetrical periodic orbits. *If the masses and motions of the finite bodies are such that there can exist orbits of the particle about one of the bodies having a line of symmetry, we have established the existence of periodic orbits having this line of symmetry.*

As in the problem treated in § 198, the period of the solution in t may be assigned arbitrarily (that is, when the finite bodies form a fixed configuration in the rotating plane). There exist then two, and only two, symmetrical closed orbits having the required period; in one the motion is direct, and in the other it is retrograde.

203. Construction of Symmetrical Periodic Orbits.—The method of constructing symmetrical periodic solutions is similar to that explained in § 199. There is a slight difference in the conditions which determine the constants of integration, and the calculation is simpler because ρ contains only cosines of multiples of τ, while w contains only sines. It has been proved that symmetrical periodic solutions of equations (50) exist, and that they are expressible in the form (18).

Since the solution is periodic for a continuous range of values of μ, each coefficient ρ_i and w_i is periodic with period 2π in τ. Also, since the initial conditions $\rho'(0) = 0$, $w(0) = 0$ hold identically in μ, it follows that $\rho_i'(0) = 0$, and $w_i(0) = 0$ for every i.

The left members of equations (50) are the same as the left members of equations (9), and therefore, when the solution (18) is substituted in equations (50), the terms of the left members have the form (19). The right members have no terms independent of μ^2, and the equations for the determination of the coefficients of the first power of μ are

$$\frac{d^2\rho_1}{d\tau^2} - 2\frac{dw_1}{d\tau} - 3\rho_1 = 2, \qquad \frac{d^2w_1}{d\tau^2} + 2\frac{d\rho_1}{d\tau} = 0,$$

of which the solution is

$$\begin{aligned} \rho_1 &= 2(1 + c_1^{(1)}) + c_2^{(1)}\cos\tau + c_3^{(1)}\sin\tau, \\ w_1 &= c_4^{(1)} - (4 + 3c_1^{(1)})\tau - 2c_2^{(1)}\sin\tau + 2c_3^{(1)}\cos\tau. \end{aligned}$$

Since w_1 is periodic the coefficient of τ must be zero; whence $c_1^{(1)} = -4/3$. The constants $c_3^{(1)}$ and $c_4^{(1)}$ are determined by the conditions

$$\rho_1'(0) = 0, \qquad w_1(0) = 0.$$

Therefore $c_3^{(1)} = c_4^{(1)} = 0$. The constant $c_2^{(1)}$ is determined in the following step of the integration [see equations (23)].

This process is applicable to all the succeeding steps. The differential equations (50) have a particular form which admits a symmetrical solution, and it can be established by complete induction that the equations for the determination of ρ_i and w_i have the form

$$\left.\begin{aligned} \frac{d^2\rho_i}{d\tau^2} - 2\frac{dw_i}{d\tau} - 3\rho_i &= A_0^{(i)} + \left(\tfrac{14}{3}c_2^{(i-1)} + A_1^{(i)}\right)\cos\tau \\ &\qquad + A_2^{(i)}\cos 2\tau + \cdots + A_i^{(i)}\cos i\tau, \\ \frac{d^2 w_i}{d\tau^2} + 2\frac{d\rho_i}{d\tau} &= \left(\tfrac{10}{3}c_2^{(i-1)} + D_1^{(i)}\right)\sin\tau + D_2^{(i)}\sin 2\tau + \cdots + D_i^{(i)}\sin i\tau, \end{aligned}\right\} \quad (56)$$

where $c_2^{(i-1)}$ is determined by the condition [compare equations (29)]

$$2c_2^{(i-1)} = A_1^{(i)} - 2D_1^{(i)}.$$

The solution of equations (56) is

$$\left.\begin{aligned} \rho_i &= a_0^{(i)} + c_2^{(i)}\cos\tau + a_2^{(i)}\cos 2\tau + \cdots + a_i^{(i)}\cos i\tau, \\ w_i &= \qquad \delta_0^{(i)}\sin\tau + \delta_2^{(i)}\sin 2\tau + \cdots + \delta_i^{(i)}\sin i\tau, \end{aligned}\right\} \quad (57)$$

where

$$\left.\begin{aligned} &a_0^{(i)} = -\tfrac{1}{3}A_0^{(i)}, \qquad a_j^{(i)} = \frac{1}{j(1-j^2)}\left(jA_j^{(i)} - 2D_j^{(i)}\right), \\ &a_j^{(i)} = \frac{1}{1-j^2}\left(A_j^{(i)} - \frac{2D_j^{(i)}}{j}\right), \qquad \delta_j^{(i)} = -\frac{1}{j^2}\left(D_j^{(i)} + 2ja_j^{(i)}\right) \qquad (j=2, \ldots, i). \end{aligned}\right\} \quad (58)$$

204. Numerical Example 2.—As a first example of a symmetrical periodic orbit we consider three finite bodies revolving in circles according to the straight-line solution of Lagrange. We suppose that the mass M, about which the particle revolves, is between M_2 and M_1. Choosing M for the unit of mass, we select $M_1 = 10$, $M_2 = 5$. The unit of distance is MM_1; and it follows from the solution of the quintic equation of Lagrange* that the distance M_2M is $R_2 = 0.77172\ \ldots$. The unit of time is selected so that $N = 1$, and the period of the solution is assigned so that $\nu = 5$; whence $\mu = m = N/\nu = 0.2$.

*See Moulton, *Celestial Mechanics* (second edition), p. 312.

The differential equations of relative motion of the particle are

$$
\left.
\begin{aligned}
&\frac{d^2\rho}{d\tau^2}-\rho\Big(\frac{dw}{d\tau}+\mu\Big)^2+\frac{1}{\rho^2}=k^2M_1\mu^2\rho\Big[\frac{1}{2}\Big\{1+3\cos 2w\Big\}\\
&\quad+\frac{3}{8}a\rho\Big\{3\cos w+5\cos 3w\Big\}+\frac{1}{16}a^2\rho^2\Big\{9+20\cos 2w+35\cos 4w\Big\}+\cdots\Big]\\
&\quad+\frac{k^2M_2}{R_2^3}\mu^2\rho\Big[\frac{1}{2}\Big\{1+3\cos 2(w-\pi)\Big\}+\frac{3}{8}\frac{a}{R_2}\rho\Big\{3\cos(w-\pi)+5\cos 3(w-\pi)\Big\}\\
&\quad+\frac{1}{16}\Big(\frac{a}{R_2}\Big)^2\rho^2\Big\{9+20\cos 2(w-\pi)+35\cos 4(w-\pi)\Big\}+\cdots\Big],\\
&\rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\Big(\frac{dw}{d\tau}+\mu\Big)=-k^2M_1\mu^2\rho\Big[\frac{3}{2}\sin 2w+\frac{3}{8}a\rho\Big\{\sin w+5\sin 3w\Big\}\\
&\quad+\frac{5}{16}a^2\rho^2\Big\{2\sin 2w+7\sin 4w\Big\}+\cdots\Big]\\
&\quad-\frac{k^2M_2}{R_2^3}\mu^2\rho\Big[\frac{3}{2}\sin 2(w-\pi)+\frac{3}{8}\frac{a}{R_2}\rho\Big\{\sin(w-\pi)+5\sin 3(w-\pi)\Big\}\\
&\quad+\frac{5}{16}\Big(\frac{a}{R_2}\Big)^2\rho^2\Big\{2\sin 2(w-\pi)+7\sin 4(w-\pi)\Big\}+\cdots\Big],
\end{aligned}
\right\}\quad(59)
$$

where a is given by the relation $\nu^2a^3=k^2$. The constant k^2 is determined by

$$N^2=\frac{M_1+M+M_2}{M+M_2+M_2R_2}\Big(M+\frac{M_2}{(1+R_2^2)}\Big)k^2;$$

whence

$$k^2=0.23763,\qquad k^2M_1=2.37630,\qquad \frac{k^2M_2}{R_2^3}=2.58518,$$

$$a=1.05914\,\mu,\qquad \frac{a}{R_2}=1.37222\,\mu.$$

The differential equations of motion become

$$
\left.
\begin{aligned}
&\frac{d^2\rho}{d\tau^2}-\rho\Big(\frac{dw}{d\tau}+\mu\Big)^2+\frac{1}{\rho^2}=(2.48074+7.44222\cos 2w)\rho\mu^2\\
&\qquad-(1.15938\cos w+1.93230\cos 3w)\rho^2\mu^3\\
&\qquad+(4.23765+9.41700\cos 2w+16.47975\cos 4w)\rho^3\mu^4+\cdots,\\
&\rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\Big(\frac{dw}{d\tau}+\mu\Big)=-(7.44222\sin 2w)\rho\mu^2\\
&\qquad+(0.38646\sin w+1.93230\sin 3w)p^2\mu^3\\
&\qquad-(4.70850\sin 2w+16.47975\sin 4w)\rho^3\mu^4+\cdots.
\end{aligned}
\right\}\quad(60)
$$

The periodic solution of equations (60) is

$$
\left.
\begin{aligned}
\rho&=1-\frac{2}{3}\mu-(0.27136+0.96615\cos\tau+4.96148\cos 2\tau)\mu^2\\
&\quad+(0.62584-19.13740\cos\tau-2.47963\cos 2\tau+0.40256\cos 3\tau)\mu^3+\cdots,\\
w&=\tau+(1.93230\sin\tau+6.82204\sin 2\tau)\mu^2\\
&\quad+(41.10885\sin\tau+10.74793\sin 2\tau-0.48307\sin 3\tau)\mu^3+\cdots.
\end{aligned}
\right\}\quad(61)
$$

On substituting the value $\mu = 0.2$, the orbit represented by equations (61) is shown in Fig. 12. The points which are numbered 1, 2, . . . , 24 represent positions of the particle in the periodic orbit at intervals of $\tau = \pi/12$. The corresponding positions in the comparison circle are indicated by the numbers 1′, 2′, . . . , 24′.

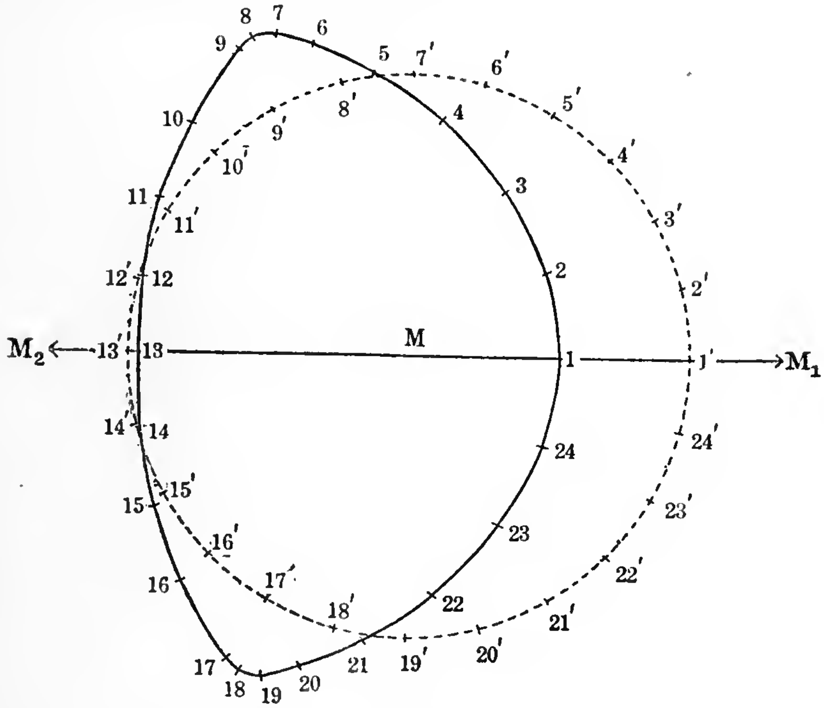

FIG. 12.

205. Numerical Example 3.—For a second example of a symmetrical periodic orbit we use the configuration (B), §201, of nine finite bodies, the numerical values being those given. The unit of time is selected so that $N = 1$, and the period of the solution is assigned so that $V = 5$, whence $\mu = 0.2$.

The differential equations [corresponding to equations (50)] of relative motion of the particle are

$$\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau} + \mu\right)^2 + \frac{1}{\rho^2} = \sum_{\iota=1}^{8} \frac{k^2 M_\iota}{R_\iota^3} \mu^2 \rho \left[\frac{1}{2}\left\{1 + 3\cos 2(w - W_\iota)\right\}\right.$$
$$+ \frac{3}{8}\frac{a}{R_\iota}\rho\left\{3\cos(w - W_\iota) + 5\cos 3(w - W_\iota)\right\}$$
$$\left. + \frac{1}{16}\left(\frac{a}{R_\iota}\right)^2 \rho^2 [9 + 20\cos 2(w - W_\iota) + 35\cos 4(w - W_\iota)] + \cdots\right],$$

$$\rho\frac{d^2 w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau} + \mu\right) = \sum_{\iota=1}^{8} - \frac{k^2 M_\iota}{R_\iota^3} \mu^2 \rho \left[\frac{3}{2}\sin 2(w - W_\iota)\right.$$
$$+ \frac{3}{8}\frac{a}{R_\iota}\rho\left\{\sin(w - W_\iota) + 5\sin 3(w - W_\iota)\right\}$$
$$\left. + \frac{5}{16}\left(\frac{a}{R_\iota}\right)^2 \rho^2 \left\{2\sin 2(w - W_\iota) + 7\sin 4(w - W_\iota)\right\} + \cdots\right].$$

On taking account of the relations (44), and choosing the unit of distance so that $A=1$, the preceding set of equations takes the form

$$\frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\mu\right)^2+\frac{1}{\rho^2}=k^2M'\mu^2\rho\left[2+\tfrac{1}{4}a^2\rho^2\left\{9+35\cos 4w\right\}+\cdots\right]$$
$$+\frac{k^2M''}{\kappa^3}\mu^2\rho\left[2+\tfrac{1}{4}\left(\frac{a}{\kappa}\right)^2\rho^2[9-35\cos 4w]+\cdots\right],$$

$$\rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right)=-k^2M'\mu^2\rho\left[\tfrac{35}{4}a^2\rho^2\sin 4w+\cdots\right]$$
$$+\frac{k^2M''}{\kappa^2}\mu^2\rho\left[\tfrac{35}{4}\left(\frac{a}{\kappa}\right)^2\rho^2\sin 4w+\cdots\right].$$

From §201 we have the following values:

$$K=2,\qquad M'=1,\qquad M''=8.2526,\qquad N^2=1.6399\ k^2.$$

Since $N=1$, the last equation gives $k^2=0.60994$. It follows that

$$\frac{k^2M''}{K^3}=0.62920,\qquad a^2=2.10300\ \mu^2,\qquad \left(\frac{a}{K}\right)^2=0.52575\ \mu^2.$$

On substituting these numerical values, the differential equations become

$$\frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\mu\right)^2+\frac{1}{\rho^2}=2.47828\ \rho\mu^2+[3.63039+8.32914\cos 4w]\rho^3\mu^4+\cdots,$$
$$\rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right)=[-8.32914\sin 4w]\rho^3\mu^4+\cdots.$$

The periodic solution of these differential equations is

$$\rho=1-\tfrac{2}{3}\mu-0.27054\mu^2+1.70909\mu^3-3.43127\mu^4-0.83291\mu^4\cos 4\tau+\cdots,$$
$$w=\tau+0.93702\mu^4\sin 4\tau+\cdots,$$

On substituting the value $\mu=0.2$, the final result is found to be

$$\rho=0.86403-0.00133\cos 4\tau+\cdots,$$
$$w=\tau+0.00150\sin 4\tau+\cdots.$$

The orbit has four axes of symmetry, namely, the lines connecting the central body with the others. It differs from the orbits of the other numerical examples in one respect—that is, it lies entirely inside the comparison circle (see §200). In terms of ρ the radius of the comparison circle is about 0.8855. Fig. 13 is not drawn to scale, but the characteristic properties of the orbit, which are readily seen from the numerical values of ρ and w, are exaggerated to make them apparent in a small drawing. The inner circle is drawn merely to indicate the direction of the deviation of the orbit from a circle.

206. The Undisturbed Orbit Must be Circular.—In the proofs of the existence of periodic orbits (§§ 198–201) it was assumed that the undisturbed orbit is circular. It remains to be shown that this assumption is necessary. The proof will be made for the case of symmetrical orbits [equations (50)] and is applicable also to the orbits of § 198. The undisturbed orbit is given by the solution of equations (50) when $\mu=0$. For $\mu=0$, the equations are the equations of motion of a particle subject to the attraction of a central force varying inversely as the square of the distance. The undisturbed orbit is therefore a conic; and, since we are concerned only with closed orbits, must be an ellipse. Since the period in τ is 2π, the major semi-axis of the ellipse must be unity (in ρ). The eccentricity, which will be denoted by $\bar{e}$, is, however, arbitrary; that is, for $\mu=0$ the differential equations admit an infinite number* of symmetrical periodic solutions. Starting now with an ellipse for the undisturbed orbit, it will be shown that the eccentricity must be zero in order to fulfill the conditions of periodicity.

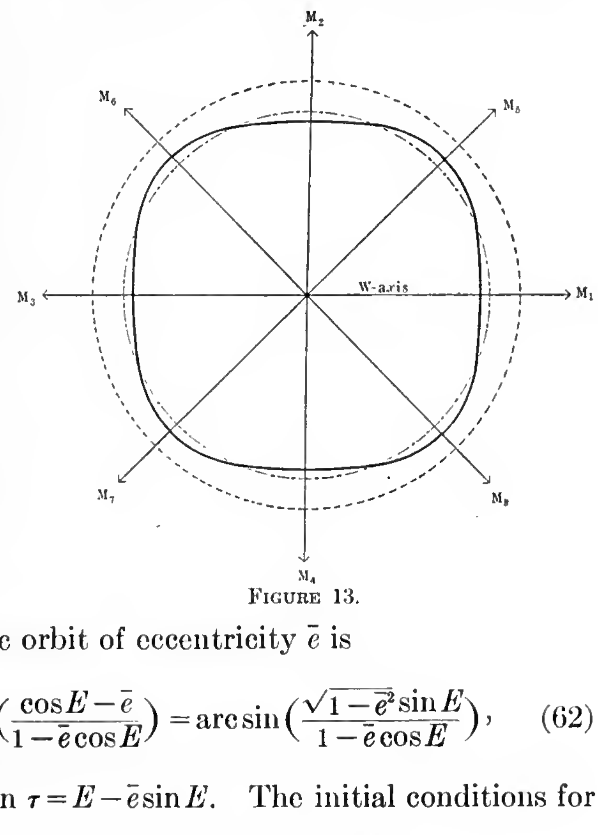

FIGURE 13.

For $\mu=0$ the solution of equations (50) representing an elliptic orbit of eccentricity $\bar{e}$ is

$$\rho=1-\bar{e}\cos E, \qquad w=\arccos\left(\frac{\cos E-\bar{e}}{1-\bar{e}\cos E}\right)=\arcsin\left(\frac{\sqrt{1-\bar{e}^2}\sin E}{1-\bar{e}\cos E}\right), \tag{62}$$

where E is defined by the relation $\tau=E-\bar{e}\sin E$. The initial conditions for $\tau=0$ are

$$\rho=1-\bar{e}, \qquad \rho'=0, \qquad w=0, \qquad w'=\frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}.$$

Consider the solution for values of μ different from zero, but sufficiently small, and let the initial conditions be

$$\rho=1-\bar{e}+\beta_1=(1+\alpha)[1-(\bar{e}+e)], \qquad \rho'=0, \qquad w=0,$$

$$w'=\frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}+\beta_4=\frac{\sqrt{1-(\bar{e}+e)^2}}{(1+\alpha)^{3/2}[1-(\bar{e}+e)]^2}-\mu.$$

If α and e can be determined in terms of μ, vanishing with μ, so that the conditions (52) are satisfied, then the solution will be periodic with the

*The general case when the differential equations (for $\mu=0$) admit a periodic solution containing an arbitrary parameter has been mentioned by Poincaré, *loc. cit.*, vol. 1, p. 84.

period 2π. All terms of the solution which are independent of μ^2 may be obtained from the two-body problem by making the substitution $w = u - \mu\tau$. These terms are given in finite form by the expressions

$$\rho = (1+\alpha)[1-(\bar{e}+e)\cos E],$$

$$u = \arccos\left(\frac{\cos E-(\bar{e}+e)}{1-(\bar{e}+e)\cos E}\right) = \arcsin\left(\frac{\sqrt{1-(\bar{e}+e)^2}\sin E}{1-(\bar{e}+e)\cos E}\right),$$

where E is defined by the relation

$$\frac{\tau}{(1+\alpha)^{3/2}} = E - (\bar{e}+e)\sin E.$$

On returning to the variable w, writing the terms in α and e as power series by Taylor's expansion, and applying the conditions (52), we obtain the equations

$$\left.\begin{aligned} 0 &= -\tfrac{3}{2}p\pi\frac{\bar{e}}{(1-\bar{e})^2}\alpha - \tfrac{3}{4}p\pi\frac{1+\bar{e}}{(1-\bar{e})^3}\alpha e + \cdots, \\ 0 &= -\tfrac{3}{2}p\pi\frac{\sqrt{1-\bar{e}^2}}{(1-\bar{e})^2}\alpha - p\pi\mu + \cdots. \end{aligned}\right\} \quad (63)$$

It follows from the known properties of the series that there are no terms in e alone, and there are no terms involving μ to the first degree except the term $-p\pi\mu$. Hence the second of equations (63) can be solved for α as a power series in e and μ in which μ is contained as a factor; the result is

$$\alpha = \mu\left\{-\frac{2}{3}\frac{(1-\bar{e})^2}{\sqrt{1-\bar{e}^2}} + \cdots\right\}.$$

When this value of α is substituted in the first equation, a factor μ can be divided out, leaving

$$0 = p\pi\frac{\bar{e}}{\sqrt{1-\bar{e}^2}} + \tfrac{1}{2}p\pi\frac{1+\bar{e}}{(1-\bar{e})\sqrt{1-\bar{e}^2}}e + \cdots.$$

This equation can be solved for e as a power series in μ, which vanishes with μ, if and only if $\bar{e}=0$. Since only those solutions are under consideration which are the analytic continuations with respect to μ of those for $\mu=0$, the condition $\bar{e}=0$ must be imposed. *The condition $e=0$ means that the undisturbed orbit must be circular.*

207. More General Types of Motion for the Finite Bodies.—This section contains some remarks upon possible extensions of the analysis which will permit applications to practical problems of celestial mechanics, and is followed by an illustrative example.

The particular problems treated in the preceding articles have no application in nature because the configurations assumed for the finite bodies do not exist. But a glance at the details shows that these configurations are not essential to the proofs. The possible generalizations of the motion of the finite bodies can be made in three ways:

(1) In §198 the existence proof depends only upon certain terms of the disturbing function which are due to the body M_1. If M_1 retains the motion there prescribed, we may add other bodies to the fixed configuration in the rotating plane provided the operations with the power series are valid. This merely increases the number of terms in the second members of the equations of motion; the existence proof and method of construction are unchanged.

(2) In the examples treated the finite bodies form a fixed configuration in a plane rotating with constant angular velocity. This is not necessary for the type of analysis used. If M_1 moves in a circle with uniform angular velocity, the other bodies can have any periodic motion, provided always that the convergence conditions hold. In this case the differential equations of motion of the particle involve τ explicitly and are periodic in τ. Two points of difference occur in the analysis: (a) Suppose the period in τ of the differential equations is T; then the assigned period of the motion of the particle must be a multiple of T. (b) The differential equations do not admit the integral of Jacobi, and hence no use can be made of this in the existence proof. This is equivalent to saying that at $\tau=0$ we can not assume $w=0$, but must determine the initial longitude of the particle by the conditions of periodicity. The method of determining the constants of integration in the construction of the solutions is explained in a paper in the *Transactions of the American Mathematical Society*, vol. 8 (1907), pp. 177–181.

(3) A further generalization of the motion of the finite bodies is possible by permitting M_1 to move in a path which is not circular. It is possible to show that the analysis can be used if the motion of M_1 is subject only to the mild restrictions that the expression for the radius vector shall contain only cosines of multiples of τ while that for the longitude shall contain only sines. The case when the orbit of M_1 is an ellipse is treated in the article referred to above. For this generalized motion of the finite bodies there may exist symmetrical orbits of the particle. In equations (50) the first contains only cosines of multiples of w, and the second only sines of multiples of w. The periodic orbit of the particle may be symmetrical if the first equation contains also sines of multiples of w multiplied by odd functions of τ, and cosines of multiples of w multiplied by even functions of τ, and the second contains cosines of multiples of w multiplied by odd functions of τ, and sines of multiples of w multiplied by even functions of τ.

From these remarks it is apparent that the treatment can be made sufficiently general to permit applications in the problems presented by the motions of the solar system. For example, suppose P is a satellite of one of the planets M, and that M_1 is the sun. This implies that the disturbing effects of the satellite upon the other bodies are neglected, since we assume that its mass is infinitesimal. The conditions upon the motion of M_1 are fulfilled if we neglect the perturbations of the other planets upon M; that is, if we suppose the orbit of M_1 relative to M is an ellipse. If we neglect the inclinations of the orbits of the other planets, and suppose that their motion is

periodic (that is, we assign a periodic motion which is approximately correct), it is possible by the methods given to treat the periodic motion of the satellite in the plane of the planetary orbit, when subject to the attraction of the sun and all the planets. The following numerical example is a simple illustration of the general idea.

208. Numerical Example 4.—The mass of M is taken as the unit of mass and M_1, of mass 10, is supposed to revolve about M in a circle of unit radius with uniform angular velocity N. A third mass, $M_2 = M = 1$, is supposed to revolve about M_1 in a circle of radius A_2 with uniform angular velocity N_2. The unit of time is chosen so that $N=1$, and the period of the motion of the particle is assigned so that $\nu=5$, whence

$$\mu = m = \frac{N}{\nu} = 0.2.$$

With reference to M as origin and an axis passing always through M_1, the coördinates of M_1, M_2, and P are, respectively $(1, 0)$, (R_2, W_2), and (r, w). The differential equations of relative motion of the particle [corresponding to equations (50)] are

$$\left.\begin{aligned}
&\frac{d^2\rho}{d\tau^2} - \rho\left(\frac{dw}{d\tau}+\mu\right)^2 + \frac{1}{\rho^2} = k^2M_1\mu^2\rho\Big[\tfrac{1}{2}\Big\{1+3\cos 2w\Big\} + \tfrac{3}{8}a\rho\Big\{3\cos w + 5\cos 3w\Big\}\\
&\quad + \tfrac{1}{16}a^2\rho^2\Big\{9+20\cos 2w + 35\cos 4w\Big\} + \cdots\Big]\\
&\quad + \frac{k^2M_2}{R_2^3}\mu^2\rho\Big[\tfrac{1}{2}\Big\{1+3\cos 2(w-W_2)\Big\} + \tfrac{3}{8}\frac{a}{R_2}\rho\Big\{3\cos(w-W_2)+5\cos 3(w-W_2)\Big\}\\
&\quad + \tfrac{1}{16}\left(\frac{a}{R_2}\right)^2\rho^2\Big\{9+20\cos 2(w-W_2)+35\cos 4(w-W_2)\Big\} + \cdots\Big],\\
&\rho\frac{d^2w}{d\tau^2} + 2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right) = -k^2M_1\mu^2\rho\Big[\tfrac{3}{2}\sin 2w + \tfrac{3}{8}a\rho\Big\{\sin w + 5\sin 3w\Big\}\\
&\quad + \tfrac{5}{16}a^2\rho^2\Big\{2\sin 2w + 7\sin 4w\Big\} + \cdots\Big]\\
&\quad - \frac{k^2M_2}{R_2^3}\mu^2\rho\Big[\tfrac{3}{2}\sin 2(w-W_2) + \tfrac{3}{8}\frac{a}{R_2}\rho\Big\{\sin(w-W_2)+5\sin 3(w-W_2)\Big\}\\
&\quad + \tfrac{5}{16}\left(\frac{a}{R_2}\right)^2\rho^2\Big\{2\sin 2(w-W_2)+7\sin 4(w-W_2)\Big\} + \cdots\Big].
\end{aligned}\right\} \quad (64)$$

The constant k^2 is given by the relation $N^2 = k^2(M+M_1)$, whence

$$k^2 = 0.09091, \qquad k^2M_1 = 0.90909, \qquad k^2M_2 = 0.09091.$$

From the relation $\nu^2a^3 = k^2M$, it follows that

$$a = 0.76910\mu.$$

The angular velocity of M_2 about M_1 will be selected so that its period with respect to the rotating axis MM_1 is one-half the period assigned for P. Hence

$$N_2 - N = 2\nu, \quad \text{or} \quad N_2 = 11.$$

The radius, A_2, of the circular orbit of M_2 with respect to M_1 is determined by the relation $N_2^2A_2^3 = k^2(M_1+M_2)$, whence $A_2 = 1.01200\mu$. On assuming that

at $\tau=0$ the finite bodies are in conjunction in the order M, M_1, M_2, the coördinates (R_2, W_2) of M_2 with respect to M are given by the expressions

$$R_2=\sqrt{1+A_2^2+2A_2\cos 2\nu t}=\sqrt{1+A_2^2+2A_2\cos 2\tau},$$

$$\sin W_2=\frac{A_2\sin 2\tau}{2R_2}, \qquad \cos W_2=\frac{R_2^2+1-A_2^2}{2R_2}.$$

On substituting the values of the constants and the coördinates R_2, W_2 in equations (64), we obtain for the numerical differential equations of relative motion

$$\left.\begin{aligned}
&\frac{d^2\rho}{d\tau^2}-\rho\left(\frac{dw}{d\tau}+\mu\right)^2+\frac{1}{\rho^2}=(0.50000+1.50000\cos 2w)\rho\mu^2\\
&\quad+(0.86523\cos w+1.44205\cos 3w)\rho^2\mu^3\\
&\quad+(0.33273+0.73940\cos 2w+1.29395\cos 4w)\rho^3\mu^4\\
&\quad+(0.27600\sin 2\tau\sin 2w-0.13800\cos 2\tau-0.41400\cos 2\tau\cos 2w)\rho\mu^3\\
&\quad+(0.10474+0.17456\cos 4\tau+0.31422\cos 2w+0.10471\cos 4\tau\cos 2w\\
&\quad-0.55864\sin 4\tau\sin 2w)\rho\mu^4\\
&\quad+(0.07960\sin 2\tau\sin w-0.31840\cos 2\tau\cos w-0.53068\cos 2\tau\cos 3w\\
&\quad+0.39801\sin 2\tau\sin 3w)\rho^2\mu^4+\cdots,\\
&\rho\frac{d^2w}{d\tau^2}+2\frac{d\rho}{d\tau}\left(\frac{dw}{d\tau}+\mu\right)=-(1.50000\sin 2w)\rho\mu^2\\
&\quad-(0.28841\sin w+1.44205\sin 3w)\rho^2\mu^3\\
&\quad-(0.36970\sin 2w+1.29395\sin 4w)\rho^3\mu^4\\
&\quad+(0.27600\sin 2\tau\cos 2w+0.41400\cos 2\tau\sin 2w)\rho\mu^3\\
&\quad-(0.31422\sin 2w+0.10741\cos 4\tau\sin 2w+0.55864\sin 4\tau\cos 2w)\rho\mu^4\\
&\quad+(0.02653\sin 2\tau\cos w+0.10613\cos 2\tau\sin w+0.39801\sin 2\tau\cos 3w\\
&\quad+0.53068\cos 2\tau\sin 3w)\rho^2\mu^4+\cdots.
\end{aligned}\right\}\quad (65)$$

The right member of the first equation contains only, (1) *cosines* of multiples of w, (2) *cosines* of multiples of w multiplied by *cosines* of multiples of τ, and (3) *sines* of multiples of w multiplied by *sines* of multiples of τ. The first equation is then unchanged if we replace w by $-w$, and τ by $-\tau$. The right member of the second equation contains only, (1) *sines* of multiples of w, (2) *sines* of multiples of w multiplied by *cosines* of multiples of τ, and (3) *cosines* of multiples of w multiplied by *sines* of multiples of τ. Hence the second equation is also unchanged if we replace w by $-w$ and τ by $-\tau$. Now let us suppose that

$$\rho=\psi_1(\tau), \qquad w=\psi_2(\tau)$$

is a solution of equations (65) satisfying the conditions $\rho'(0)=w(0)=0$. It follows from the form of the differential equations that ψ_1 is an even function, and ψ_2 is an odd function of τ. When $\tau=0$ the finite bodies are in conjunction in the order M, M_1, M_2. Therefore, *if the particle P crosses the line MM_1 orthogonally when the finite bodies are in conjunction in the order M, M_1, M_2, the orbit in the rotating plane is symmetrical with respect to this line and this epoch.*

On constructing the solution of equations (65) by the formulas (58), we get

$$
\left.
\begin{aligned}
\rho = {} & 1 - .66667\,\mu + (0.38889 + 0.72102\cos\tau - \cos 2\tau)\mu^2 \\
& + (-0.02616 + 2.09168\cos\tau - 0.45400\cos 2\tau - 0.30043\cos 3\tau \\
& + 0.03450\cos 4\tau)\mu^3 + \cdots, \\
w = {} & \tau + (-1.44204\sin\tau + 1.37500\sin 2\tau)\mu^2 \\
& + (-6.29838\sin\tau + 2.12066\sin 2\tau + 0.36051\sin 3\tau \\
& - 0.03881\sin 4\tau)\mu^3 + \cdots.
\end{aligned}
\right\} \quad (66)
$$

The orbit represented by equations (66) is shown in Fig. 14. The points which are numbered $1, 2, \ldots, 8$ represent the positions of the particle in the periodic orbit at intervals of $\tau = \pi/4$. The corresponding positions in the comparison circle are indicated by the numbers $1', 2', \ldots, 8'$.

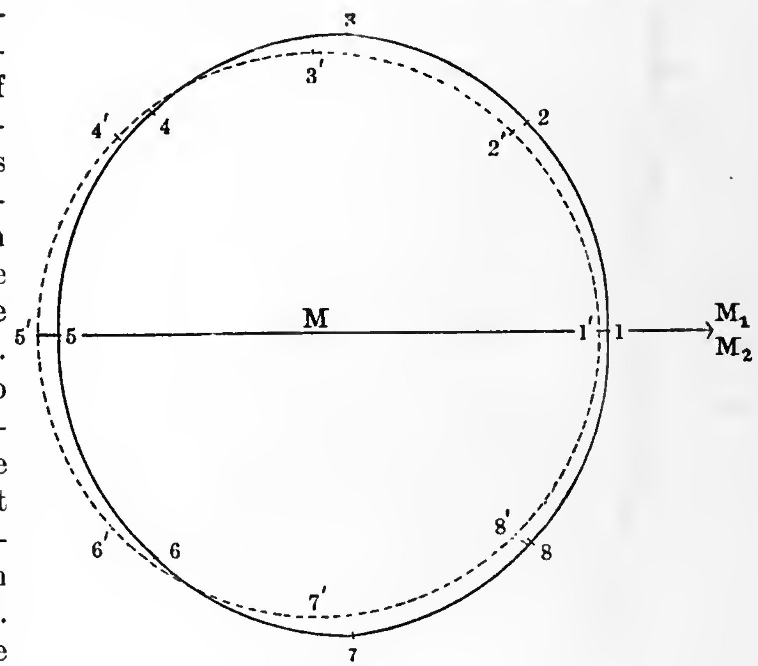

Fig. 14.

With reference to the differential equations (65) we can make the same statement concerning the uniqueness of the solution that was made in §198. These equations were written on the assumption that, at $\tau = 0$, the finite bodies are in conjunction in the order M, M_1, M_2. Without this assumption the expressions for R_2 and W_2 contain a parameter indicating the position of M_2 in its orbit at the origin of time. With reference to the physical problem, therefore, we can not affirm in this case that, for a preassigned period, there exists one and only one direct periodic orbit. It is necessary here to add a condition on the form of the configuration of the finite bodies and particle at the origin of time. For example, we might obtain another symmetrical periodic orbit having the preassigned period if the particle crosses the line M, M_1 when the finite bodies are in conjunction in the order M, M_2, M_1; and we might have still other orbits with the preassigned period if P crosses this line when the finite bodies are not in conjunction.

CHAPTER XIV.

CERTAIN PERIODIC ORBITS OF k FINITE BODIES REVOLVING ABOUT A RELATIVELY LARGE CENTRAL MASS.

BY FRANK LOXLEY GRIFFIN.

209. The Problem.—For a given system of k finite bodies, moving in a given plane relative to another given body, there is a $4k$-fold infinitude of possible orbits—the variations which the configuration of the system undergoes and its orientation in the plane being determined jointly by the mutual attractions of the bodies according to the Newtonian law, and by the values at any instant of the $2k$ relative coördinates and their first derivatives with respect to the time. The differential equations admit no algebraic or uniform transcendental integrals,* aside from the two fundamental integrals of energy and areas, even when the masses of all the bodies except one are very small; nevertheless, by restricting the initial values of the coördinates and their derivatives, in a manner to be shown below, it is possible to find an extensive class of periodic solutions.

In fact, for arbitrary values of the masses (save that one of them, M, shall be large in comparison with the others, $M_1, M_2, \ldots, M_k$),† *there exists a k-fold infinitude of distinct periodic orbits of the system, having an arbitrarily preassigned period T.* In these orbits (which, for small finite values of $M_1, M_2, \ldots, M_k$, depart but little from a set of concentric circles about the planet) the k satellites come periodically into a "symmetrical conjunction," that is, they are all momentarily in one straight line with the planet and moving at right angles to that line. These conjunctions may, or may not, always occur at the same absolute longitude; in the latter case the motion is periodic with reference to a uniformly rotating line.

Besides the demonstration of the existence of such periodic orbits, this chapter contains: A method of constructing the solutions without integration, a single application of that process having provided formulas which reduce the problem to one of algebraic computation; a numerical application to the case of Jupiter's satellites I, II, and III; a proof of the non-existence of certain other types of orbits; and a brief consideration of some related questions.

*See memoirs by Bruns and Poincaré, in *Acta Mathematica*, vols. 11 and 13.

†In other words, the distribution of masses is such as is presented by the sun and any number of planets, or by a planet and any number of satellites. For convenience, in what follows, a single expression, planet and satellites, will be used with the understanding that it covers also sun and planets.

The problem may be formulated thus: Let quantities μ, $\beta_1, \ldots, \beta_k$ be defined by

$$M\beta_i\mu = M_i \qquad (i=1, \ldots, k), \qquad (1)$$

where one of the β's is to be selected arbitrarily. Let a system of positive or negative integers without common divisor, $p_i (i=1, \ldots, k-1)$, and a number $q_k \neq 0$ be selected arbitrarily, save for the restriction mentioned below, and let ν, n_i, and a_i be defined by

$$\nu = \frac{2\pi}{T} = \frac{n_k}{q_k} = \frac{n_i - n_k}{p_i} \qquad (i=1, \ldots, k-1), \qquad (2)$$

$$n_i^2 a_i^3 = \kappa^2 M \qquad (i=1, \ldots, k), \qquad (3)$$

where κ^2 denotes the gravitational constant, and where, of the three values of a_i satisfying (3), that one is to be selected which is real. Also let the notation be so selected that $a_1, \ldots, a_k$ are in ascending order of magnitude, the p_i being so selected that no two of the a_i are equal and no n_i vanishes.

If μ were zero—that is, if the satellites were "infinitesimal"—possible orbits would be circles about the planet with $a_1, \ldots, a_k$ as radii; from (3) it follows that the angular velocities would be $n_1, \ldots, n_k$. It is quite immaterial whether any of the n_i are negative; the results obtained hold irrespective of retrograde motion of some of the bodies. The configuration of the infinitesimal system would undergo periodic variations with the period T; for, it follows from (2) that

$$\frac{2\pi}{n_i - n_k} = \frac{T}{p_i},$$

or each synodic period is a sub-multiple of T. This condition being satisfied,* the motion of the infinitesimal system would be periodic with respect to a line through M, rotating with uniform angular velocity—that of M_k, or, indeed, that of any other body† M_j—though whether or not the system ever returns to the same position in space depends upon whether q_k is rational or irrational.

In describing the orbits mentioned, the infinitesimal satellites would be subject to certain initial conditions, the $2k$ coördinates and their derivatives with respect to the time having at the instant $t=t_0$ certain values, say $c_{ij} (i=1, \ldots, k;\ j=1, \ldots, 4)$; but if the k *finite* satellites are subjected to these same initial conditions, their mutual disturbances in general destroy periodicity. The first problem is, then, to determine what, if any, increments Δc_{ij} can be given to the former initial values c_{ij} to preserve the periodicity when all the satellites are finite.

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*Poincaré, treating three satellites (*Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, pp. 154–6), states the condition thus: Integers α, β, γ, mutually prime, exist such that $\alpha+\beta+\gamma=0$ and $\alpha n_1+\beta n_2+\gamma n_3=0$. Evidently, in the case of three satellites, this condition is equivalent to (2), since $(n_1-n_3)/\beta=(n_2-n_3)/(-\alpha)$; but for a greater number it is not so. Thus, if $n_1=7$, $n_2=5$, $n_3=3\sqrt{2}$, $n_4=\sqrt{2}$, the integers 5, -7, -1, 3 satisfy a condition similar to Poincaré's, but periodicity is impossible. For the general case a re-formulation such as (2) is necessary.

†The commensurability of $n_i-n_k (i=1, \ldots, k-1)$ evidently involves that of n_i-n_j $(i=1, \ldots, j-1, j+1, \ldots, k)$. For from $n_i-n_k=p_i\nu$ and $n_j-n_k=p_j\nu$ it follows that $n_i-n_j=(p_i-p_j)\nu$.

210. The Differential Equations.—Let the common plane of relative motion of the k bodies be selected as the ΞH-plane, the origin being at M, and $M\Xi$ and MH being rectangular axes which rotate in the plane with the uniform angular velocity N. Let the coördinates of M_i referred to these axes be ξ_i and η_i; then the differential equations of motion are

$$\left.\begin{aligned}
&(a) \qquad \frac{d^2\xi_i}{dt^2} - 2N\frac{d\eta_i}{dt} - N^2\xi_i + \kappa^2(M+M_i)\frac{\xi_i}{r_i^3} + \sum_j{}' \kappa^2 M_j\left(\frac{\xi_i-\xi_j}{\rho_{ij}^3} + \frac{\xi_j}{r_j^3}\right) = 0,\\
&(b) \qquad \frac{d^2\eta_i}{dt^2} + 2N\frac{d\xi_i}{dt} - N^2\eta_i + \kappa^2(M+M_i)\frac{\eta_i}{r_i^3} + \sum_j{}' \kappa^2 M_j\left(\frac{\eta_i-\eta_j}{\rho_{ij}^3} - \frac{\eta_j}{r_j^3}\right) = 0,
\end{aligned}\right\} \qquad (4)$$

where

$$r_i^2 = \xi_i^2 + \eta_i^2, \qquad \rho_{ij}^2 = (\xi_j - \xi_i)^2 + (\eta_j - \eta_i)^2,$$

and $\sum_j{}'$ means $\Sigma_{j=1}^{j=\kappa}$ $(j \neq i)$. Except in proving a certain symmetry theorem, these coördinates are less convenient than polar coördinates referred to rotating reference lines. Besides (1), (2), and (3), let the following definitions be made:

$$\left.\begin{aligned}
&a_{ij} = a_i/a_j, && \nu q_i = n_i && (i = 1, \ldots, k),\\
&a_j\sigma_{ij} = \rho_{ij}, && \delta_{ij} = \beta_j q_1^2 a_{1i} a_{1j}^2 && (j \neq i),\\
&\nu t = \tau, && \phi_{ji} = (p_j - p_i)\tau + (\lambda_j - \lambda_i), &&
\end{aligned}\right\} \qquad (5)$$

where the λ_i are arbitrary constants, later to be taken as the longitudes of the M_i at the origin of time for $\mu = 0$.

Let polar coördinates be introduced by the equations

$$\xi_i = r_i \cos u_i, \qquad \eta_i = r_i \sin u_i, \qquad r_i = a_i x_i, \qquad u_i = w_i + p_i\tau + \lambda_i, \qquad (6)$$

and N be taken equal to n_k, so that w_i is the longitude of M_i referred to a line rotating with uniform speed n_i. The differential equations become

$$\left.\begin{aligned}
&(a) \qquad x_i w_i'' + 2x_i'(w_i' + q_i) + \mu \sum_j{}' \delta_{ij} x_j \sin(\phi_{ji} + w_j - w_i)\left(\frac{1}{x_j^3} - \frac{1}{\sigma_{ij}^3}\right) = 0,\\
&(b) \qquad x_i'' - x_i(w_i' + q_i)^2 + \frac{q_i^2(1+\beta_i\mu)}{x_i^2}\\
&\qquad\qquad + \mu \sum_j{}' \delta_{ij}\left[\frac{a_{ij}x_i}{\sigma_{ij}^3} + x_j\cos(\phi_{ji} + w_j - w_i)\left(\frac{1}{x_j^3} - \frac{1}{\sigma_{ij}^3}\right)\right] = 0,
\end{aligned}\right\} \qquad (7)$$

where $a_j^2\sigma_{ij}^2 = a_i^2 x_i^2 + a_j^2 x_j^2 - 2a_i a_j x_i x_j \cos(\phi_{ji} + w_j - w_i)$, and where the accents on the variables indicate derivatives with respect to τ.

211. Symmetry Theorem.—*If a symmetrical conjunction occurs at any instant $t = t_0$, then the orbit of each satellite before and after the conjunction is symmetrical, both with regard to geometric equality of figures and with regard to intervals of time.* A proof will be given only for the case $t_0 = 0$, which does not limit the generality since any other case is reduced to this one by the substitution $t = t_1 + t_0$.

The differential equations (4) are invariant under the substitution

$$\bar{\xi}_i=\xi_i, \qquad \bar{\eta}_i=-\eta_i, \qquad \bar{t}=-t. \tag{8}$$

Consequently, every solution of (4) is transformed by (8) into some solution of (4). Moreover, the initial conditions

$$\xi_i=a_i, \qquad \eta_i=0, \qquad \frac{d\xi_i}{dt}=0, \qquad \frac{d\eta_i}{dt}=b_i \tag{9}$$

are transformed into

$$\bar{\xi}_i=a_i, \qquad \bar{\eta}_i=0, \qquad \frac{d\bar{\xi}_i}{d\bar{t}}=0, \qquad \frac{d\bar{\eta}_i}{d\bar{t}}=b_i. \tag{9'}$$

Therefore, that solution of (4) which satisfies the initial conditions (9) is transformed by (8) into itself. Hence, if that solution is

$$\xi_i=\Phi_i(t), \qquad \eta_i=\Psi_i(t) \qquad (i=1, \ldots, k), \tag{10}$$

then

$$\Phi_i(t)=\Phi_i(\bar{t})=\Phi_i(-t), \qquad \Psi_i(t)=-\Psi_i(\bar{t})=-\Psi_i(-t),$$

whence also

$$\xi_i(\eta_1, \ldots, \eta_k)=\xi_i(-\eta_1, \ldots, -\eta_k).$$

It will be noted that the proof holds, whatever the value of N. It is also geometrically evident that the symmetry, if present at all, is independent of the rate of rotation of the reference line.

212. Conditions for Periodic Solutions.—Since the differential equations (7) are unchanged if τ is replaced by $\tau+2n\pi$, or t by $t+nT$ (n being an integer), it follows that if

$$x_i=x_i(\tau), \qquad w_i=w_i(\tau) \qquad (i=1, \ldots, k) \tag{11}$$

is a solution, then so is

$$x_i=x_i(\tau+2n\pi), \qquad w_i=w_i(\tau+2n\pi). \tag{12}$$

These two will be the same solution if the coördinates and their derivatives have the same values at $\tau=\tau_0$; that is, if

$$\left.\begin{aligned} x_i(\tau_0+2n\pi)&=x_i(\tau_0), \qquad & w_i(\tau_0+2n\pi)&=w_i(\tau_0), \\ x_i'(\tau_0+2n\pi)&=x_i'(\tau_0), \qquad & w_i'(\tau_0+2n\pi)&=w_i'(\tau_0). \end{aligned}\right\} \tag{13}$$

If these conditions are satisfied, then, for all values of τ,

$$x_i(\tau+2n\pi)=x_i(\tau), \qquad w_i(\tau+2n\pi)=w_i(\tau);$$

that is, (13) are sufficient conditions for the periodicity of the solutions. That they are also necessary is obvious, if the period is to be $2n\pi$.

Special case.—In the case of a symmetrical conjunction at $\tau=0$, other sufficient conditions can be formulated. For, if $x_i'(0)=w_i(0)=0$ $(i=1,\ldots,k)$, and if every λ_i is a multiple of π, it follows from the symmetry theorem that

$$\left.\begin{aligned} x_i(\pi)&=+x_i(-\pi), & w_i(\pi)&=-w_i(-\pi),\\ x_i'(\pi)&=-x_i'(-\pi), & w_i'(\pi)&=+w_i'(-\pi). \end{aligned}\right\}\quad(14)$$

But, by equations (13), if τ_0 is put equal to $-\pi$, the conditions for periodicity of $x_i(\tau)$ and $w_i(\tau)$ are

$$\left.\begin{aligned} &(a)\quad x_i(\pi)=x_i(-\pi), & &(c)\quad w_i(\pi)=w_i(-\pi),\\ &(b)\quad x_i'(\pi)=x_i'(-\pi), & &(d)\quad w_i'(\pi)=w_i'(-\pi). \end{aligned}\right\}\quad(15)$$

Of these conditions (a) and (d) are satisfied by virtue of (14), while (b) and (c) are also satisfied if $x_i'(\pi)=w_i(\pi)=0$. It may then be stated that sufficient conditions for the periodicity of x_i and w_i (with period in t equal to T) are

$$\left.\begin{aligned} &(a)\quad x_i'(0)=0, \qquad w_i(0)=0, \qquad \lambda_i=0 \text{ or } \pi,\\ &(b)\quad x_i'(\pi)=0, \qquad w_i(\pi)=0. \end{aligned}\right\}\quad(16)$$

Moreover, after conditions $(16a)$ have been imposed, conditions (b) are necessary as well as sufficient.

213. Nature of the Periodicity Conditions.—For $\mu=0$ the differential equations (7) admit the solution with period 2π (or T in t)

$$x_i=1, \qquad w_i=0 \qquad\qquad (i=1,\ldots,k),$$

giving the circular orbits $r_i=a_i$, $u_i=\lambda_i+p_i\tau$, in which at $\tau=0$, $x_i=1$, $x_i'=w_i=w_i'=0$. If these initial values are given increments $\Delta c_{ij}(i=1,\ldots,k;\ j=1,2,3,4)$, then the solutions of the differential equations (7) for $\mu\neq 0$ are developable as power series in μ and the Δc_{ij}, which converge throughout a preassigned interval of τ for sufficiently small values of those parameters.* Such solutions are in general non-periodic; in fact, the periodicity conditions (13) or (16) impose the condition that $4k$ power series in these $4k+1$ parameters shall vanish. In the cases to be considered these $4k$ equations will determine the Δc_{ij} as unique functions of μ, holomorphic in the vicinity of $\mu=0$ and vanishing with μ; so that, for sufficiently small values of μ, there exist initial conditions (depending upon T, q_k, the p_i, μ, and the β_i) such that the orbits described are periodic with the required period.

Evidently for smaller and smaller values of μ, smaller and smaller deviations from the initial conditions of undisturbed motion are sufficient in order to get periodic orbits. These orbits for $\mu\neq 0$ may be said to "grow out of" the undisturbed circular orbits as μ grows from zero. Of course, for any given masses, μ and the β_i being fixed, the possible orbits of this sort can

*See §§14–16.

vary only with T, q_k, or the p_i; but to a *range* of values of μ there corresponds a *class* of orbits.

In what follows it will be inquired whether the conditions for periodicity can be satisfied by such values of the Δc_{ij} as to prove the existence of a class of periodic orbits of each of the following types:

Type I. The finite system has a symmetrical conjunction.

Type II. The infinitesimal system has a symmetrical conjunction, but the finite system has none.

Type III. Neither system has a symmetrical conjunction.

214. Integration of the Differential Equations as Power Series in Parameters.—It will be necessary to obtain the first few terms of the developments mentioned in the preceding article. Instead of increments Δc_{ij} to the initial undisturbed values of the coördinates it will be more convenient, in finding the properties of the solutions, to employ parameters Δn_i, e_i, ω_i, τ_i, defined as follows. At $\tau=0$ let

$$\left.\begin{array}{ll} x_i=(1+\Delta c_{i1})=(1+\Delta n_i)^{-\frac{2}{3}}(1-e_i\cos\theta_i), & \nu x_i'=\Delta c_{i2}=n_i(1+\Delta n_i)^{\frac{1}{3}}\dfrac{e_i\sin\theta_i}{1-e_i\cos\theta_i}, \\ v_i-\lambda_i=w_i=\Delta c_{i3}=\omega_i+\cos^{-1}\dfrac{\cos\theta_i-e_i}{1-e_i\cos\theta_i} & (i=1,\ .\ .\ .\ ,k), \\ \nu w_i'=n_i\Delta c_{i4}=\dfrac{n_i(1+\Delta n_i)\sqrt{1-e_i^2}}{(1-e_i\cos\theta_i)^2}-n_i, & -q_i(1+\Delta n_i)\tau_i=\theta_i-e_i\sin\theta_i, \end{array}\right\}\quad(17)$$

the v_i being equal to $u_i+q_k\tau$, the true longitudes from a fixed reference line.

It is evident that the Δc_{ij} are holomorphic functions of the Δn_i, e_i, ω_j, and τ_i for sufficiently small values of the latter quantities. Consequently, solutions of (7) exist also as power series in the new parameters. Further, since the real positive values of the radicals and the smallest values of the inverse cosines are to be taken in (17), the Δc_{ij} are given uniquely in terms of the Δn_i, e_i, ω_i, and τ_i. From these two facts it follows that, if the latter quantities can be determined as unique power series in μ, satisfying the conditions for periodicity, then also there exist for the Δc_{ij} unique power series in μ, satisfying the conditions. Conversely, while the Jacobian of the Δc_{ij} with respect to the new parameters is zero for $\Delta n_i=e_i=\omega_i=\tau_i=0$, yet, in the only case where discussion will be necessary (viz., for $\Delta c_{i2}=\Delta c_{i3}=0$, whence $\omega_i=\tau_i=0$), the solution for the Δn_i and e_i in terms of the Δc_{i1} and Δc_{i4} is unique; for the Jacobian of the Δc_{i1} and Δc_{i4} with respect to Δn_i and e_i is distinct from zero for $\Delta n_i=e_i=0$. Hence, in this case, if the Δc_{i1} and Δc_{i4} exist as unique series in μ, satisfying the periodicity conditions, so also must the Δn_i and e_i exist as such series.

In the developments of the coördinates as power series in μ and the new parameters all those terms independent of μ may be obtained, together with a knowledge of their properties, in the following simple manner: The terms

in question are those remaining when μ is put equal to zero, and are therefore the solution of the problem of k infinitesimal satellites when the initial conditions are (17)—in other words, the solutions of k two-body problems. The dynamical meaning of the new parameters is then evident. The orbits of the infinitesimal system, subjected to the initial conditions (17), are ellipses in which the mean angular motions, major semi-axes, eccentricities, longitudes of pericenter, and times of pericenter passage are respectively

$$n_i(1+\Delta n_i), \qquad a_i(1+\Delta n_i)^{-\frac{2}{3}}, \qquad e_i, \qquad \lambda_i+\omega_i, \qquad \frac{\tau_i}{\nu}.$$

If in the development of x_i the coefficient of $\Delta n_i^f\, e_i^g\, \omega_i^h\, \tau_i^k$ be denoted by $x_{i,fghk}$ then, by applying Taylor's theorem to the well-known developments of the coördinates in elliptic motion as power series in the eccentricity,* it is found that the following coefficients of first and second degree terms do not vanish:

$$\left.\begin{aligned}
&x_{i,0000}=1, \qquad x_{i,1000}=-\frac{2}{3}, \qquad x_{i,0100}=-\cos q_i\tau, \qquad x_{i,2000}=\frac{5}{9},\\
&x_{i,0200}=\sin^2 q_i\tau, \quad x_{i,0101}=-q_i\sin q_i\tau, \quad x_{i,1100}=\frac{2}{3}\cos q_i\tau+q_i\tau\sin q_i\tau,\\
&w_{i,1000}=q_i\tau, \quad w_{i,0100}=2\sin q_i\tau, \quad w_{i,0010}=1, \quad w_{i,0001}=q_i,\\
&w_{i,0200}=\frac{5}{4}\sin 2q_i\tau, \quad w_{i,0101}=-2q_i\cos q_i\tau, \quad w_{i,1100}=2q_i\tau\cos q_i\tau.
\end{aligned}\right\} \qquad (18)$$

From simple dynamical considerations the following important properties can be established. Let $x_{i,fghk}$ be written $x_{i,fghk}^{(q_i\tau)}$ to indicate its dependence upon τ. Then

$$x_{i,0g00}^{\prime(m\pi)}=w_{i,0g00}^{(m\pi)}=0 \qquad (i=1,\ \ldots,\ k), \qquad (19)$$

where m is any integer; for, the coefficients $x'_{i,0g00}$, etc., are those of the terms which do not involve Δn_i, ω_i, and τ_i, these terms being obtained by putting $\Delta n_i=\omega_i=\tau_i=0$ in the developments. But for these parameters equal to zero the initial positions are apses and the periods (in t) are $2\pi/n_i$. Hence, at $\tau=m\pi/q_i$, M_i is at an apse and $x'_i=w_i=0$, whatever the value of e_i. Since this is true for a range of values of e_i, it follows that the coefficient of each power of e_i in x'_i and in w_i is zero at $\tau=m\pi/q_i$.

It is evident that in the terms independent of μ only those parameters appear whose subscript is the same as that of the coördinate developed; the terms involving μ introduce, however, the other $4(k-1)$ parameters.

Terms involving μ.—The only terms involving μ whose coefficients are needed in the sequel are μ and μe_j $(j=1,\ \ldots,\ k)$. Let the coefficient of μ in the development of x_i be $x_i(0;\ \tau)$, and that of μe_j be $x_i(j;\ \tau)$; let the coefficients of the same quantities in w_i be respectively $w_i(0;\ \tau)$ and $w_i(j;\ \tau)$ $(i=1,\ \ldots,\ k;\ j=1,\ \ldots,\ k)$. The process of finding these depends as follows upon two properties of the solutions:

*Moulton, *Introduction to Celestial Mechanics* (new edition), p. 171.

(*a*) Since the solutions must satisfy the differential equations identically in the parameters, the equating of coefficients of corresponding powers on both sides furnishes sets of differential equations for the successive coefficients in the solutions.

(*b*) The arbitrary constants which the successive coefficients carry are determined by the conditions that the solutions shall reduce identically to equations (17) at $\tau=0$.

For each pair of coefficients $x_i(f;\ \tau)$ and $w_i(f;\ \tau)$ $(f=0, \ldots, k)$, equations (7) give two simultaneous differential equations of the second order. The one from (7*a*) can be integrated once immediately, and its integral combined with the equation from (7*b*) renders the latter a well-known type,

$$x_i''(f;\tau)+q_i^2x_i(f;\tau)+\sum_{m=0}^{\infty}(\gamma_m\cos m\tau+\delta_m\sin m\tau)+\alpha\tau=0. \qquad (20)$$

Its solution, $x_i(f;\ \tau)$, when substituted into the first integral, permits the final integration for $w_i(f;\ \tau)$. The initial conditions are

$$x_i(f;0)=x_i'(f;0)=w_i(f;0)=w_i'(f;0)=0 \qquad (i=1, \ldots, k;\ f=0, \ldots, k), \quad (21)$$

for the conditions (17) do not involve μ at all.

Now the form of the solution varies greatly according as a term $\cos q_i\tau$ or $\sin q_i\tau$ is or is not present in (20). In the former case the solutions contain a so-called Poisson term, $\tau\cos q_i\tau$ or $\tau\sin q_i\tau$, and in the latter case they do not. In all the $x_i(f;\tau)$ and $w_i(f;\tau)$ $(f=1, \ldots, k)$, a Poisson term is present; they are present in the $x_i(0;\tau)$ if, and only if, for some pair of the n_i, say n_f and n_g, there exists an integer J such that

$$J\cdot(n_f-n_g)=n_g. \qquad (22)$$

The meaning and consequences of such a relation will be discussed in §219.

In performing the integrations it is necessary to expand

$$(1-2\epsilon_{ij}\cos\phi_{ji}+\epsilon_{ij}^2)^{-\frac{s}{2}} \qquad (s=3,\ 5)$$

as a cosine series, where, for the sake of a uniform notation, the following definitions are made:

$$\epsilon_{ij}=a_{ji} \quad \text{and} \quad \eta_{ij}=a_{ji}, \text{ if } j<i; \qquad \epsilon_{ij}=a_{ij} \quad \text{and} \quad \eta_{ij}=1, \text{ if } j>i. \qquad (23)$$

Then

$$\left.\begin{aligned}(1-2\epsilon_{ij}\cos\phi_{ji}+\epsilon_{ij}^2)^{-3/2}&=\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji},\\(1-2\epsilon_{ij}\cos\phi_{ji}+\epsilon_{ij}^2)^{-5/2}&=\sum_{m=0}^{\infty}G_m(\epsilon_{ij})\cos m\phi_{ji},\end{aligned}\right\} \qquad (24)$$

where the F_m and G_m are well-known power series in ϵ_{ij}, beginning with ϵ_{ij}^m.

Finally, the desired coefficients of the functions x_i and w_i are, for $i=1, \ldots, k$ and for $f=0, 1, \ldots, k$

$$
\left.\begin{aligned}
x_i(f;\tau) &= A_{if}+B_{if}\cos q_i\tau+C_{if}\sin q_i\tau+D_{if}\tau \\
&\quad +E_{if}\tau\cos q_i\tau+H_{if}\tau\sin q_i\tau+J_{if}\tau\cos 2q_i\tau \\
&\quad +K_{if}\tau\sin 2q_i\tau+\sum_{m=1}^{\infty}(a_{if}^{(m)}\cos m\tau+c_{if}^{(m)}\sin m\tau), \\
w_i(f;\tau) &= L_{if}+N_{if}\tau-q_iD_{if}\tau^2-2\Big[B_{if}+\frac{H_{if}}{q_i}\Big]\sin q_i\tau \\
&\quad +2\Big[C_{if}-\frac{E_{if}}{q_i}\Big]\cos q_i\tau+2H_{if}\tau\cos q_i\tau \\
&\quad -2E_{if}\tau\sin q_i\tau-\tfrac{5}{2}J_{if}\tau\sin 2q_i\tau \\
&\quad +\tfrac{5}{2}K_{if}\tau\cos 2q_i\tau+\sum_{m=1}^{\infty}(b_{if}^{(m)}\sin m\tau+d_{if}^{(m)}\cos m\tau),
\end{aligned}\right\} \tag{25}
$$

where $D_{io}=0$ and $J_{if}=K_{if}=0$ for $f\neq i$, while if every $\lambda_i=0$ or π, then

$$
J_{ii}=0,\ C_{if}=D_{if}=E_{if}=L_{if}=c_{if}^{(m)}=d_{if}^{(m)}=0 \qquad (f=0, \ldots, k); \tag{26}
$$

but if no relation (22) holds, then

$$
E_{io}=H_{io}=J_{ii}=K_{ii}=0, \qquad D_{if}=0 \qquad (f=0, \ldots, k).
$$

Those constants whose values are needed in the proofs are

$$
\left.\begin{aligned}
H_{if} &= -\frac{\delta_{if}\cos(\lambda_i-\lambda_f)\eta_{if}^3}{2q_i}\Big[\tfrac{9}{2}F_0+\tfrac{1}{4}F_2-\tfrac{9}{2}\eta_{if}^2(1+a_{if}^2)G_0 \\
&\quad +\tfrac{21}{8}a_{if}\eta_{if}^2G_1+\tfrac{3}{4}\eta_{if}^2(1+a_{if}^2)G_2-\tfrac{3}{8}a_{if}\eta_{if}^2G_3+X_{if}\Big], \\
H_{ii} &= N_{io}+\sum_j{}'\frac{\delta_{ij}\eta_{ij}^3}{2q_i}\Big[3a_{ij}F_0+F_1-a_{ij}\eta_{ij}^2\Big(\tfrac{15}{2}+3a_{ij}^2\Big)G_0 \\
&\quad +3a_{ij}^2\eta_{ij}^2G_1+\tfrac{9}{4}a_{ij}\eta_{ij}^2G_2+Y_{ij}\Big], \\
C_{io} &= \sum_i{}'\delta_{ij}\sum_{m=1}^{\infty}\Theta_m(\epsilon_{ij})\sin m(\lambda_j-\lambda_i),
\end{aligned}\right\} \tag{27}
$$

where

$$
\left.\begin{aligned}
\Theta_1(\epsilon_{ij}) &= \frac{q_j-q_i}{2q_i(q_j-q_i^2-q_i^2)}\Big[2+\eta_{ij}^3(2a_{ij}F_1-2F_0+F_2)-\frac{2q_i}{q_j-q_i}(2-\eta_{ij}^3 2F_0-F_2)\Big], \\
\Theta_m(\epsilon_{ij}) &= \frac{m\eta_{ij}^3(q_j-q_i)}{2q_i(m^2q_j-q_i^2-q_i^2)}\Big[2a_{ij}F_m-F_{m-1}+F_{m+1} \\
&\quad +\frac{2q_i}{m(q_j-q_i)}F_{m-1}-F_{m+1}\Big] \qquad (m\neq 1),
\end{aligned}\right\} \tag{28}
$$

where the X_{ij} and Y_{ij} vanish, except for special relations among the q_j.

215. Existence of Periodic Orbits of Type I.—When k finite satellites are subjected to the initial conditions (17), the solutions of the differential equations are expressible as power series—whose first coefficients have been tabulated in equations (18) and (25)—in μ and the Δn_i, e_i, ω_i, and τ_i, converging throughout an arbitrarily preassigned time interval for sufficiently small values of these $4k+1$ quantities. And, although the orbits are in general non-periodic (as shown by the non-periodic terms in the tabulated coefficients), it will now be proved that the conditions for periodicity can nevertheless be satisfied, provided that every $\lambda_i=0$ or π, by assigning to the ω_i and τ_i the value zero (identically as to μ) and to the Δn_i and e_i certain values dependent upon μ, that is, at $\tau=0$ there is to be a symmetrical conjunction in which the velocities and distances must be properly chosen with reference to the masses if periodic motion is to result.

In this case the conditions for periodicity are (16), of which equations (a) are already satisfied. The necessary and sufficient conditions are then

$$\left.\begin{aligned}
&(a)\qquad 0=\Delta n_i(q_i\pi)+e_i(2\sin q_i\pi)+\mu w_i(0;\pi)+\cdots,\\
&(b)\qquad 0=e_i(q_i\sin q_i\pi)+\mu(x_i'(0;\pi))+\Delta n_i e_i x'_{i,1100}(\pi)\\
&\qquad\qquad +\mu\sum_{f=1}^{k} e_f x_i'(f;\pi)+\cdots \qquad (i=1,\ldots,k).
\end{aligned}\right\}\qquad(29)$$

The problem of solving these $2k$ equations has been treated in Chapter I. Here the functional determinant, taken at $\Delta n_i=e_i=\mu=0$, is merely the determinant of the linear terms, whose value is

$$\Delta_1=\pi^k\prod_{i=1}^{k} q_i^2\sin q_i\pi,\qquad(30)$$

so that its vanishing depends upon the q_i. Since, from (2) and (5),

$$q_i=q_k+p_i\qquad (i=1,\ldots,k-1),$$

it follows that Δ_1 vanishes if, and only if, the arbitrary q_k is selected as an integer.

Case I: q_k is not an integer.—Since $\Delta_1\neq 0$, the Δn_i and e_i exist* as unique power series in μ, vanishing with μ, converging for μ sufficiently small, and satisfying (29). Thus, for all sufficiently small masses of the k finite satellites there exist initial conditions for which the resulting orbits are periodic; and the presence of q_k and of the arbitrary integers p_i shows that for given masses there is a k-fold infinitude of orbits of this type. A family of such orbits exists "growing out of" *any set of circles* for which the infinitesimal system would have commensurable synodic periods, whose least common multiple is not divisible by the sidereal period of satellite k. In Case I consecutive symmetrical conjunctions do not occur at the same absolute longitude; nor will *any* later ones occur at the same longitude if q_k is irrational.†

*See §§1, 2. †See §219.

Case II: q_k is an integer.—Here $\Delta_1 = 0$. Nevertheless the Jacobian of the $w_i(\pi)$ with respect to the Δn_i, taken at $\Delta n_i = e_i = \mu = 0$, is

$$\Delta_2 = \pi^k \prod_{i=1}^{k} q_i \neq 0. \tag{31}$$

(The only possibility for $q_i = 0$ is $p_i = -q_k$ which requires $n_i = 0$, and such a selection for p_i has been excluded.) Hence (29a) can be solved for the Δn_i* as power series in the e_i and μ, converging for sufficiently small values of the latter quantities. Now, by (19), every term in (29a, b) has either μ or some Δn_i as a factor; hence the solutions have the form

$$\Delta n_i = \mu P_i(e_j, \mu) \qquad (i=1, \ldots, k;\ j=1, \ldots, k). \tag{32}$$

If the power series (32) are substituted in (29b), the resulting series converge for sufficiently small values of μ and e_j and contain μ as a factor. (This merely means that $\Delta n_i = \mu = 0$ satisfy the periodicity conditions, whatever may be the values of the e_j). If μ is divided out,† relations are obtained among the e_j and μ of the form

$$0 = x_i'(0;\pi) - e_i \frac{1}{q_i \pi} x'_{i,1100}(\pi) \cdot w_i(0;\pi) + \sum_{j=1}^{k} e_j x_i'(j;\pi) + \mu \{\ \}_i + \cdots \tag{33}$$

At this point two questions of importance arise, viz., as to the vanishing of the $x_i'(0;\pi)$ and as to the vanishing of the determinant of the coefficients of the linear terms in e_j. By (25) and (27) every $x_i'(0;\pi)$ is zero unless the relation (22) holds for some pair of the n_i. When such a relation does hold, the equations (33) are not satisfied by $e_j = \mu = 0$, so that solutions for the e_j, vanishing with μ, do not exist. Hence, periodic orbits of Type I, "growing out of the circular orbits," do not exist, if, for any n_f and n_g, $J \cdot (n_f - n_g) = n_g$, where J is an integer. When no such relation exists,‡ the equations (33) are satisfied by $e_j = \mu = 0$. It remains to examine the determinant Δ_3 of the first degree terms in the e_j. This involves, in each of its elements, power series in the ϵ_{ij}, or a_i/a_j, a_j/a_i; and it is unknown whether there are any sets of values of the ϵ_{ij} for which $\Delta_3 = 0$. It will, however, be shown that there is an infinite number of values for which the determinant is distinct from zero.

Let P_{if} be any element of Δ_3, the first subscript indicating the row and the second the column; then, by equations (33) and (25),

$$\left.\begin{aligned} P_{if} &= x_i'(f;\pi) = q_i(-1)^{q_i} \pi H_{if} && (f \neq i), \\ P_{ii} &= x_i'(i;\pi) - q_i(-1)^{q_i} w_i(0;\pi) = q_i(-1)^{q_i} \pi [H_{ii} - N_{i0}]. && \end{aligned}\right\} \tag{34}$$

The ϵ_{ij} depend upon ν, q_k, the β_i, and the p_i which were arbitrarily chosen. It will now be shown that for a fixed selection of the β_i, ν, and q_k there is an

*This case is not essentially different from that treated in §4. For although Δ_1 has no minors of order less than k distinct from zero, yet, so far as the linear terms alone are concerned, the equations (29) may be regarded as k independent pairs.

†It is precisely this step which makes the selection of the parameters Δn_i, e_i especially advantageous.

‡That q_k an integer does not involve the existence of such a relation is shown in §219.

infinite number of selections of the p_i (viz., all for which the ϵ_{ij} are "sufficiently small") for which $\Delta_{ij} \neq 0$. For convenience, let all the ϵ_{ij} be expressed in terms of a single parameter α by

$$a_i = b_i^2 \alpha^{2(k-i)} a_k \qquad (i=1, \ldots, k-1), \qquad (35)$$

where the b_i, ν, and q_k (hence also a_k and n_k) are constants independent of α. Every element of Δ_3 is, then, a power series in α; for by (35), (34), and (27)

$$\left.\begin{array}{lll}
\epsilon_{ij} = \dfrac{b_j^2}{b_i^2}\alpha^{2(i-j)}, & \eta_{ij} = \dfrac{b_j^2}{b_i^2}\alpha^{2(i-j)}, & \text{if } j<i, \\
\epsilon_{ij} = \dfrac{b_i^2}{b_j^2}\alpha^{2(j-i)}, & \eta_{ij} = 1, & \text{if } j>i, \\
q_i = \dfrac{q_k}{b_i^3}\alpha^{3(i-k)}, & \delta_{ij} = \beta_j \dfrac{q_k^2}{b_i^2 b_j^4}\alpha^{2i+4j-6k}, & \\
P_{11} = \alpha^{12-6k} Q_{11}(\alpha), & P_{ii} = \alpha^{(6i+4-6k)} Q_{ii}(\alpha) & \text{if } i>1, \\
P_{if} = \alpha^{(12i-6f-6k)} \cdot Q_{if}(\alpha), & & \text{if } f<i, \\
P_{if} = \alpha^{(8f-2i-6k)} \cdot Q_{if}(\alpha), & & \text{if } f>i,
\end{array}\right\} \qquad (36)$$

where the $Q_{if}(\alpha)$ $(f=1, \ldots, k)$ are power series in α, *beginning with a constant term.** In P_{ii} under the sign $\sum_j{}'$, the lowest power of α for $j<i$ is $10i-4j-6k$, this exponent having its smallest value $6i+4-6k$, when $j=i-1$; while for $j>i$ the lowest power is $6j-6k$, whose smallest value is $6i+6-6k$. Hence in the i^{th} row of Δ_3 the lowest power of α in any of the P_{if} is

for $f<i$,	$6i+6-6k$,	viz., for $f=i-1$,
for $f=1$,	$6i+4-6k$,	except when $i=1$,
for $f>i$,	$6i+8-6k$,	viz., for $f=i+1$.

But, in the first row P_{11} carries α^{12-6k} as against α^{14-6k} in P_{12}, which is the next lowest power. Evidently, then, in every row of Δ_3 the lowest power of α occurs in the element of the main diagonal; and if that lowest power of α is removed from the row as a factor of Δ_3, a new determinant Δ_4 is obtained, all of whose main diagonal elements begin with a constant term, while the series in every other element carries a positive power of α as a factor. Thus the development of Δ_4 as a single series in α begins with a constant term (the product of those in the principal diagonal) and is distinct from zero both for $\alpha=0$ and for all values of α up to some finite value. Hence Δ_3 also must be distinct from zero for all values of α sufficiently small, and vanishes, if at all, at a finite number of points.

In case some $q_j = 3q_i$, the special terms X_{ij} and Y_{ij} (which may be present in the P_{ij} and P_{ii} for other special relations also) carry powers of α lower than some of those considered above. In P_{ij} the power is lowered to $\alpha^{10i-4j-6k}$, and in P_{ii} to $\alpha^{2i+4-6k}$; in P_{ji} to α^{6i-6k}, and in P_{jj} remains unchanged as $\alpha^{6j+4-6k}$. But in every case it is easily seen that the power is lower in the

*If X_{ij} and Y_{ij} are present in P_{ij} and P_{ii}, certain changes must be made.

main diagonal element than elsewhere in the same row, except in the first row, where P_{12} may now carry α^{12-6k}, as does P_{11}. And even this exception is immaterial, since, when the rows have been factored as before, the constant term in the element Q_{12} is to be multiplied by a minor whose first column contains α^8 as a factor, and hence can not destroy the constant term in the main diagonal product. Therefore, it is true without exception that $\Delta_3 \neq 0$ for all values of α sufficiently small.

Therefore, equations (33) can be solved for the e_i in terms of μ, vanishing with μ. The substitution of these solutions in (32) gives the Δn_i, as well as the e_i, as holomorphic functions of μ, for μ sufficiently small. Hence there exist initial conditions giving periodic orbits of Type I even when q_k is an integer, provided that no relation (22) holds and that α is sufficiently small.

In drawing this last conclusion, however, a point of delicacy arises. The p_i are functions of α in the foregoing argument, and obviously the p_i are not integers (as the formulation of the problem requires them to be) for all values of α on any interval. The question arises as to whether there are, indeed, any values of α, "sufficiently small," for which the p_i are integers. From (36) and $p_i = q_i - q_k$, it follows that

$$p_i = q_k\left(\frac{1}{b_i^3\,\alpha^{3(k-i)}} - 1\right) \qquad (i=1,\ldots,k-1). \qquad (37)$$

The present discussion will be confined to exhibiting a selection of the b_i such that there is an infinite number of values of α less than any assigned quantity, for each of which the p_i are integers without common divisor. Let the assigned value be α_0 and let the b_i be defined by

$$b_i = \frac{1}{\alpha_0^{k-i}}\sqrt[3]{\frac{q_k}{q_k+k-i}} \qquad (i=1,\ldots,k-1); \qquad (38)$$

and consider (37) for $\alpha = \alpha_0\sqrt[3]{1/n}$, where n is an integer. Evidently, since q_k is an integer, and since

$$p_i = (q_k+k-i)\,n^{k-i} - q_k \qquad (i=1,\ldots,k-1), \qquad (39)$$

the p_i are integers. Consider the possibility of a common factor. If p_{k-1} and p_{k-2} have a common factor, their difference has the same factor. Thus, if there is a factor common to $(q_k+1)n - q_k$ and $(q_k+2)n^2 - q_k$, it is also a factor of $n^2 + n(n-1)(q_k+1)$. Hence if n is prime and greater than q_k, such a factor must divide $n+(n-1)(q_k+1)$ and also the difference between this number and p_{k-1}, or $(n-1)$. But, as n and $n-1$ are mutually prime, there is no factor of $n-1$ which divides $n+(n-1)(q_k+1)$, and hence no factor common to p_{k-1} and p_{k-2} if n is chosen a prime number greater than q_k. There is an infinite number of primes; hence the p_i have the stated property.

The periodic solutions exist, then, and might be obtained as series in μ alone (convergent for sufficiently small values) by substituting in the original series in μ and the Δn_i and e_i the values of the latter $2k$ parameters, as obtained in terms of μ from the periodicity conditions. A far more advantageous method is, however, available.

216. Method of Construction of Solutions. Type I.—It has been shown that, for μ sufficiently small, there exist series

$$x_i(\tau)=1+\sum_{n=1}^{\infty}x_{i,n}(\tau)\mu^n, \qquad w_i(\tau)=\sum_{n=1}^{\infty}w_{i,n}(\tau)\mu^n \qquad (i=1,\ldots,k), \tag{40}$$

which (*a*) converge for $0\leqq\tau\leqq 2\pi$, (*b*) satisfy the differential equations (7), (*c*) satisfy $x_i'(0)=0$, $w_i(0)=0$ identically in μ, and (*d*) satisfy $x_i(\tau+2\pi)-x_i(\tau)=w_i(\tau+2\pi)-w_i(\tau)=0$ identically in μ.

The permanent convergence of series (40) follows from (*d*) and (*a*). From (*c*) and (*d*) follow respectively

$$x_{i,n}'(0)=w_{i,n}(0)=0, \tag{41}$$

$$x_{i,n}(\tau+2\pi)-x_{i,n}(\tau)=w_{i,n}(\tau+2\pi)-w_{i,n}(\tau)=0. \tag{42}$$

These equations (41) and (42) will determine the constants of integration arising at each step.

First order terms.—Since (40) must satisfy (7) identically in μ, the $x_{i,1}(\tau)$ and $w_{i,1}(\tau)$ must satisfy

$$\left.\begin{aligned}
&(a) \quad w_{i,1}''+2q_i x_{i,1}'+\sum_j{}'\delta_{ij}\sin\phi_{ji}\left(1-\eta_{ij}^3\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji}\right)=0,\\
&(b) \quad x_{i,1}''-2q_i w_{i,1}'-3q_i^2x_{i,1}+q_i^2\beta_i\\
&\qquad +\sum_j{}'\delta_{ij}\left[\cos\phi_{ji}+\eta_{ij}^3(a_{ij}-\cos\phi_{ji})\sum_{m=0}^{\infty}F_m(\epsilon_{ij})\cos m\phi_{ji}\right]=0.
\end{aligned}\right\} \tag{43}$$

Since every ϕ_{ji} is a multiple of τ plus a multiple of π, equations (43) are of the type

$$\left.\begin{aligned}
&(a) \quad w_{i,1}''+2q_i x_{i,1}'+\sum_{m=1}^{\infty}D_{i,1}^{(m)}\sin m\tau=0,\\
&(b) \quad x_{i,1}''-2q_i w_{i,1}'-3q_i^2x_{i,1}+E_{i,1}^{(0)}+\sum_{m=1}^{\infty}E_{i,1}^{(m)}\cos m\tau=0,
\end{aligned}\right\} \tag{44}$$

where the $D_{i,1}^{(m)}$ and $E_{i,1}^{(m)}$ are linearly related to the F_m, and can be expressed in terms of the latter as soon as the p_i are chosen. The solutions are

$$\left.\begin{aligned}
x_{i,1}(\tau)&=-\frac{1}{q_i^2}(2q_ic_{i,1}^{(1)}+E_{i,1}^{(0)})+c_{i,1}^{(2)}\cos q_i\tau+c_{i,1}^{(3)}\sin q_i\tau+\sum_{m=1}^{\infty}A_{i,1}^{(m)}\cos m\tau,\\
w_{i,1}(\tau)&=\frac{1}{q_i}(3q_ic_{i,1}^{(1)}+2E_{i,1}^{(0)})\tau+c_{i,1}^{(4)}-2c_{i,1}^{(2)}\sin q_i\tau+2c_{i,1}^{(3)}\cos q_i\tau+\sum_{m=1}^{\infty}B_{i,1}^{(m)}\sin m\tau,
\end{aligned}\right\} \tag{45}$$

where the $c_{i,1}^{(j)}$ $(j=1,\ldots,4;\ i=1,\ldots,k)$ are the constants of integration and

$$(m^2-q_i^2)A_{i,1}^{(m)}=E_{i,1}^{(m)}-\frac{2q_i}{m}D_{i,1}^{(m)}, \qquad m^2B_{i,1}^{(m)}=D_{i,1}^{(m)}-2mq_iA_{i,1}^{(m)}. \tag{46}$$

Poisson terms do not appear in (45); for, since no relation (22) holds, no term in $\cos q_i\tau$ or $\sin q_i\tau$ is present in (44). Now, by (41) and (42),

$$c_{i,1}^{(3)}=c_{i,1}^{(4)}=0, \qquad c_{i,1}^{(1)}=-\frac{2}{3q_i}E_{i,1}^{(0)},$$

so that the k constants $c_{i,1}^{(2)}$ alone remain to be determined. And here arise two cases, just as in the existence proof:

Case I. q_k *is not an integer.*—Here, q_i not being an integer, $\cos q_i\tau$ does not have the period 2π; consequently, by (42), $c_{i,1}^{(2)}=0$ $(i=1, \ldots, k)$.

Case II. q_k *is an integer.*—Here $\cos q_i\tau$ has the period 2π, and (42) is satisfied for the arbitrary $c_{i,1}^{(2)}\neq 0$. These k constants remain undetermined until the second-order terms are found, when the $c_{i,1}^{(2)}$ are uniquely determined in destroying Poisson terms.

Terms of any order. Case I.—Assume that for $n=1, \ldots, h-1$, the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ have been found, the constants being determined, and have the form

$$x_{i,n}(\tau)=\sum_{m=0}^{\infty} A_{i,n}^{(m)}\cos m\tau, \qquad w_{i,n}(\tau)=\sum_{m=0}^{\infty} B_{i,n}^{(m)}\sin m\tau \qquad (i=1, \ldots, k). \tag{47}$$

An induction will show that $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ have the form (47); that, moreover, the differential equations for $x_{i,h}$ and $w_{i,h}$ are of the type (44); and that the constants of integration are determined just as in the preceding case for $n=1$. The differential equations are [see (7)]

$$\left.\begin{aligned}
&(a) \quad \sum_{k+l=h} x_{i,k}w''_{i,l}+2\sum_{k+l=h} x'_{i,k}w'_{i,l}+2q_i x'_{i,h}\\
&\qquad +\sum_j{}'\delta_{ij}\sum_{k+l+m=h-1} x_{j,k}(\sin\overline{\phi_{ji}+w_j-w_i})_l[x_j^{-3}-\sigma_{ij}^{-3}]_m=0,\\
&(b) \quad x''_{i,h}-\sum_{k+l+m=h} x_{i,m}(w'_{i,k}w'_{i,l}+2q_i w'_{i,k})-q_i^2x_{i,h}+q_i^2(x_i^{-2})_h\\
&\qquad +q_i^2\beta_i(x_i^{-2})_{h-1}+\sum_j{}'\delta_{ij}\Big\{a_{ij}\sum_{k+l=h-1} x_{i,k}(\sigma_{ij}^{-3})_l\\
&\qquad +\sum_{k+l+m=h-l} x_{j,k}(\cos\overline{\phi_{ji}+w_j-w_i})_l[x_j^{-3}-\sigma_{ij}^{-3}]_m\Big\} \quad =0,
\end{aligned}\right\} \tag{48}$$

where an expression in parenthesis, having a subscript l outside, denotes the sum of all those terms in the expression which involve μ^l. It is to be shown (α) that the variables in (48) whose second subscript is h enter in the same form as the $x_{i,1}$ and $w_{i,1}$ enter (44), and (β) that the remaining terms of (48) (a) and (b) reduce respectively to a sine series and a cosine series in multiples of τ.

Evidently in (48a) the only terms involving the $x_{i,h}$ and $w_{i,h} (i=1, \ldots, h)$ are $w''_{i,h}+2q_i x'_{i,h}$; and in (48b), aside from $(x_i^{-2})_h$, they are $x''_{i,h}-2q_i w'_{i,h}-q_i^2 x_{i,h}$. Now it can be shown easily by induction that

$$\left(\frac{d^h x_i^{-2}}{d\mu^h}\right)_{\mu=0}=\sum_\nu N_\nu(-1)^{\nu_0}\prod_{f=0}^{h} x_i^{-(2+\nu_0)}\left(\frac{d^f x_i}{d\mu^f}\right)_{\mu=0}^{\nu_f},$$

where the N_ν are positive numbers and the ν_f are positive integers (or zero) satisfying the conditions

$$\sum_{f=1}^{h}\nu_f=\nu_0, \qquad \sum_{f=1}^{h} f\nu_f=h. \tag{49}$$

Now $x_{i,h}$ enters only through $(d^h x_i/d\mu^h)^{\nu_h}$, and for this term $N_\nu=2$, $\nu_0=1$, $\nu_h=1$, and $\nu_f=0\,(f=1, \ldots, h-1)$; hence, in $(x_i^{-2})_h$, $x_{i,h}$ appears with the coefficient -2. Therefore, in (48b), the terms involving $x_{i,h}$ and $w_{i,h}$ are $x''_{i,h}-2q_i w'_{i,h}-3q_i^2 x_{i,h}$, and this establishes statement (α) above.

On using the notation $F^c(\tau)$ and $F^s(\tau)$ to designate respectively a cosine series and a sine series in multiples of τ, it is evident that

$$F_1^s \cdot F_2^s = F^c, \qquad F_1^c \cdot F_2^c = F^c, \qquad F^c \cdot F^s = F^s,$$
$$(F^c)^n = F^c, \qquad (F^s)^{2n} = F^c, \qquad (F^s)^{2n+1} = F^s.$$

Hence, as every $x_{i,n}$ and $w'_{i,n} (n=1, \ldots, h-1)$ is a $F^c(\tau)$, so also are all sums of products of these quantities, and also all polynomials in the $x_{i,n}$, *e. g.* $(x_i^{-\lambda})_l$ and parts of $(\sigma_{ij}^{-3})_l$. Similarly, the sums of all products $x_{i,j} w''_{i,k}$ and $x'_{i,j} w'_{i,l}$ are $F^s(\tau)$.

There remain in (48) only the terms $(\sin \overline{m_{ij}+w_j-w_i})$, where $m_{ij} = \phi_{ji}$ in (48*a*) and $m_{ij} = \phi_{ji} + \pi/2$ in (48*b*). Let, for the moment, $z_{ij} = m_{ij} + w_j - w_i$. Then, since $[d^f z_{ij}/d\mu^f]_{\mu=0} = w_{j,f} - w_{i,f}$, it can be shown by a simple induction that

$$\left[\frac{d^l \sin z_{ij}}{d\mu^l}\right]_{\mu=0} = \sum_{\nu} N_\nu \prod_{f=0}^{l} \sin\left(m_{ij} + \nu_0 \frac{\pi}{2}\right)\left(w_{j,f} - w_{i,f}\right)^{\nu_f},$$

where the N_ν are numbers and the ν_f satisfy (49) after h is replaced by l. Now since every $(w_{j,f} - w_{i,f})$ $(f=1, \ldots, h-1)$ is a $F^s(\tau)$, the product $\Pi_{f=1}^{f=l}(w_{j,f} - w_{i,f})^{\nu_f}$ is a $F^s(\tau)$ or $F^c(\tau)$ according as $\Sigma_{f=1}^{f=l} \nu_f$ is odd or even. But when this sum is odd, ν_0 is odd; and when even, ν_0 is even. The entire product $\Pi_{f=0}^{f=l}$ is then always a $F^s(\tau)$ if $m_{ij} = \phi_{ji}$, or a $F^c(\tau)$ if $m_{ij} = \phi_{ii} + \pi/2$.

The differential equations (48) are, therefore, of the form

$$\left.\begin{aligned}
&(a) \qquad w''_{i,h} + 2q_i x'_{i,h} + \sum_{m=1}^{\infty} D_{i,h}^{(m)} \sin m\tau = 0, \\
&(b) \qquad x''_{i,h} - 2q_i w'_{i,h} - 3q_i^2 x_{i,h} + E_{i,h}^{(0)} + \sum_{m=1}^{\infty} E_{i,h}^{(m)} \cos m\tau = 0,
\end{aligned}\right\} \qquad (50)$$

where no term in $\cos q_i\tau$ or $\sin q_i\tau$ occurs under the summation sign, since q_i is not an integer. Obviously the integration of (50) is the same problem as that of (44), so that the solutions are

$$\left.\begin{aligned}
x_{i,h}(\tau) &= -\frac{1}{q_i^2}(2q_i c_{i,h}^{(1)} + E_{i,h}^{(0)}) + c_{i,h}^{(2)} \cos q_i\tau + c_{i,h}^{(3)} \sin q_i\tau + \sum_{m=1}^{\infty} A_{i,h}^{(m)} \cos m\tau, \\
w_{i,h}(\tau) &= \frac{1}{q_i}(3q_i c_{i,h}^{(1)} + 2E_{i,h}^{(0)})\tau + c_{i,h}^{(4)} - 2c_{i,h}^{(2)} \sin q_i\tau + 2c_{i,h}^{(3)} \cos q_i\tau + \sum_{m=1}^{\infty} B_{i,h}^{(m)} \sin m\tau \\
&\qquad (i=1, \ldots, k),
\end{aligned}\right\} \qquad (51)$$

where

$$(m^2 - q_i^2) A_{i,h}^{(m)} = E_{i,h}^{(m)} - \frac{2q_i}{m} D_{i,h}^{(m)}, \qquad m^2 B_{i,h}^{(m)} = D_{i,h}^{(m)} - 2mq_i A_{i,h}^{(m)}. \qquad (52)$$

Also, by (41) and (42),

$$c_{i,h}^{(1)} = -\frac{2}{3q_i} E_{i,h}^{(0)}, \qquad c_{i,h}^{(2)} = c_{i,h}^{(3)} = c_{i,h}^{(4)} = 0 \qquad (i=1, \ldots, k), \qquad (53)$$

so that the induction is completely established. Thus the successive $x_{i,n}$ and $w_{i,n}$ reduce to

$$x_{i,n}(\tau)=\frac{1}{3q_i^2}E_{i,n}^{(0)}+\sum_{m=1}^{\infty}A_{i,n}^{(m)}\cos m\tau,\quad w_{i,n}(\tau)=\sum_{m=1}^{\infty}B_{i,n}^{(m)}\sin m\tau\quad (i=1,\ldots,k),\tag{51'}$$

and may be obtained *without integration* by applying (52). It is merely necessary to compute the $D_{i,n}^{(m)}$ and $E_{i,n}^{(m)}$ from equations (48) at each step.

Terms of any order. Case II.—Assume, as in Case I, that for $n=1,\ldots,(h-1)$ the $x_{i,n}(\tau)$ and $w_{i,n}(\tau)$ have the form (47), all the constants of integration having been determined except the $c_{i,h-1}^{(2)}$. The differential equations for the $x_{i,h}$ and $w_{i,h}$ are again (50) (a) and (b), where, however, since q_i is an integer, $\cos q_i\tau$ and $\sin q_i\tau$ may occur under the summation sign. These terms arise from two sources: from the terms $c_{i,h-1}^{(2)}\cos q_i\tau$ and $c_{i,h-1}^{(2)}\sin q_i\tau$ in the $x_{i,h-1}$ and $w_{i,h-1}$, and from similar terms in the earlier $x_{i,n}$ and $w_{i,n}$, as well as (usually) from combinations of the ϕ_{ji} in the coefficients.

When (50a) are integrated and combined with (50b), equations for the $x_{i,h}$ are obtained; to avoid Poisson terms in the $x_{i,h}$, the coefficients of the terms in $\cos q_i\tau$ in these last equations must be made to vanish. These coefficients, $E_{i,h}^{(q_i)}-2D_{i,h}^{(q_i)}$, involve the $c_{i,h-1}^{(2)}$ and various known constants; *e. g.*, the $c_{i,n}^{(2)}$ $(n=1,\ldots,h-2)$; and the vanishing of the $E_{i,h}^{(q_i)}-2D_{i,h}^{(q_i)}$ will usually determine the $c_{i,h-1}^{(2)}$.

In the first place, the only terms of (48) in which the $x_{i,h-1}$ and $w_{i,h-1}$ appear are*

$$\left.\begin{aligned}
(a)\quad & x_{i,h-1}w''_{i,1}+x_{i,1}w''_{i,h-1}+2x'_{i,h-1}w'_{i,1}+2x'_{i,1}w'_{i,h-1}\\
&+\sum_j{}'\,\delta_{ij}\Big\{\overline{w_{j,h-1}-w_{i,h-1}}\cos\phi_{ji}-2x_{j,h-1}\sin\phi_{ji}\\
&-\eta_{ij}^3\sigma_{ij,0}^{-3}(x_{j,h-1}\sin\phi_{ji}+\overline{w_{j,h-1}-w_{i,h-1}}\cos\phi_{ji}+3\eta_{ij}^5\sigma_{ij,0}^{-5}\sin\phi_{ji}(\alpha_{ij}^2x_{i,h-1}\\
&+x_{j,h-1}-\alpha_{ij}x_{j,h-1}+x_{i,h-1}\cos\phi_{ji}+\alpha_{ij}\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})\Big\},\\
(b)\quad & -2q_ix_{i,h-1}w'_{i,1}-2q_ix_{i,1}w'_{i,h-1}-2w'_{i,1}w'_{i,h-1}+6q_i^2x_{i,1}x_{i,h-1}-2q_i^2\beta_ix_{i,h-1}\\
&+\sum_j{}'\,\delta_{ij}^4\Big\{-(2x_{j,h-1}\cos\phi_{ji}+\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})\\
&+\eta_{ij}^3\sigma_{ij,0}^{-3}(\alpha_{ij}x_{i,h-1}-x_{j,h-1}\cos\phi_{ji}+\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})\\
&-3\eta_{ij}^5\sigma_{ij,0}^{-5}(\alpha_{ij}-\cos\phi_{ji})(\alpha_{ij}^2x_{i,h-1}+x_{j,h-1}-\alpha_{ij}\overline{x_{j,h-1}+x_{i,h-1}\cos\phi_{ji}}\\
&+\alpha_{ij}\overline{w_{j,h-1}-w_{i,h-1}}\sin\phi_{ji})\Big\},
\end{aligned}\right\}\tag{54}$$

*For $h=2$ the first four terms in (a) and in (b) are to be divided by 2.

where $\sigma_{ij,0}=(1-2\epsilon_{ij}\cos\phi_{ji}+\epsilon_{ij}^2)^{1/2}$. Whence it is seen that the $c_{f,h-1}^{(2)}$ enter the $E_{i,h}^{(q_i)}-2D_{i,h}^{(q_i)}$ linearly; and, denoting their coefficients by R_{if} $(i=1, \ldots, k;$ $f=1, \ldots, k)$, it is found that

$$R_{if}=(-1)^{q_i}\frac{2}{\pi}P_{if} \qquad (i=1, \ldots, k; f=1, \ldots, k), \tag{55}$$

where the P_{if} are the elements of the determinant Δ_3, discussed in the existence proof. Hence, when $\Delta_3 \neq 0$, the determinant of the coefficients of the $c_{f,h-1}^{(2)}$ in the equations $E_{i,h}^{(q_i)}-2D_{i,h}^{(q_i)}$ is zero. The constants $c_{f,h-1}^{(2)}$ $(f=1, \ldots, k)$ are then uniquely determined.

The values of the $x_{i,h}(\tau)$ and $w_{i,h}(\tau)$ are given by (51) and (52); equations (53) are, however, replaced by

$$c_{i,h}^{(1)}=-\frac{2}{3q_i}E_{i,h}^{(0)}, \qquad c_{i,h}^{(3)}=c_{i,h}^{(4)}=0 \qquad (i=1, \ldots, k), \tag{56}$$

the $c_{i,h}^{(2)}$ remaining undetermined at this step. The induction is thus established. In this case, too, the successive $x_{i,n}$ and $w_{i,n}$ have the form (51′), the $c_{i,n}^{(2)}$ appearing as $A_{i,n}^{(q_i)}$, and are obtained without integration. But, besides computing the $D_{i,n}^{(m)}$ and $E_{i,n}^{(m)}$ and applying (52), it is necessary to obtain also the $D_{i,n+1}^{(q_i)}$ and $E_{i,n+1}^{(q_i)}$ and to determine the $c_{i,n}^{(2)}$ by solving the equations $E_{i,n+1}^{(q_i)}-2D_{i,n+1}^{(q_i)}=0$.

Remarks: (1) In constructing the solutions it has been tacitly assumed that the Fourier series representation of the $x_{i,n}$ and $w_{i,n}$ and the attendant manipulations of these series are valid. This is justified by the consideration that the well-known functions encountered in the first step are representable, together with their derivatives, by uniformly convergent Fourier series. But the products of two such series is another of the same type, and also the integral of a uniformly convergent Fourier series is uniformly convergent. At every step the convergence remains uniform.

(2) The question naturally arises as to whether, in any orbits of Type I, the smallest period is a multiple of the period of the infinitesimal system, namely, mT. An examination of the method of constructing the solutions furnishes the answer. If mq_k is not an integer, the constants are determined precisely as in Case I; if mq_k is an integer, then by the method used in Case II. In any event, for a given value of μ and a given set of p_i and q_k, there is a *unique* solution satisfying (41) and (42) (with $2m\pi$ replacing 2π). The solutions found above, however, satisfy these conditions; hence there are no orbits of this type whose smallest period is a multiple of the period of the infinitesimal system.

217. Concerning Orbits of Type II.—For Type II, as for Type I, all the λ_i are multiples of π; but it is proposed to ascertain whether the ω_i and τ_i exist as functions of μ (not identically zero, but vanishing with μ) so as to satisfy the periodicity conditions (13). The question can be studied most easily by examining the method of constructing the solutions.

Let the origin of time be selected as the instant when satellite k has an apsidal passage, and the origin of longitude at that satellite's apsidal position so that $\omega_k=\tau_k=0$; and let it be assumed for the moment that the ω_i and τ_i $(i=1, \ldots, k-1)$ exist as functions of μ satisfying (13); then there exist solutions of the type (40) satisfying (42), but not satisfying (41), for all values of n except for $i=k$. Because of the absence of conditions (41) some of the constants in each step are left undetermined until terms of higher orders are found.

For the first-order terms the differential equations are again (44), and the solutions are (45), the coefficients being determined by (46). Evidently the $c_{i,1}^{(1)}$ are determined as in Type I, but for the other $c_{i,1}^{(j)}$ two cases arise.

Case I. q_k is not an integer.—Here, by (42), $c_{i,1}^{(2)}=c_{i,1}^{(3)}=0$ $(i=1, \ldots, k)$, but the $c_{i,1}^{(4)}$ are at present undetermined, except $c_{k,1}^{(4)}=0$.

Case II. q_k is an integer.—Here the terms $\cos q_i\tau$ and $\sin q_i\tau$ have the required period 2π, and hence the $c_{i,1}^{(2)}$ and $c_{i,1}^{(3)}$, together with the $c_{i,1}^{(4)}$, remain undetermined, except $c_{k,1}^{(4)}=c_{k,1}^{(3)}=0$.

Terms of any order. Case I.—Consider next the terms of order two. The differential equations are still (48), using $h=2$; however, because of the $c_{j,1}^{(4)}$ these equations now reduce not to (50), but to the form

$$\left.\begin{aligned}
&(a) && w_{i,2}''+2q_i x_{i,2}'+H_{i,2}^{(0)}+\sum_{m=1}^{\infty}(D_{i,2}^{(m)}\sin m\tau+H_{i,2}^{(m)}\cos m\tau)=0,\\
&(b) && x_{i,2}''-2q_i w_{i,2}'-3q_i^2 x_{i,2}+E_{i,2}^{(0)}+\sum_{m=1}^{\infty}(E_{i,2}^{(m)}\cos m\tau+J_{i,2}^{(m)}\sin m\tau)=0,
\end{aligned}\right\}\quad(57)$$

where the $H_{i,2}^{(m)}$ and $J_{i,2}^{(m)}$ vanish with the $c_{j,1}^{(4)}$. The first integration apparently introduces non-periodic terms $H_{i,2}^{(0)}\tau$; but computation shows that these coefficients vanish. Thus

$$H_{i,2}^{(0)}=\sum_j{}'(c_{j,1}^{(4)}-c_{i,1}^{(4)})\delta_{ij}\eta_{ij}^3\left[\tfrac{3}{4}a_{ij}\eta_{ij}^2(2G_0-G_2)-\tfrac{1}{2}F_1\right],$$

and by Le Verrier's relations among the $F_n(\epsilon_{ij})$, and $G_n(\epsilon_{ij})$,* it is found that the bracketed expression vanishes identically in the ϵ_{ij}. Hence the solutions of (57) have the form

$$\left.\begin{aligned}
x_{i,2}&=\frac{1}{3q_i^2}E_{i,2}^{(0)}+\sum_{m=1}^{\infty}(A_{i,2}^{(m)}\cos m\tau+P_{i,2}^{(m)}\sin m\tau),\\
w_{i,2}&=c_{i,2}^{(4)}\qquad+\sum_{m=1}^{\infty}(B_{i,2}^{(m)}\sin m\tau+Q_{i,2}^{(m)}\cos m\tau),
\end{aligned}\right\}\quad(58)$$

where the $P_{i,2}^{(m)}$ and $Q_{i,2}^{(m)}$ vanish with the $c_{j,1}^{(4)}$, which (together with the $c_{j,2}^{(4)}$) remain at present undetermined, the other constants arising in the $x_{i,2}$ and

*Tisserand: *Mécanique Céleste*, vol. 1, pp. 276, 278.

the $w_{i,2}$ having been determined as were the corresponding ones in the first-order terms.

For the terms of order three, the differential equations are of the form (57), and the $H^{(0)}_{i,3}$ do not vanish identically. For, while the $c^{(4)}_{j,2}$ drop out just as the $c^{(4)}_{j,1}$ did in the preceding step, the $c^{(4)}_{j,1}$ are now present (linearly only), entering both through the $P^{(m)}_{j,2}$ and $Q^{(m)}_{j,2}$ and directly from the $w_{j,1}(\tau)$. The destruction of the non-periodic terms $H^{(0)}_{i,3}\,\tau$, by setting the $H^{(0)}_{i,3}=0$, will be found to require the vanishing of these $c^{(4)}_{j,1}$ which have been previously undetermined.

The $H^{(0)}_{i,3}$ consist of two sorts of terms: those which arise identically by combining trigonometric functions of a single ϕ_{ji} and its multiples, and those which may appear because of relations among the various ϕ_{ji}. Terms of the latter sort can not be collected, unless the numerical values of the p_i have been chosen; but their possible presence will be shown to be immaterial so far as the conclusions are concerned.

In equations (48a), using $h=3$, the terms which reduce identically to constants may be selected by fixing upon some one ϕ_{ji} and expressing the coefficients of $\cos m\phi_{ji}$ in $x_{i,1}$ and $w_{i,2}$, and of $\sin m\varphi_{ji}$ in $w_{i,1}$ and $x_{i,2}$. Incidentally it may be noted that the series in each $x_{i,n}$ and $w_{i,n}$, as given by (47) or (58), is in reality a rearrangement of several Fourier series in the various ϕ_{ji}. The resulting complicated constant, involving several series in the $F_n(\epsilon_{ij})$ and $G_n(\epsilon_{ij})$, can be treated advantageously by expressing all the a_i, ϵ_{ij}, q_i, and δ_{ij} in terms of a single parameter α, just as in (35) and (36). In the $H^{(0)}_{i,3}$ the coefficient of each $c^{(4)}_{j,1}$, say M_{ij}, becomes then a power series in α; and it is found that $M_{ii}=-\sum\limits_j{}' M_{ij}$, and

$$M_{ij}=\alpha^{4i+2j-6k}\,.\,N_{ij}\qquad (j>i),\qquad\qquad M_{ij}=\alpha^{6j-6k}\,.\,N_{ij}\qquad (j<i),\qquad (59)$$

where the N_{ij} are infinite series in their respective ϵ_{ij}, beginning with constant terms.

Now the vanishing of the $H^{(0)}_{i,3}$ requires that the $c^{(4)}_{j,1}$ satisfy k linear equations, which must be homogeneous, since the vanishing of all of the unknowns would reduce the $H^{(0)}_{i,3}$ to their values in Type I, viz., zero. That is,

$$\sum_{j=1}^{k} M_{ij}c^{(4)}_{j,1}=0\qquad\qquad (i=1,\ .\ .\ .\ ,k).\qquad (60)$$

But, from the value of M_{ii} above, it is evident that the determinant of the M_{ij} is identically zero, so that any equation is a consequence of the others and may be suppressed. Let the first equation be the one which is dropped. Then the remaining $k-1$ equations $(i=2,\ .\ .\ .\ ,k)$ will determine the $c^{(4)}_{j,1}$ $(j=1,\ .\ .\ .\ ,k-1)$ uniquely in terms of $c^{(4)}_{k,1}$ if their determinant Δ_5, of order $k-1$ in the M_{ij}, is distinct from zero. From (59) the lowest powers of

α in the various elements of any row of Δ_5 may be ascertained. Evidently the exponent $4i+2j-6k$ takes its smallest value $4i+2-6k$ when $j=1$, and its largest value $6i-2-6k$ when $j=i-1$; while $6j-6k$ takes its smallest value $6i+6-6k$ when $j=i+1$. As this last is greater than the highest value of the former exponent, it is clear that the lowest power of α in any of the $M_{ij}(j=1, \ldots, i-1, i+1, \ldots, k-1)$ is $4i+2-6k$, which occurs in M_{i1} and also in M_{ii}. Thus, in the r^{th} row of Δ_5 (where $i=r+1$), the lowest power is $4r+6-6k$; and this appears for $r=1, \ldots, k-2$ in two columns, the first and $(r+1)^{th}$. But, in the last row, where $r=k-1$, the lowest power can appear only in the first column. Hence if the factor $\alpha^{4r+6-6k}$ is removed from the elements of the r^{th} row $(r=1, \ldots, k-1)$, a new determinant Δ_6 is obtained in whose first column the series of each element begins with a constant term, as does also the series of one other element in each row except the last row.

Now, if Δ_6 is developed by the minors of its last row, it is clear that a constant term can not be lacking when the development is rearranged as a single power series in α. For the only element of the last row which can contribute to the constant term is that in the first column; and in the minor of this element constant terms are present in all the elements of the main diagonal, and nowhere else.

Thus $\Delta_6 \neq 0$ at $\alpha=0$; and hence Δ_6 and likewise Δ_5 are distinct from zero for all values of α sufficiently small. Therefore the vanishing of the $H^{(0)}_{i,3}$ determines the $c^{(4)}_{j,1}$ $(j=1, \ldots, k-1)$ uniquely and homogeneously in terms of $c^{(4)}_{k,1}$. But this latter constant is to be put equal to zero by reason of the choice of the origin of time, as noted above in discussing the first-order terms. Hence every $c^{(4)}_{j,1}=0$. Thus the $w_{i,1}(\tau)$ reduce to the values which they would have for orbits of Type I. Then also every $P^{(m)}_{i,2}=Q^{(m)}_{i,2}=0$; and the $x_{i,2}(\tau)$ and $w_{i,2}(\tau)$ reduce to the values in Type I, except for the $c^{(4)}_{j,2}$, which remain as yet undetermined for $j=1, \ldots, k-1$.

In the next step these $c^{(4)}_{j,2}$ will enter the $H^{(0)}_{i,4}$ (by identity) in precisely the same way as the $c^{(4)}_{j,1}$ entered the $H^{(0)}_{i,3}$, and must likewise vanish. Similarly, the $c^{(4)}_{j,n}$ in the terms of order n will remain undetermined until the $H^{(0)}_{i,n+2}$ are set equal to zero, when they must vanish together with the $P^{(m)}_{i,n+1}$ and $Q^{(m)}_{i,n+1}$, to which they will in the meantime have given rise. Thus the final determination of the constants arising at any step reduces the terms of that order to the values which they would have for Type I.

If various relations among the p_j give rise to other terms in the $H^{(0)}_{i,3}$ than those which are present identically in some one ϕ_{ji}, —even if these terms introduce into various elements lower powers of α than have been treated in Δ_5, — these new terms can not affect the argument in general; for, if they introduce into the development lower powers of α than were previously present, with non-vanishing coefficients, then this new development is equally as useful as the former value of Δ_6; while, if they do not furnish

terms of lower order, neither can they in general destroy the terms treated in Δ_6, since they must involve new β's in their coefficients. Any cancellation could occur, then, only for a few special relations among the masses.

The conclusion is not yet warranted that no orbits of Type II exist in Case I; but there can be none when a is below some "sufficiently small" finite value, unless possibly for a few very special relations among the masses.

Terms of any order. Case II.—The differential equations for the second-order terms $x_{i,2}(\tau)$, $w_{i,2}(\tau)$ are again (48); but because of the $c_{j,1}^{(3)}$ and $c_{j,1}^{(4)}$ the equations reduce to (57), as in Case I above, where, however, the $H_{i,2}^{(m)}$ and $J_{i,2}^{(m)}$ now involve them (linearly) and vanish with these *two* sets of constants.

In order that $x_{i,2}$ and $w_{i,2}$ shall be periodic, it is necessary, just as for Type I, that $E_{i,2}^{(q_i)}-2D_{i,2}^{(q_i)}$, the coefficient of $\cos q_i\tau$ in the final differential equation for $x_{i,2}(\tau)$, shall vanish. It is equally necessary that $J_{i,2}^{(q_i)}+2H_{i,2}^{(q_i)}$, the coefficient of $\sin q_i\tau$ in the same equation, shall vanish. Further, the "secular terms" $H_{i,2}^{(0)}\tau$ must vanish. The $H_{i,2}^{(0)}$ are free from the $c_{j,1}^{(4)}$, just as in Case I above; but they now contain the $c_{j,1}^{(2)}$ and $c_{j,1}^{(3)}$, vanishing with the latter set.

Now, from (54) and the fact that no relation (22) holds, it is evident that the $E_{i,2}^{(q_i)}-2D_{i,2}^{(q_i)}$ do not involve the $c_{j,1}^{(3)}$ nor the $c_{j,1}^{(4)}$; also the $J_{i,2}^{(q_i)}+2H_{i,2}^{(q_i)}$ involve neither the $c_{j,1}^{(2)}$ nor the $c_{j,1}^{(4)}$, and vanish with the $c_{j,1}^{(3)}$. Further, since the $x_{i,1}(\tau)$ and $w_{i,1}(\tau)$ may be written

$$\left.\begin{aligned} x_{i,1}(\tau) &= c_{i,1}^{(2)}\sin\left(q_i\tau+\frac{\pi}{2}\right)+c_{i,1}^{(3)}\sin q_i\tau+\sum_{m=0}^{\infty}A_{i,1}^{(m)}\cos m\tau, \\ w_{i,1}(\tau) &= c_{i,1}^{(4)}+2c_{i,1}^{(2)}\cos\left(q_i\tau+\frac{\pi}{2}\right)+2c_{i,1}^{(3)}\cos q_i\tau+\sum_{m=1}^{\infty}B_{i,1}^{(m)}\sin m\tau, \end{aligned}\right\} \tag{61}$$

it is found that the $c_{j,1}^{(3)}$ enter the $J_{i,2}^{(q_i)}+2H_{i,2}^{(q_i)}$, which are the coefficients of $\sin q_i\tau$, in precisely the same way as the $c_{j,1}^{(2)}$ enter the $E_{i,2}^{(q_i)}-2D_{i,2}^{(q_i)}$, which are the coefficients of $\sin(q_i\tau+\pi/2)$.

In the treatment of Case II for Type I it was shown that, when $\Delta_4\neq 0$, the equations $E_{i,2}^{(q_i)}-2D_{i,2}^{(q_i)}=0$ admit a unique solution for the $c_{j,1}^{(2)}$ ($j=1, \ldots, k$). The same conclusion is evidently valid here; and it also follows at once that the equations

$$J_{i,2}^{(q_i)}+2H_{i,2}^{(q_i)}=0 \qquad (i=1, \ldots, k) \tag{62}$$

admit a unique solution for the $c_{j,1}^{(3)}$ ($j=1, \ldots, k$), namely, $c_{j,1}^{(3)}=0$. In (62), $c_{k,1}^{(3)}$ does not appear, being already zero; but this does not affect the conclusion, since there are obviously no more solutions for $k-1$ of the unknowns, after the last one has been determined, than there are for all k. The $H_{i,2}^{(0)}$ now vanish identically.

The $c_{j,1}^{(1)}$, $c_{j,1}^{(2)}$, $c_{j,1}^{(3)}$ are now determined as for Type I; but the $c_{j,1}^{(4)}$ remain undetermined until the next step, and the solutions are at present given by

(58), as in Case I of Type II, where, however, the $c_{j,2}^{(2)}$ and $c_{j,2}^{(3)}$, as well as the $c_{j,2}^{(4)}$, remain undetermined.

In getting the third-order terms certain constants will be determined: the $c_{j,2}^{(2)}$ and $c_{j,2}^{(3)}$ by the vanishing of the $E_{i,3}^{(qi)}-2D_{i,3}^{(qi)}$ and $J_{i,3}^{(qi)}+2H_{i,3}^{(qi)}$ respectively; and the $c_{j,1}^{(4)}$ by the vanishing of the $H_{i,3}^{(0)}$, which reduce to their values in Case I when the $c_{j,2}^{(3)}=0$.

Likewise, the $c_{j,n}^{(2)}$, $c_{j,n}^{(3)}$ of any order are determined in the next order, $(n+1)$, and the $c_{j,n}^{(4)}$ in the order $(n+2)$. Each set is obtained from linear equations, whose determinant remains the same for each successive order. Thus, with the same possible exceptions as in Case I, it is impossible in Case II to determine the constants otherwise than as for Type I.

Remarks: (1) If either of the determinants Δ_3 or Δ_5 vanishes for some value of α, there may still be no new values for the constants of integration which would satisfy later conditions in subsequent differential equations and render the solutions periodic in form. Whether the series converge for this value of α would be unknown; so that the mere vanishing of Δ_3 or Δ_5 would not warrant the conclusion that orbits of Type II exist.

(2) The foregoing conclusions extend beyond a denial of the possibility of obtaining equations of periodic orbits by a certain method of analysis, or solving the differential equations in a certain way: *the non-existence of a class of physical orbits* of a certain type is asserted, though possibly there exist numerous individual orbits of the type. All orbits, periodic or not, arising from any set of Δc_{ij} and μ [see (17) and (7)], can be represented by power series in these parameters converging through the interval $0 \leqq \tau \leqq 2\pi$, provided that the parameters are sufficiently small. And from the existence of a class of periodic orbits of the type sought would follow the existence of a range of values of the Δc_{ij} and μ (including zero) satisfying the periodicity conditions. These equations obviously could not be satisfied for a range of values of μ by arbitrary values of the Δc_{ij}, and therefore they would define the Δc_{ij} as functions of μ, holomorphic for μ sufficiently small and vanishing with μ. The substitution of these values of the Δc_{ij} into the original developments of the coördinates would render the latter series in μ alone, having properties (40*a*, *b*, *d*). If these series are impossible save for Type I, then there does not exist a class of physical orbits of Type II growing out of the circles named.

218. Concerning Orbits of Type III.—It may be inquired whether there exists a class of periodic orbits growing out of circular orbits of an infinitesimal system which has no "grand conjunctions"; that is, whether there are periodic solutions of (7) when some of the λ_i have other values than are possible in Type I.

Let the initial conditions be (17), let some instant when M_k is at an apse be selected as the origin of time, and let the origin of longitude be the

apsidal position of M_k, both for $\mu=0$ and for $\mu\neq 0$.* Then $\lambda_k=0$, and $w\equiv\tau\equiv 0$ identically in k and μ; and the conditions for periodicity are (13). But equations (7) admit two integrals, an examination of which shows that two equations of (13), namely $x_k(2\pi)=x_k(0)$ and $x_k'(2\pi)=x_k'(0)$, are a consequence of the other $4k-2$ equations. Hence equations (13) become

$$
\left.\begin{aligned}
&(a) && 0=e_i(1-\cos 2q_i\pi)+\mu X_{i,0}+e_i\tau_i(-\sin 2q_i\pi)\\
& && \quad +\mu\sum_{f=1}^{f=k}\left\{\Delta n_f X_{i,f}^{(1)}+eX_{f\,i,f}^{(2)}+\omega_f X_{i,f}^{(3)}+\tau_f X_{i,f}^{(4)}\right\}+\cdots,\\
&(b) && 0=e_i(q_i\sin 2q_i\pi)+\mu X'_{i,0}+e_i\tau_i(q_i^2\,\overline{1-\cos 2q_i\pi})\\
& && \quad +\mu\sum_{f=1}^{f=k}\left\{\Delta n_f X_{i,f}^{\prime(1)}+e_f X_{i,f}^{\prime(2)}+\omega_f X_{i,f}^{\prime(3)}+\tau_f X_{i,f}^{\prime(4)}\right\}+\cdots,\\
&(c) && 0=\Delta n_i(2q_i\pi)+e_i(2\sin 2q_i\pi)+\mu W_{i,0}+e_i\tau_i(2q_i\,\overline{1-\cos 2q_i\pi})\\
& && \quad +\mu\sum_{f=1}^{f=k}\left\{\Delta n_f W_{i,f}^{(1)}+e_f W_{i,f}^{(2)}+\omega_f W_{i,f}^{(3)}+\tau_f W_{i,f}^{(4)}\right\}+\cdots,\\
&(d) && 0=e_i(-2q_i\,\overline{1-\cos 2q_i\pi})+\mu W'_{i,0}+e_i\tau_i(2q_i\sin 2q_i\pi)\\
& && \quad +\mu\sum_{f=1}^{f=k}\left\{\Delta n_f W_{i,f}^{\prime(1)}+e_f W_{i,f}^{\prime(2)}+\omega_f W_{i,f}^{(3)}+\tau_f W_{i,f}^{\prime(4)}\right\}+\cdots\\
& && \qquad\qquad\qquad\qquad (i=1,\ldots,k-1),
\end{aligned}\right\}\qquad (63)
$$

where

$$
X_{i,0}=X_i(0;2\pi)-X_i(0;0),\ \ldots,\qquad W'_{i,0}=W'_i(0;\,2\pi)-W'_i(0;0),
$$

$$
X_{i,f}^{(2)}=X_i(f;2\pi)-X_i(f;0),\ \ldots,\qquad W_{i,f}^{\prime(2)}=W'_i(f;\,2\pi)-W'_i(f;0),
$$

and the $X_{i,f}^{(j)}$ $(j=1,\,3,\,4)$, etc., are constants whose values will not be needed here.

*In treating orbits of Type I, the instant of a symmetrical conjunction was regarded as the beginning of a period and was taken as the origin of time; but this was merely for simplicity, since in periodic motion any other instant could be so regarded. By taking as $\tau=0$ the instant when M_k of the infinitesimal system is at arbitrarily selected longitude, and choosing that longitude as a new origin of longitude (so that $\lambda_k=0$), it is clear that the longitudes which the other infinitesimal satellites have at that instant constitute a set of λ's, not all of which are multiples of π. Each family of orbits of Type I may thus be said to arise from any one of an infinitude of sets of the λ_i other than multiples of π, though all the sets have $\lambda_k=0$. By reason of the w_i (τ), which are in general distinct from zero, the absolute longitude v_i of any finite satellite at the new $\tau=0$ would vary with μ; (but, since families of orbits of Type I exist for every position of the line of conjunction, one may obtain a family in which the initial longitude of M_k is zero, identically as to μ, by selecting orbits for different μ's from different families, taking the conjunction line as needed for the μ used.) For such an origin of time M_k would not in general be at an apse for $\tau=0$.

Hence if the origins of time and longitude be chosen at an apsidal position of M_k for all values of μ, the sets of λ_i other than multiples of π, which can give rise to orbits of Type I, are largely excluded. (Whether *any* such sets remain, depends upon whether, in Type I, M_k has any apses other than at the symmetrical conjunctions of the system; and this has not been ascertained.)

While the determinant of the linear terms of the Δn_i, e_i, ω_i, and τ_i in (63) is zero, yet, when q_k is not an integer, solutions of (63) (c) and (d) exist for the Δn_i and e_i as power series in μ and the ω_i and τ_i. Properties of the series (63), similar to (19), are easily established, which show that the solutions for the Δn_i and e_i, and also the series obtained by substitution of these solutions into (63) (a) and (b), carry μ as a factor. After this substitution and a division by μ, (63) (a) and (b) become

$$\left.\begin{aligned}
&(a) \qquad 0=X_{i,0}+\frac{1}{2q_i}W'_{i,0}+\sum_{j=1}^{k-1}\Big\{\omega_j\Big[X^{(3)}_{i,j}+\frac{1}{2q_i}W'^{(3)}_{i,j}\Big] \\
&\qquad\qquad +\tau_j\Big[X^{(4)}_{i,j}+\frac{1}{2q_i}W'_{i,j}\Big]\Big\}+\cdots, \\
&(b) \qquad 0=X'_{i,0}+\frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)}W'_{i,0}+\sum_{j=1}^{k-1}\Big\{\omega_j\Big[X'^{(3)}_{i,j} \\
&\qquad\qquad +\frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)}W'^{(3)}_{i,j}\Big]+\tau_j\Big[X'^{(4)}_{i,j}+\frac{\sin 2q_i\pi}{2(1-\cos 2q_i\pi)}W'^{(4)}_{i,j}\Big]\Big\} \\
&\qquad\qquad +\tau_i\frac{q_iW'_{i,0}}{1-\cos 2q_i\pi}+\cdots.
\end{aligned}\right\} \quad (64)$$

The constant term in ($64a$) vanishes, since for q_k not an integer no relation (22) holds; the constant term in ($64b$) reduces to $-2q_iC_{i,0}$. If any of the $C_{i,0}$ are distinct from zero, (64) are not satisfied by $\omega_i=\tau_i=\mu=0$; hence ω_i and τ_i do not exist as holomorphic functions of μ vanishing with μ, or periodic orbits of the type sought do not exist. The necessary condition for periodicity, namely, that the $C_{i,0}$ vanish, is, by (28),

$$\sum_j{}'\delta_{ij}\sum_{m=1}^{\infty}\Theta_m(\epsilon_{ij})\sin m(\lambda_j-\lambda_i)=0 \qquad (i=1,\ .\ .\ .\ ,\ k). \qquad (65)$$

Of these k-equations in the quantities $\lambda_j(j=1,\ .\ .\ .\ .\ ,\ k-1)$, one is evidently a consequence of the others; for, before λ_k is put equal to zero, the Jacobian of the $C_{i,0}$ with respect to the $\lambda_j(i=1,\ .\ .\ .\ k;\ j=1,\ .\ .\ .\ .\ ,\ k)$ is identically zero. Let the equation for $i=1$ be suppressed.

If particular values be assigned to all, save one, of the λ_j, the last λ can still be given an infinitude of values for which any one $C_{i,0}$ is distinct from zero; hence equations (65) impose very special conditions upon the λ_i. Whether there are any sets of λ's, other than those of Type I which satisfy (65), is unknown.* It will, however, be shown that there are no others "in the vicinity of" multiples of π, provided a is sufficiently small.

*The limitations upon any such sets seem fully as severe here as in Type I.

If the $C_{i,0}$ are developed as power series in the $\lambda_f - J_f\pi$ $(J_f = 0, 1)$, the coefficients of the linear terms are simply $[\partial C_{i,0}/\partial\lambda_f]$ for $\lambda_j = J_j\pi$. Evidently, from (28),

$$\left.\begin{aligned}\left[\frac{\partial C_{i,0}}{\partial\lambda_f}\right]_{\lambda_j=J_j\pi} &= \delta_{if}\sum_{m=1}^{\infty} m\Theta_m(\epsilon_{if})\cos m(J_f - J_i)\pi \qquad (f \neq i),\\ \left[\frac{\partial C_{i,0}}{\partial\lambda_f}\right]_{\lambda_j=J_j\pi} &= \delta_{if} = -\sum_j{}'\delta_{ij}\sum_{m=1}^{\infty} m\Theta_m(\epsilon_{ij})\cos m(J_j - J_i)\pi \quad (f \neq i).\end{aligned}\right\} \quad (66)$$

Denoting by S_{if} the coefficient of λ_f in the i^{th} equation of (65), and introducing α by (35), the S_{if} are obtained as power series in α. It is found that the lowest exponent of α present in S_{if} is $(f-i)$ if $f<i$ and is $5(f-i)$ if $f>i$. Hence, of all the S_{if} $(f<i)$, S_{i1} carries the lowest power of α, namely, the $(1-i)^{\text{th}}$; and of all the $S_{if} (f>i)$, $S_{i,i+1}$ carries the lowest, namely, the fifth. Consequently, if $\alpha^{(1-i)}$ be removed as a factor from all the S_{if} $(f=1, \ldots, k-1;\ i=2, \ldots, k)$, a determinant Δ_7 is obtained (equal to the determinant of the coefficients S_{if} multiplied by a power of α), in whose r^{th} row all elements save those of the first and $(r+1)^{th}$ columns begin with a power of α. Therefore Δ_7 is of precisely the same type as Δ_6, and the discussion of the latter shows also that Δ_7 (and hence the determinant of the S_{if}) is distinct from zero for all values of α sufficiently small. Therefore quantities Λ_f $(f=1, \ldots, k-1)$ exist such that equations (65) are not satisfied for any values of the λ_f for which $|\lambda_f - J_f\pi| < \Lambda_f$ except $\lambda_f = 0$ or π.

Moreover, the existence of a set of λ_f satisfying (65) would not prove the existence of ω_i and τ_i as functions of μ satisfying (64). If it is assumed for the moment that orbits of Type III exist, and the method of construction is examined, equations related to (65) are encountered. Thus, in finding the second-order terms, the $H_{i,2}^{(0)}$ must be made to vanish, and each of these involves Fourier series in the various $\lambda_f - \lambda_i$. When q_k is an integer, somewhat different difficulties arise in the existence proof; the same difficulty is, however, encountered in the construction.

A general negative conclusion is not yet warranted; but it is evident that if any orbits of Type III exist, they must satisfy very special conditions. In every case, however, periodic orbits of the type sought do not exist if (22) holds.

219. Concerning Lacunary Spaces.—The relation (22) may be expressed in the form $J(p_f - p_g) = p_g + q_k$; hence this relation can hold only if q_k is an integer. But the converse holds true only when $k=2$. For example, in the following selections q_k is an integer, but no relation (22) holds:

$$q_k = 2, \qquad p_{k-1} = 3, \qquad p_{k-2} = 5, \qquad \ldots, \qquad p_1 = 2k-1,$$

$$q_k = 2, \qquad p_{k-1} = 3, \qquad p_{k-2} = 7, \qquad \ldots, \qquad p_1 = 4k-1, \text{ etc.}$$

When $k=2$, (22) always holds* if q_k is an integer; for then $p_1=1$ (otherwise T would not be the smallest synodic period of the infinitesimal system), and hence $Jp_1=q_2$ is satisfied by giving J the integral value q_2.

Since $n_k T=2q_k\pi$, the case q_k an integer is the one where the consecutive conjunctions of the infinitesimal system occur at the same absolute longitude; and, denoting the synodic period of the two infinitesimal satellites M_f and M_g by T_{fg}, since

$$n_g T_{fg}=n_g \frac{2\pi}{n_f-n_g},$$

it follows that $n_g T_{fg}=2\pi J$ when (22) holds; or the consecutive conjunctions of the infinitesimal pair M_f and M_g occur all in the same absolute longitude.

Moreover, since all the w_i vanish at the beginning and end of each period, all the "grand conjunctions" of the finite system occur at the same longitudes as those of the infinitesimal system, and intermediate conjunctions of any finite pair occur at very nearly the same longitudes as those of the corresponding infinitesimal pair.

These facts suggest a physical reason for the non-existence of periodic orbits under certain circumstances. The greater part of the mutual disturbances of two bodies occur while they are near conjunction; and, if the consecutive grand conjunctions occur at exactly the same longitude, the perturbations of the elements would tend to be cumulative. Nevertheless, if there are more than two bodies, the mutual disturbances may so balance each other as to yield periodic orbits, especially if the bodies are far apart (*i. e.*, a sufficiently small) unless (22) holds. But if two bodies have conjunctions between the grand conjunctions, all occurring very near the same longitude, the other bodies can not counterbalance the large perturbations of the two.

More exactly, there exists a range of values of the masses and the e_i, including zero, for which periodic orbits are impossible; so that, unless the orbits for $\mu=0$ are eccentric rather than circular, there are for small values of μ no periodic plane orbits of k satellites when (22) holds.

So far as this result extends, it would indicate that no asteroids having nearly circular orbits would be found, whose periods compared to Jupiter's are in the ratios $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, etc. Those whose periods are nearly in any such ratio should be found subject to very great perturbations.

It is well known that lacunary spaces of the sort just mentioned do occur among the asteroids. That there are such spaces also when the ratio of the periods is $\frac{3}{5}$, and other such values, is not surprising, as in any case the slightest deviation from the correct initial values destroys periodicity, there being Poisson terms in the solutions.

*In Poincaré's discussion of the problem of three bodies, therefore, the case where q_k is an integer without such a relation as (22) holding does not arise.

220. Jupiter's Satellites I, II, and III.—Of Jupiter's longer known satellites, the innermost three move almost exactly in a plane, apparently in periodic orbits having symmetrical conjunctions; and their masses with respect to that of the planet are very small. Since for orbits of Type I the increase in the longitude of M_i during a period is independent of μ, being equal to $n_i T$, the average angular velocity of each finite satellite for a period may be taken as the corresponding n.

The unit of time being the sidereal day, and the unit of mass being the mass of Jupiter, the observational data are:*

$$\left.\begin{array}{lll} \lambda_1=\pi, & \lambda_2=0, & \lambda_3=0, \\ M_1=0.000017, & M_2=0.000023, & M_3=0.000088, \\ n_1=3.551552261, & n_2=1.769322711, & n_3=.878207937. \end{array}\right\} \quad (67)$$

Since $(n_1-n_3)/3=.891114775$, and $n_2-n_3=.891114774$, the n_i of (67) satisfy, far beyond observational accuracy, the equations (2), where $\nu=.891114774$. Then the period T is 7.0509271 days, and

$$\left.\begin{array}{lll} p_1=3, & p_2=1, & q_3=.985516077, \\ \phi_{12}=\pi+2\tau, & \phi_{13}=\pi+3\tau, & \phi_{23}=\tau, \end{array}\right\} \quad (68)$$

so that satellite III advances 354°.785788 during each period, while satellites I and II advance 1080° and 360° respectively more than this. If μ is taken arbitrarily as .0001, then

$$\beta_1=.17, \qquad \beta_2=.23, \qquad \beta_3=.88.$$

It seems desirable, however, to retain the β's in the computations, inasmuch as a new determination of the masses may render it necessary to use other values than those given above.

The $D_{i,1}^{(m)}$ and $E_{i,1}^{(m)}$ of (46) are obtained by writing equations (43) in the form

$$\left.\begin{array}{r} w_{i,1}''+2q_i x_{i,1}'+\sum_j{}' \dfrac{\delta_{ij}\eta_{ij}^3}{2}\sum_{m=1}^{\infty} U_{ij}^{(m)}\sin m\phi_{ji}=0, \\ x_{i,1}''-2q_i w_{i,1}'-3q_i^2 x_{i,1}+q_i^2\beta_i+\sum_j{}' \dfrac{\delta_{ij}\eta_{ij}^3}{2}\sum_{m=0}^{\infty} V_{ij}^{(m)}\cos m\phi_{ji}=0, \end{array}\right\} \quad (69)$$

where, rearranging according to multiples of τ,

$$\begin{array}{ll} & V_{ij}^{(0)}=2\alpha_{ij}F_0-F_1, \\ U_{ij}^{(1)}=\dfrac{2}{\eta_{ij}^3}+F_2-2F_3, & V_{ij}^{(1)}=\dfrac{2}{\eta_{ij}^3}+2\alpha_{ij}F_1-(F_2+2F_3), \\ U_{ij}^{(m)}=F_{m-1}-F_{m+1}, & V_{ij}^{(m)}=2\alpha_{ij}F_m-(F_{m-1}+F_{m+1}) \qquad (m>1). \end{array}$$

*Tisserand, *Traité de Mécanique Céleste*, vol. 4, p. 2. The n_i given there (203°.48895528, 101°.37472396, and 50°.31760833) are here reduced to circular measure.

Evidently the $U_{1,2}^{(m)}$, $U_{1,3}^{(m)}$, and $U_{1,3}^{(m)}$ enter respectively the $D_{1,2}^{(2m)}$, $D_{1,3}^{(3m)}$, and $D_{2,3}^{(m)}$, etc. From (23), (5), and the relation $(a_i/a_j)^3=(n_j/n_i)^2$, the ϵ_{ij} and δ_{ij} are found to be

$$\left.\begin{array}{lll}
\epsilon_{12}=\epsilon_{21}=.6284333, & \epsilon_{13}=\epsilon_{31}=.3939606, & \epsilon_{23}=\epsilon_{32}=.6268932, \\
\dfrac{\delta_{12}\eta_{12}^3}{2}=3.1366\,\beta_2, & \dfrac{\delta_{21}\eta_{21}^3}{2}=1.23875\,\beta_1, & \dfrac{\delta_{31}\eta_{31}^3}{2}=.19132\,\beta_1, \\
\dfrac{\delta_{13}\eta_{13}^3}{2}=1.2326\,\beta_3, & \dfrac{\delta_{23}\eta_{23}^3}{2}=.77464\,\beta_3, & \dfrac{\delta_{32}\eta_{32}^3}{2}=.30443\,\beta_2.
\end{array}\right\} \quad (70)$$

The $F_m(\epsilon_{ij})$, and also the $G_m(\epsilon_{ij})$, etc., encountered in the higher orders are readily computed by using the tables of coefficients given by LeVerrier.*

In obtaining the successive $A_{i,n}^{(m)}$ and $B_{i,n}^{(m)}$ from (52), the smallest divisors introduced are $16-q_1^2$, $4-q_2^2$, and $1-q_3^2$, or .1156616, .05772591, and .02875806 respectively. These divisors decrease materially the effectiveness of the small value of μ; nevertheless the terms above those of the second order seem relatively unimportant and will not be computed. The coefficients of μ in the $x_i(\tau)$ and $w_i(\tau)$ are found to be:†

$$\begin{aligned}
x_{1,1}(\tau)=&(.3\beta_1-.1\beta_2)-.8\beta_2\cos 2\tau-.1\beta_3\cos 3\tau-203.7\beta_2\cos 4\tau \\
&+(.7\beta_2-.2\beta_3)\cos 6\tau-.2\beta_2\cos 8\tau+.1\beta_2\cos 10\tau+\cdots,
\end{aligned}$$

$$\begin{aligned}
w_{1,1}(\tau)=&+.3\beta_2\sin 2\tau+.4\beta_3\sin 3\tau+406.3\beta_2\sin 4\tau \\
&-(1.1\beta_2-.3\beta_3)\sin 6\tau+.3\beta_2\sin 8\tau-3\beta_3\sin 9\tau-.1\beta_2\sin 10\tau+\cdots,
\end{aligned}$$

$$\begin{aligned}
x_{2,1}(\tau)=&(.5\beta_1+.3\beta_2-.1\beta_3)+.8\beta_3\cos\tau-(58.4\beta_1+100.1\beta_3)\cos 2\tau-.7\beta_3\cos 3\tau \\
&+(.7\beta_1-.2\beta_3)\cos 4\tau-.1\beta_3\cos 5\tau-.2\beta_1\cos 6\tau+.1\beta_1\cos 8\tau+\cdots,
\end{aligned}$$

$$\begin{aligned}
w_{2,1}(\tau)=&-2.9\beta_3\sin\tau+(114.3\beta_1+199.2\beta_3)\sin 2\tau+1.1\beta_3\sin 3\tau \\
&-(.8\beta_1-.2\beta_3)\sin 4\tau+.1\beta_3\sin 5\tau+.2\beta_1\sin 6\tau-.1\beta_1\sin 8\tau\cdots,
\end{aligned}$$

$$\begin{aligned}
x_{3,1}(\tau)=&(.4\beta_1+.5\beta_2+.3\beta_3)+28.9\beta_2\cos\tau+.6\beta_2\cos 2\tau-(.4\beta_1-.2\beta_2)\cos 3\tau \\
&+.1\beta_2\cos 4\tau\cdots,
\end{aligned}$$

$$w_{3,1}(\tau)=-55.2\beta_2\sin\tau-.8\beta_2\sin 2\tau-(.4\beta_1+.2\beta_2)\sin 3\tau-.1\beta_2\sin 4\tau\cdots.$$

For the second-order terms the $D_{i,2}^{(m)}$ and $E_{i,2}^{(m)}$ are found from (54), where it is convenient first to rearrange the coefficients of $\sigma_{ij,0}^{-3}$ and $\sigma_{ij,0}^{-5}$ according to multiples of τ. The $x_{i,2}$ and $w_{i,2}$ involve all multiples of τ; but, as they

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†The writer regrets to state that the values formerly published (*Transactions of the American Mathematical Society*, vol. 9, pp. 29–33) are practically all erroneous, the factor q_i of the last terms of (46) having been overlooked in making the calculation.

carry the factor μ^2, all save the following terms fall below the limit of accuracy in the $x_{\iota,1}$ and $w_{\iota,1}$ above. To facilitate comparison, the second-order terms are shown multiplied by $\mu=.0001$. They are

$$.0001\,x_{1,2}(\tau)=-.6\beta_2^2+(.1\beta_1\beta_2+.1\beta_2^2+.1\beta_2\beta_3)\cos 2\tau \\ -(2.9\beta_1\beta_2-12.2\beta_2^2-9.2\beta_2\beta_3)\cos 4\tau-2.0\beta_2^2\cos 8\tau+\cdots,$$

$$.0001\,w_{1,2}(\tau)=.3\beta_2\beta_3\sin\tau-(.3\beta_1\beta_2+.2\beta_2^2+.3\beta_2\beta_3)\sin 2\tau+(5.8\beta_1\beta_2-24.4\beta_2^2 \\ -18.3\beta_2\beta_3)\sin 4\tau+.1\beta_2\beta_3\sin 6\tau+5.1\beta_2^2\sin 8\tau\cdots,$$

$$.0001\,x_{2,2}(\tau)=(.2\beta_1^2+.6\beta_1\beta_3+.5\beta_3^2)-(.1\beta_1\beta_3+.1\beta_3^2)\cos\tau \\ +(6.3\beta_1^2+12.6\beta_1\beta_2+12.6\beta_1\beta_3+3.1\beta_3^2)\cos 2\tau \\ -(.2\beta_1^2+.1\beta_1\beta_2+6\beta_1\beta_3+.5\beta_3^2)\cos 4\tau+\cdots,$$

$$.0001\,w_{2,2}(\tau)=(.2\beta_1\beta_3+.2\beta_2\beta_3+.4\beta_3^2)\sin\tau-(12.5\beta_1^2+25.0\beta_1\beta_2+25.1\beta_1\beta_3 \\ +1.3\beta_2\beta_3+6.2\beta_3^2)\sin 2\tau+(.1\beta_3^2\sin 3\tau+.4\beta_1^2+.1\beta_1\beta_2+1.4\beta_1\beta_3 \\ +1.3\beta_3^2)\sin 4\tau+\cdots,$$

$$.0001\,x_{3,2}(\tau)=-(2.3\beta_1\beta_2+2.5\beta_2^2+3.9\beta_2\beta_3)\cos\tau-.1\beta_2^2\cos 2\tau \\ -.1\beta_1\beta_2\cos 7\tau+\cdots,$$

$$.0001\,w_{3,2}(\tau)=(4.7\beta_1\beta_2+4.9\beta_2^2+7.6\beta_2\beta_3)\sin\tau+.2\beta_2^2\sin 2\tau+\cdots.$$

Since $r_\iota=a_\iota(1+x_{\iota,1}\mu+x_{\iota,2}\mu^2+\cdots)$ and $v_\iota=\lambda_\iota+q_\iota\tau+w_{\iota,1}\mu+w_{\iota,2}\mu^2+\cdots$, the radius vector and absolute longitude of each satellite are obtained by computing the a_ι from (3) and using the coefficients above; the deviations from their values in the undisturbed circular orbits are given to five significant figures, so far as the terms of the first two orders are concerned. How much these would be affected by terms of higher orders is unknown; in fact no proof has been given that the series converge for $\mu=.0001$, although they have been proved convergent for all μ sufficiently small.

To show the general shape of these orbits, the values of the v_ι and r_ι/a_ι will be given to four decimals, using the values of the β_ι tabulated above:

$$r_1/a_1=1-.0044\cos 4\tau,\qquad v_1=\pi+3.9855\tau+.0089\sin 4\tau,$$

$$r_2/a_2=1-.0093\cos 2\tau,\qquad v_2=1.9855\tau-.0003\sin\tau+.0185\sin 2\tau \\ +.0001\sin 3\tau+.0001\sin 4\tau,$$

$$r_3/a_3=1+.0006\cos\tau,\qquad v_3=.9855\tau-.0011\sin\tau.$$

Hence if these orbits are thought of as ellipses rotating in the plane, the major semi-axes would be the respective a_ι, the several eccentricities would be .0044, .0093, .0006, and the axes would rotate forward at rates whose average values are the n_ι. The three satellites are in line with Jupiter at the beginning and middle of each period, II and III being on the same side of the planet at $\tau=0$, and I and III on the same side at $\tau=\pi$. Whenever

II is in conjunction with I or III, the inner of the pair is near a perijove and the outer is near an apojove, which decreases the amount of their mutual perturbations.

No radius vector or longitude differs very widely at any time from its value in a circular orbit (a_i and $\lambda_i+q_i\tau$, respectively). The largest departures are for satellite II, as r_2/a_2 reaches a minimum of .9907 at $\tau=0$ and a maximum of 1.0093 at about $\tau=\pi/2$ and every half-period thereafter, v_2 meanwhile ranging from 64′ more to 64′ less than the mean longitude of II. Similarly, satellites I and III get 30′ and 4′ respectively ahead of and behind their mean positions, and the r_i/a_i at such instants closely approximate their mean value, unity. For satellite I the maxima of r/a occur at intervals of a quarter-period, and for satellite III they occur at intervals of a period.

Finally, it may be noted that, for this system of bodies, the increments Δc_{ij} (see § 209) which have been given to the initial values of the coördinates and their time-derivatives to preserve periodicity when the bodies are finite are approximately

r_i	r_i'	v_i	v_i'
$-.0044\,a_1$	0	0	.0310 (radians per day)
$-.0093\,a_2$	0	0	.0329 (radians per day)
$+.0006\,a_3$	0	0	$-.0010$ (radians per day)

221. Orbits About an Oblate Central Body.*—If the central body is an oblate spheroid and the satellites are spheres moving in its equatorial plane, periodic orbits of Type I still exist, the successive grand conjunctions falling at the same or a different longitude, according as q_k is or is not an integer.

The differential equations for this case are obtained from (4) by simply multiplying each $1/r_j^3$ $(j=1,\ldots,k)$ by $f(r_j)$, where

$$f(r_j)\equiv 1+\tfrac{3}{10}\left(\frac{a\epsilon}{r_j}\right)^2+\tfrac{9}{56}\left(\frac{a\epsilon}{r_j}\right)^4+\cdots+\frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}\,\frac{3}{2n+3}\left(\frac{a\epsilon}{r_j}\right)^{2n}+\cdots,$$

a being the equatorial radius and ϵ the eccentricity of the spheroid.†

Let $a^2\epsilon^2/a_i^2=\gamma_i\mu$; then equations (7) are replaced by

$$\left.\begin{aligned}
&(a)\quad x_iw_i''+2x_i'(w_i'+q_i)+\mu\sum_j{}'\delta_{ij}x_j\sin(\phi_{ji}+w_j-w_i)\\
&\qquad\qquad\times\left(\frac{1}{x_j^3}\left[1+\tfrac{3}{10}\gamma_j\frac{\mu}{x_j^2}+\cdots\right]-\frac{1}{\sigma_{ij}^3}\right)=0,\\
&(b)\quad x_i''-x_i(w_i'+q_i)^2+q_i^2(1+\beta_i\mu)\left[\frac{1}{x_i^2}+\frac{3}{10}\frac{\gamma_i\mu}{x_i^4}+\cdots\right]+\mu\sum_j{}'\delta_{ij}\left\{\frac{a_{ij}x_i}{\sigma_{ij}^3}\right.\\
&\qquad\qquad\left.+x_j\cos(\phi_{ji}+w_j-w_i)\left(\frac{1}{x_j^3}\left[1+\frac{3}{10}\frac{\gamma_j\mu}{x_j^2}+\cdots\right]-\frac{1}{\sigma_{ij}^3}\right)\right\}=0.
\end{aligned}\right\}\quad(71)$$

*This section takes a first step in a direction suggested by Professor K. Laves, particularly with reference to Jupiter's satellites.

†Moulton, *Celestial Mechanics*, p. 122, where $a^2+\nu=r_j^2$, and $2\pi\sigma k^2\sqrt{1-\epsilon^2}=3k^2M/2a^3$.

This substitution requires the flattening to vanish with the masses $M_1, \ldots, M_k$, so that the central body becomes spherical if the others become infinitesimal; but the amount of flattening corresponding to any given set of finite masses remains quite arbitrary, even if the p_i and q_k are specified; for the values of the a_i merely determine the ratios of the γ_i, and one γ may be taken at pleasure. In the solutions of (71) satisfying initial conditions (17), the terms independent of μ are the same as formerly. Hence in Case I where q_k is not integral, the Δn_i and e_i still exist as convergent power series in μ satisfying (29 *a*, *b*), though of course their values in terms of μ are now different because of the γ_i. Thus periodic orbits exist.

In case q_k is an integer the γ_i enter the $x_i(0;\ \tau)$ and $x_i(i;\ \tau)$, but do not appear in the P_{ij} or P_{ij} of Δ_3. Thus the argument in Case II is likewise unaltered, and periodic orbits of Type I exist under the same conditions as when the bodies are all spherical.

The numerical results for Jupiter's satellites given above are affected but slightly in the first and second orders by including the flattening of Jupiter. The corrections to be added are in fact

$$
\begin{array}{lll}
\text{to } x_{1,1}(\tau) & \text{add} & .1\gamma_1 \cos 4\tau \\
\text{to } .0001\, x_{1,2} & \text{add} & 1.8\,\beta_2\gamma_1 \cos 4\tau \\
\text{to } .0001\, w_{1,2} & \text{add} & -3.6\,\beta_2\gamma_1 \sin 4\tau \\
\text{to } x_{2,1}(\tau) & \text{add} & .1\gamma_2 \cos 2\tau \\
\text{to } .0001\, x_{2,2} & \text{add} & (.2\beta_1 + .4\beta_3)\gamma_2 \cos 2\tau \\
\text{to } .0001\, w_{2,2} & \text{add} & -(.5\beta_1 + .8\beta_3)\gamma_2 \sin 2\tau \\
\text{to } x_{3,1}(\tau) & \text{add} & .1\gamma_3 \cos \tau \\
\text{to } .0001 x_{3,2} & \text{add} & -\ .1\beta_2\gamma_3 \cos \tau \\
\text{to } .0001\, w_{3,2} & \text{add} & .1\beta_2\gamma_3 \sin \tau
\end{array}
$$

And since $\gamma_1 = .36$, $\gamma_2 = 14.3$, and $\gamma_3 = 5.7$, the values of the r_i/a_i and v_i given above are changed as follows:

$$
\begin{array}{ll}
\text{to } r_1/a_1 \text{ add } .0018 \cos 4\tau, & \text{to } r_2/a_2 \text{ add } .0007 \cos 2\tau, \\
\text{to } v_1 \text{ add} -.0030 \sin 4\tau, & \text{to } v_2 \text{ add } -.0011 \sin 2\tau;
\end{array}
$$

then

$$
\begin{array}{ll}
r_1/a_1 = 1 - .0026 \cos 4\tau, & v_1 = \pi + 3.99855\tau + .0059 \sin 4\tau, \\
r_2/a_2 = 1 - .0086 \cos 2\tau, & v_2 = 1.9855\tau - .0003 \sin\tau + .0174 \sin 2\tau \\
 & \qquad + .0001 \sin 3\tau + .0001 \sin 4\tau, \\
r_3/a_3 = 1 + .0006 \cos \tau, & v_3 = .9855\tau - .0011 \sin \tau.
\end{array}
$$

CHAPTER XV.

CLOSED ORBITS OF EJECTION AND RELATED PERIODIC ORBITS.

222. Introduction.—In the problem of two bodies there is in no sense continuity between circular orbits revolving in the forward and retrograde directions, except where their dimensions shrink to zero or become infinitely great. But in the restricted problem of three bodies the deviations from the circular forms of the orbits are such that there is geometrical continuity in some classes between those which revolve in the forward direction and those which are retrograde; and the limit between the two types is an orbit passing through one of the finite bodies. If the infinitesimal body leaves one of the finite bodies, its orbit is called an *orbit of ejection*; and if it strikes a finite mass, it is called *an orbit of collision.*

In certain cases orbits of ejection are also orbits of collision, or closed orbits of ejection. When the direction of collision is exactly opposite to that of ejection, they are the limits of two classes of periodic orbits, in one of which the motion is direct and in the other of which it is retrograde. The closed orbits of ejection are not themselves periodic orbits, even if the physical impossibility be disregarded and the problem considered purely from the mathematical point of view; for, if the expressions for the coördinates are followed, in the sense of analytic continuity, beyond the values of t for which a collision occurs they become complex, and never become real again for increasing real values of t. Those orbits in which the ejection and collision are not in opposite directions are not the limits of periodic orbits, or at least of orbits which re-enter after a single revolution.

The object of the investigations of this chapter is to determine the limiting types of certain classes of periodic orbits, and thus partially to prepare the way for the discussion of the evolution of the various classes of periodic orbits with varying values of the parameters on which they depend, and to show the relations among these various classes. The existence of the closed orbits of ejection will be established, some of their properties will be derived, and it will be proved that each one in which the direction of ejection and collision is opposite is the limit of two series of periodic orbits.

223. Ejectional Orbits in the Two-Body Problem.—As preliminary to the general problem, the special case in which there is only one finite mass will first be treated. Let the mass of the finite body be $1-\mu$ and let the units be so chosen that the gravitational constant is unity. Then the motion of

the infinitesimal body projected along the fixed ξ-axis satisfies the differential equation

$$\frac{d^2\xi}{dt^2} = \mp\frac{1-\mu}{\xi^2}, \tag{1}$$

where the sign is $-$ or $+$ according as the motion is in the positive or negative direction from the origin.

Suppose $\zeta = \zeta_0$ and $d\xi/dt = \xi' = \xi_0'$ at $\tau = \tau_0$. Then the first integral of equation (1) becomes

$$\left(\frac{d\xi}{dt}\right)^2 = \xi'^2 = \pm\frac{2(1-\mu)}{\xi} \mp \frac{2(1-\mu)}{\xi_0} + \xi_0'^2 = \pm\frac{2(1-\mu)}{\xi} + c_1. \tag{2}$$

If c_1 is negative, $|\xi|$ has a finite maximum for which ξ' vanishes; if c_1 is zero, ξ' approaches zero as $|\xi|$ becomes infinitely large; if c_1 is positive, ξ' is finite for $|\xi|$ infinite. It will be assumed that c_1 is negative in order to get orbits of ejection which are closed. Then, without loss of generality, it can be supposed that ξ_0 is the greatest value of ξ for projection in the positive direction, or the least for projection in the negative direction. Then

$$\xi_0' = 0, \qquad c_1 = \mp\frac{2(1-\mu)}{\xi_0}. \tag{3}$$

With the initial values (3), the integral (2) becomes

$$\sqrt{\xi_0\xi - \xi^2} - \frac{\xi_0}{2}\sin^{-1}\left(-1 + \frac{2\xi}{\xi_0}\right) = \sqrt{\frac{2(1-\mu)}{\pm\xi_0}}\left(t - t_0\right) - \left(\frac{1}{4} + j\right)\pi\xi_0, \tag{4}$$

where j is an integer.

Now consider ξ as a function of $(t-t_0)$. Since the right members of (1) and (2) are regular for all values of t and all values of ξ except $\xi = 0$, it follows that ξ is a regular function of t for all values of t except those for which ξ vanishes. These values of t are easily determined from (4), and are found to be

$$t_j - t_0 = \left(\frac{1}{2} + j\right)\pi\,\xi_0\sqrt{\frac{\pm\xi_0}{2(1-\mu)}}, \tag{5}$$

where j takes all integral values.

The character of ξ as a function of $(t-t_0)$ in the vicinity of $t = t_j$ is easily determined from (4). The left member is expansible as a power series in $\sqrt{\pm\xi} = \eta$, and the equation can be written in the form

$$F(\eta) = \sqrt{\xi_0\xi - \xi^2} - \frac{\xi_0}{2}\sin^{-1}\left(-1 + \frac{2\xi}{\xi_0}\right) - \left(\frac{1}{4} + j\right)\pi\,\xi_0 = \sqrt{\frac{2(1-\mu)}{\pm\xi_0}}\,(t - t_0).$$

It is found that

$$F(0) = j\pi\xi_0, \qquad \frac{\partial F(0)}{\partial\eta} = 0, \qquad \frac{\partial^2 F(0)}{\partial\eta^2} = 0, \qquad \frac{\partial^3 F(0)}{\partial\eta^3} = -4(\pm\xi_0)^{-1/2}.$$

Therefore η is expansible as a power series in $(t-t_j)^{1/3}$, starting with a term of the first degree in $(t-t_j)^{1/3}$. Since $\pm\xi = \eta^2$, it follows that ξ is expansible as a power series in $(t-t_j)^{1/3}$, starting with a term of the second degree in $(t-t_j)^{1/3}$. It is easily seen from (4) that $F(\eta)$ is an odd function of η. Therefore η is an odd series in $(t-t_j)^{1/3}$, and ξ is an even series in $(t-t_j)^{1/3}$. Since the only singularities are given by (5), the radius of the circle of convergence for the series for both η and ξ is $\pi|\xi_0|\sqrt{\dfrac{\pm\xi_0}{2(1-\mu)}}$.

The form of the solution in the vicinity of $t=t_j$ being known, the coefficients of the series can easily be found from (1) by the method of undetermined coefficients. It is convenient in the computation to let

$$\tau=(t-t_j)^{1/3}, \tag{6}$$

after which (1) becomes

$$\tau\frac{d^2\xi}{d\tau^2}-2\frac{d\xi}{d\tau}=\mp\frac{9(1-\mu)}{\xi^2}\tau^5. \tag{7}$$

The solution of this equation with initial value of ξ equal to zero has the form

$$\pm\xi=a_2\tau^2+a_4\tau^4+\cdots+a_{2n}\tau^{2n}+\cdots\cdots \tag{8}$$

By direct substitution and comparison of coefficients, it is found that

$$\left.\begin{aligned}&\pm\xi=c\tau^2\left[1+\alpha\tau^2-\frac{3}{7}\alpha^2\tau^4+\frac{23}{63}\alpha^3\tau^6-\frac{1894}{4851}\alpha^4\tau^8+\frac{3293}{7007}\alpha^5\tau^{10}\cdots\right],\\ &c=\left[\frac{9}{2}(1-\mu)\right]^{1/3}, \qquad\qquad \alpha=\text{arbitrary constant.}\end{aligned}\right\} \tag{9}$$

More convenient formulas for use can be developed by eliminating the term in ξ^{-2} from (7). After the transformation (6) the integral (2) becomes

$$\left(\frac{d\xi}{d\tau}\right)^2=\pm18(1-\mu)\left(\frac{1}{\xi}-\frac{1}{\xi_0}\right)\tau^4.$$

On using this equation to eliminate ξ^{-2} from (7), the result is found to be

$$\tau\xi\frac{d^2\xi}{d\tau^2}-2\xi\frac{d\xi}{d\tau}+\frac{\tau}{2}\left(\frac{d\xi}{d\tau}\right)^2=\mp9(1-\mu)\tau^5. \tag{10}$$

Now it follows from equations (8) and (9) that

$$\pm\xi=c\tau^2\left[1+\alpha\tau^2+\sum_{j=2}^{\infty}a_{2j}\tau^{2j}\right],$$

$$\pm\frac{d\xi}{d\tau}=2c\tau\left[1+2\alpha\tau^2+\sum_{j=2}^{\infty}(j+1)a_{2j}\tau^{2j}\right],$$

$$\pm\frac{d^2\xi}{d\tau^2}=2c\left[1+6\alpha\tau^2+\sum_{j=2}^{\infty}(j+1)(2j+1)a_{2j}\tau^{2j}\right],$$

$$\begin{aligned}\tau\xi\frac{d^2\xi}{d\tau^2}=2c^2\tau^3\Bigg[&1+7\alpha\tau^2+6\alpha^2\tau^4+\sum_{j=2}^{\infty}(2j^2+3j+2)a_{2j}\tau^{2j}+\alpha\sum_{j=3}^{\infty}(2j^2-j+6)a_{2j-2}\tau^{2j}\\ &+\sum_{j=4}^{\infty}\left\{\sum_{k=2}^{j-2}(j-k+1)(2j-2k+1)a_{2k}a_{2j-2k}\right\}\tau^{2j}\Bigg],\end{aligned}$$

$$\begin{aligned}-2\xi\frac{d\xi}{d\tau}=-4c^2\tau^3\Bigg[&1+3\alpha\tau^2+2\alpha^2\tau^4+\sum_{j=2}^{\infty}(j+2)a_{2j}\tau^{2j}\\ &+\alpha\sum_{j=3}^{\infty}(j+2)a_{2j-2}\tau^{2j}+\sum_{j=4}^{\infty}\left\{\sum_{k=2}^{j-2}(j-k+1)a_{2k}a_{2j-2k}\right\}\tau^{2j}\Bigg],\end{aligned}$$

$$\begin{aligned}\frac{\tau}{2}\left(\frac{d\xi}{d\tau}\right)^2=2c^2\tau^3\Bigg[&1+4\alpha\tau^2+4\alpha^2\tau^4+2\sum_{j=2}^{\infty}(j+1)a_{2j}\tau^{2j}\\ &+4\alpha\sum_{j=3}^{\infty}ja_{2j-2}\tau^{2j}+\sum_{j=4}^{\infty}\left\{\sum_{k=2}^{j-2}(k+1)(j-k+1)a_{2k}a_{2j-2k}\right\}\tau^{2j}\Bigg].\end{aligned}$$

Hence, on equating to zero the coefficient of τ^{2j} after these series have been substituted in (10), it is found that

$$j(2j+3)a_{2j}=-\alpha(2j^2+j+2)a_{2j-2}-\sum_{k=2}^{j-2}(j-k+1)(2j-k)a_{2k}a_{2j-2k}, \tag{11}$$

which gives the coefficients very simply for all values of j greater than unity.

The result of applying (11) and reducing the coefficients to the decimal form is

$$\begin{aligned}\pm\xi=c\tau^2[1+\alpha\tau^2-0.42857\alpha^2\tau^4+0.36508\alpha^3\tau^6-0.39044\alpha^4\tau^8\\+0.46996\alpha^5\tau^{10}-0.60863\alpha^6\tau^{12}+0.82861\alpha^7\tau^{14}-1.16832\alpha^8\tau^{16}+\cdots].\end{aligned}$$

So far as this solution is written the signs of the terms alternate and a_{2j} has α^j as a factor. This is a general property which will be needed in establishing the existence of the closed orbits of ejection in the problem of three bodies (§230). It follows at once from (11) that the part of the coefficient a_{2j} which comes from the first term on the right is opposite in sign to the coefficient of the preceding term. Since the sum of the subscripts of the product terms in the right member of (11) is $2j$, their product has the same sign as the coefficient of a_{2j-2}. Therefore, a_{2j} and a_{2j-2} are opposite in sign for all j. It also follows from (11) that a_{2j} contains α as a factor to one degree higher than it appears in a_{2j-2}.

The expression for ξ has a branch-point at $t=t_j$, where three branches unite. If $t-t_j=\rho e^{\sqrt{-1}(\varphi+2n\pi)}$, the three distinct branches are

$$\xi_1=\rho e^{\sqrt{-1}\frac{2\varphi}{3}},\qquad \xi_2=\rho e^{\sqrt{-1}\frac{2}{3}(\varphi+2\pi)},\qquad \xi_3=\rho e^{\sqrt{-1}\frac{2}{3}(\varphi+4\pi)}. \tag{12}$$

If $t-t_j$ is real and negative, $\varphi=\pi$ and the second of these expressions alone is real. If $t-t_j$ is real and positive, $\varphi=0$ and the first of them alone is real. Consequently the analytic continuation of the real branch of the expression for ξ through $t=t_j$ leads to a complex value, and this value remains complex as t, remaining real, increases to $+\infty$. Therefore the motion is not strictly periodic.

In case the orbit is not one of ejection the branch-points of the functions which express the coördinates in terms of the time are not on the real axis. If the major axis is kept fixed, and if the eccentricity is varied from zero to unity, an examination of the equations from which the character of the functions can be determined shows that the singularities start from both positive and negative infinity on the lines $t_j-t_0=j\pi\xi_0\sqrt{\frac{\pm\xi_0}{2(1-\mu)}}$ and approach the real axis as a limit as e approaches unity. At these singular points two branches of the function permute except when, for $e=1$, two of them have united on the real axis, and then three branches permute. If e increases beyond unity, each of the branch-points divides into two which move equally in opposite directions along the real axis, and for $e=\infty$ one of each pair unites with another of an adjacent pair.

224. The Integral.—Equation (2) holds for all values of t, and therefore when the series (9) is substituted in it the result is an identity in τ. The conditions that the coefficients of the various powers of τ shall be identical in the left and right members furnish severe tests of the accuracy of the computation of (9). The integral also gives the relation between the arbitrary a and the greatest distance ξ_0. By direct substitution, it is found that

$$a = \frac{-c}{5\xi_0}. \tag{13}$$

The interval from ejection to collision is found from equations (5), (9), and (13) to be

$$P = \frac{\pi(+\xi_0)^{3/2}}{\sqrt{2(1-\mu)}} = \frac{\pi\left(\frac{-c}{5a}\right)^{3/2}}{\sqrt{2(1-\mu)}}. \tag{14}$$

For $a=0$, which corresponds to a parabolic orbit, P is infinite.

225. Orbits of Ejection in Rotating Axes.—In the problem of three bodies the motion of the infinitesimal body will be referred to rotating axes. In the demonstration of the existence of closed orbits of ejection in the problem of three bodies it will be necessary to use some of the properties of the orbits of ejection in that of two bodies. For this reason the orbits now under consideration will be referred to axes rotating uniformly with the period 2π.

Suppose the ejection takes place at $t=t_1$ and along the x-axis, where the rectangular coördinates are denoted by x and y. Then x and y are given by the equations

$$x = +\xi\cos(t-t_1), \qquad y = -\xi\sin(t-t_1). \tag{15}$$

Those orbits which re-enter in the direction opposite to that of ejection are of greatest interest in the present connection. The condition that they shall have this property is that their period in t from ejection to collision shall be a multiple of 2π. This condition becomes, by virtue of (14),

$$(\pm\xi_0)^{3/2} = 2j\sqrt{2(1-\mu)}, \tag{16}$$

where j is a positive integer.

Figs. 15, 16, and 17 show the curves for j equal to 1, 2, and 3, at least as to general form, in full lines for ejection along the x-axis in the positive direction, and in dotted lines for ejection in the negative direction. These curves for the three values of j are not drawn to the same scale, for it follows from (16) that their linear dimensions are proportional to $j^{2/3}$. One of the important properties of all these curves is that they intersect the x-axis perpendicularly at their mid-points.

226. Ejectional Orbits in the Problem of Three Bodies.—The differential equations of motion for the infinitesimal body when the finite masses describe circular orbits are, in canonical units,

$$\left.\begin{aligned} &\frac{d^2x}{dt^2}-2\frac{dy}{dt}=\frac{\partial U}{\partial x}, \qquad \frac{d^2y}{dt^2}+2\frac{dy}{dt}=\frac{\partial U}{\partial y}, \\ &U=\tfrac{1}{2}(x^2+y^2)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}, \quad r_1^2=(x+\mu)^2+y^2, \quad r_2^2=(x-1+\mu)^2+y^2. \end{aligned}\right\} \qquad (17)$$

These equations have the integral

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=2U-C. \qquad (18)$$

When μ is zero the problem reduces to that of two bodies, which was treated in §223. The singularity in the solution, whether μ is zero or not, comes from the fact that r_1 tends toward zero as a limit as t tends toward t_1. It is intuitionally clear that the mass μ will have only a slight influence on the motion of the infinitesimal body while r_1 is small, and it seems probable, therefore, that the nature of the singularity at $t=t_1$ is the same whether μ is distinct from zero or not. This is, indeed, the case, as was first proved by Levi-Civita in a very important memoir.*

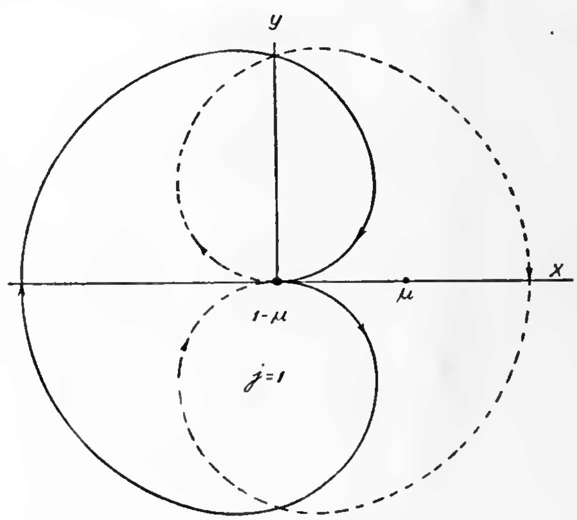


FIG. 15.

It follows from (18) that $x'^2+y'^2$ tends toward infinity as r_1 tends toward zero, but that the limit of $r_1\,[x'^2+y'^2]$, for r_1 equal to zero, is the finite quantity $2(1-\mu)$. If μ is zero, x and y are developable as power series in $(t-t_1)^{1/3}$, and this suggests defining an independent variable σ in terms of which x, y, and $t-t_1$ are expressible by series of the form

$$\left.\begin{aligned} &x+\mu=\alpha_2\sigma^2+\alpha_3\sigma^3+\cdots, \qquad y=\beta_2\sigma^2+\beta_3\sigma^3+\cdots, \\ &t-t_1=0+\gamma_3\sigma^3+\cdots, \qquad (\gamma_3\neq 0) \end{aligned}\right\} \qquad (19)$$

Since the solutions of analytic differential equations in the vicinity of points for which they are regular are themselves regular, while in the vicinity of singular points the solutions are regular or not, depending on supplementary circumstances, it is advantageous to choose such dependent variables that the equations shall be regular for $r_1=0$, $\sigma=0$. This Levi-Civita has done, preserving the canonical form, with rare skill and elegance. His dependent variables, which may be denoted here by p, q, u, and v, are related to the rectangular coördinates and their first derivatives by

*Sur la résolution qualitative du problème restreint des trois corps, *Acta Mathematica*, vol. 30 (1906), pp. 305–327.

$$\left.\begin{aligned} &x+\mu+\sqrt{-1}\,y=(p+\sqrt{-1}\,q)^2, \\ &x'+y-\sqrt{-1}(x+\mu+y')=\frac{u-\sqrt{-1}\,v}{2(p+\sqrt{-1}\,q)}, \qquad dt=(p^2+q^2)\,d\sigma=\rho^2 d\sigma. \end{aligned}\right\} \quad (20)$$

In these variables equations (17) become

$$\left.\begin{aligned} &\frac{dp}{d\sigma}=+\frac{\partial H}{\partial u}, \qquad \frac{dq}{d\sigma}=+\frac{\partial H}{\partial v}, \qquad \frac{du}{d\sigma}=-\frac{\partial H}{\partial p}, \qquad \frac{dv}{d\sigma}=-\frac{\partial H}{\partial q}, \\ &H=\frac{1}{8}\left\{(u+2\rho^2 q)^2+(v-2\rho^2 p)^2\right\}-\left\{1-\mu-C\rho^2+\frac{1}{2}\rho^6+\mu\rho^2 V\right\}, \\ &V=\frac{1}{r_2}-p^2+q^2-\mu, \qquad r_2=1-2(p^2-q^2)+(p^2+q^2)^2. \end{aligned}\right\} \quad (21)$$

It follows from (20) that $p=q=0$ if $x+\mu=y=0$, and that u and v are finite for $r_1=0$. Therefore the form of H shows that the differential equations (21) are regular in the vicinity of $p=q=0$. Consequently, p and q are developable

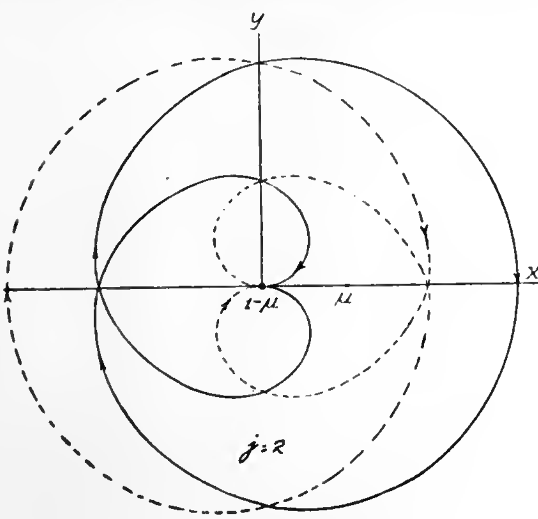


FIG. 16.

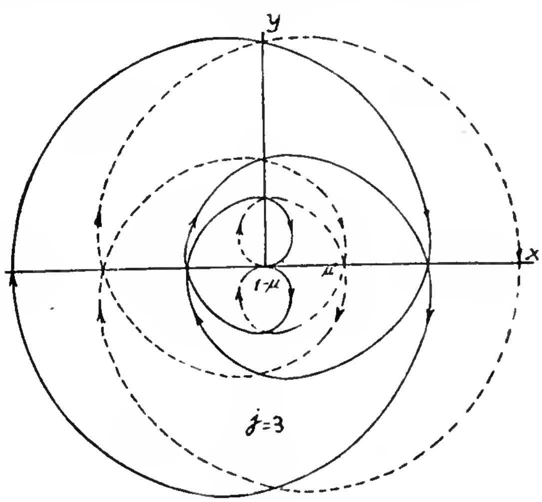


FIG. 17.

as power series in σ, vanishing with σ. It follows from (20) that x, y, and $t-t_1$, considered as functions of σ, have the form (19); and therefore x and y expressed in terms of τ, defined in (6), have the form

$$x+\mu=c[\tau^2+a_3\tau^3+\cdots]. \qquad y=c[b_2\tau^2+b_3\tau^3+\cdots]. \quad (22)$$

227. Construction of the Solutions of Ejection.—The character of the solutions in the vicinity of $(t-t_1)^{1/3}=\tau=0$ having been found, they can be obtained without difficulty from (17). Upon using τ as the independent variable and expanding the expression for r_2, these equations become

$$\left.\begin{aligned} &\tau\frac{d^2x}{d\tau^2}-2\frac{dx}{d\tau}-6\tau^3\frac{dy}{d\tau}=9\tau^5(x+\mu)-\frac{9(1-\mu)\tau^5(x+\mu)}{[(x+\mu)^2+y^2]^{3/2}} \\ &\qquad\qquad +9\mu\tau^5\left[2(x+\mu)+3(x+\mu)^2-\frac{3}{2}y^2+\cdots\right], \\ &\tau\frac{d^2y}{d\tau^2}-2\frac{dy}{d\tau}+6\tau^3\frac{dx}{d\tau}=9\tau^5 y-\frac{9(1-\mu)\tau^5 y}{[(x+\mu)^2+y^2]^{3/2}}-9\mu\tau^5 y\left[1+3(x+\mu)+\cdots\right]. \end{aligned}\right\} \quad (23)$$

It will now be supposed that the line of ejection is along the x-axis. Therefore the initial conditions are

$$x(0)+\mu=y(0)=\frac{1}{3\tau^2}\frac{dy}{d\tau}=0, \qquad \left[\frac{x+\mu-c\tau^2}{\tau^4}\right]_{\tau=0}=ca=\text{arb. const.} \tag{24}$$

With the initial conditions (24) the solutions have an important property of symmetry. Let them be written in the form

$$x+\mu=\tau^2 f(\tau), \qquad y=\tau^4 g(\tau), \qquad \frac{dx}{d\tau}=\tau\varphi(\tau), \qquad \frac{dy}{d\tau}=\tau^3\psi(\tau). \tag{25}$$

Now make the transformation of variables

$$x+\mu=x_1, \qquad y=-y_1, \qquad \tau=-\tau_1, \qquad \frac{dx}{d\tau}=-\frac{dx_1}{d\tau_1}, \qquad \frac{dy}{d\tau}=+\frac{dy_1}{d\tau_1}.$$

Equations (23) are not changed in form by this substitution. Therefore the solution of the transformed equations with the initial conditions

$$x_1(0)=y_1(0)=\frac{1}{3\tau_1^2}\frac{dy_1}{d\tau_1}=0, \qquad \left[\frac{x_1-c\tau_1^2}{\tau_1^4}\right]_{\tau_1=0}=a$$

are

$$x_1=\tau_1^2 f(\tau_1), \qquad y_1=\tau_1^4 g(\tau_1), \qquad \frac{dx_1}{d\tau_1}=\tau_1\varphi(\tau_1), \qquad \frac{dy_1}{d\tau_1}=\tau_1^3\psi(\tau_1),$$

where f, g, φ, and ψ are identical with the functions represented by the same symbols in (25). Therefore

$$\left.\begin{aligned} \tau^2 f(\tau)&=+\tau_1^2 f(\tau_1)=+\tau^2 f(-\tau), \qquad & \tau\varphi(\tau)&=-\tau_1\varphi(\tau_1)=+\tau\varphi(-\tau),\\ \tau^4 g(\tau)&=-\tau_1^4 g(\tau_1)=-\tau^4 g(-\tau), \qquad & \tau^3\psi(\tau)&=+\tau_1^3\psi(\tau_1)=-\tau^3\psi(-\tau).\end{aligned}\right\} \tag{26}$$

It follows that x and $dy/d\tau$ are even functions of τ and that $dx/d\tau$ and y are odd functions of τ. The first equation of (22) contains only even powers of τ, and the second contains only odd powers, starting with a term of the fifth degree as the lowest.

With the initial conditions (24), the solution of (23) is found to be

$$\left.\begin{aligned} \pm(x+\mu)&=c\left\{\tau^2+a\tau^4-\tfrac{3}{7}a^2\tau^6+\left[-\tfrac{1}{2}+\tfrac{23}{63}a^3+\tfrac{1}{2}\mu\right]\tau^8+\cdots\right\},\\ \pm y&=-c\left\{\tau^5-a\tau^7-\tfrac{3}{7}a^2\tau^9+\left[\tfrac{23}{63}a^3-\tfrac{3}{14}\mu\right]\tau^{11}+\cdots\right\},\\ c&=\left[\frac{9(1-\mu)}{2}\right]^{1/3}, \qquad a=\text{arbitrary constant,}\end{aligned}\right\} \tag{27}$$

where the positive or negative signs are to be used in the left members according as the initial projection is in the positive or negative direction.

The constant of the integral (18) is given by the equation

$$\left.\begin{aligned} C=&-\frac{1}{9\tau^4}\left[\left(\frac{dx}{d\tau}\right)^2+\left(\frac{dy}{d\tau}\right)^2\right]+(x^2+y^2)\\ &+\frac{2(1-\mu)}{[(x+\mu)^2+y^2]^{1/2}}+\frac{2\mu}{[1-2(x+\mu)+(x+\mu)^2+y^2]^{1/2}}.\end{aligned}\right\} \tag{28}$$

It follows from (27) that the right member of this equation can be developed as a series of the form

$$C=\frac{C_{-2}}{\tau^2}+C_0+C_2\tau^2+C_4\tau^4+\cdots.$$

Since this equation must be an identity in τ it follows that $C=C_0$, $C_{-2}=C_2=C_4=C_6=\cdots=0$. Those expressions which are zero constitute a check on the computation of the coefficients of (27). By direct substitution of (27) in (28), it is found that

$$C=C_0=-\frac{20}{9}c^2a+\mu(2+\mu). \tag{29}$$

The force function is sometimes used in the symmetrical form

$$\overline{U}=(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\mu\left(r_2^2+\frac{2}{r_2}\right)=U+\mu(2-\mu),$$

instead of in the form given in (17). Then the constant C becomes

$$\overline{C}=-\frac{20}{9}c^2a+3\mu. \tag{30}$$

228. Recursion Formulas for Solutions.—The second terms in the right members of (23) give rise to a large part of the labor of constructing the solutions. They can be eliminated by use of the integral, and relatively simple recursion formulas can be developed for the construction of the solution after the terms of lowest order have been found.

The integral (18) becomes in the notation of (23)

$$\left.\begin{aligned}\left(\frac{dx}{d\tau}\right)^2+\left(\frac{dy}{d\tau}\right)^2=9\tau^4\left[x^2+y^2\right]+\frac{18\tau^4(1-\mu)}{[(x+\mu)^2+y^2]^{1/2}}+18\tau^4\mu\Big\{1+(x+\mu)\\+(x+\mu)^2-\frac{1}{2}y^2+(x+\mu)^3-\frac{3}{2}(x+\mu)y^2\cdots\Big\}-9\tau^4C.\end{aligned}\right\} \tag{31}$$

On multiplying the first equation of (23) by $x+\mu$ and the second by y and adding the results; and then multiplying the first by y and the second by $-(x+\mu)$ and adding the results, it is found that

$$\left.\begin{aligned}\tau(x+\mu)\frac{d^2x}{d\tau^2}+\tau y\frac{d^2y}{d\tau^2}-2(x+\mu)\frac{dx}{d\tau}-2y\frac{dy}{d\tau}-6\tau^3\left[(x+\mu)\frac{dy}{d\tau}-y\frac{dx}{d\tau}\right]\\=9\tau^5\left[(x+\mu)^2+y^2\right]-\frac{9(1-\mu)\tau^5}{[(x+\mu)^2+y^2]^{1/2}}\\+9\mu\tau^5\Big\{2(x+\mu)^2-y^2+3(x+\mu)^3-\frac{9}{2}(x+\mu)y^2\cdots\Big\},\\ \tau y\frac{d^2x}{d\tau^2}-\tau(x+\mu)\frac{d^2y}{d\tau^2}-2y\frac{dx}{d\tau}+2(x+\mu)\frac{dy}{d\tau}-6\tau^3\left[y\frac{dy}{d\tau}+(x+\mu)\frac{dx}{d\tau}\right]\\=9\mu\tau^5\Big\{+3(x+\mu)y+6(x+\mu)^2y\cdots\Big\}.\end{aligned}\right\} \tag{32}$$

On eliminating the second term of the first of these equations by means of the integral (31), the simplified equations become

$$\left.\begin{aligned}
&\tau(x+\mu)\frac{d^2x}{d\tau^2}+\tau y\frac{d^2y}{d\tau^2}-2(x+\mu)\frac{dx}{d\tau}-2y\frac{dy}{d\tau}-6\tau^3\left[(x+\mu)\frac{dy}{d\tau}-y\frac{dx}{d\tau}\right]\\
&=\frac{27}{2}\tau^5\left[(x+\mu)^2+y^2\right]-\frac{\tau}{2}\left[\left(\frac{dx}{d\tau}\right)^2+\left(\frac{dy}{d\tau}\right)^2\right]+9\mu\tau^5\Big\{3(x+\mu)^2-\frac{3}{2}y^2+4(x+\mu)^3\\
&\qquad\qquad -6(x+\mu)y^2\cdots\Big\}-\frac{9}{2}\tau^5\left[C-\mu(1-\mu)\right],\\
&\tau y\frac{d^2x}{d\tau^2}-\tau(x+\mu)\frac{d^2y}{d\tau^2}-2y\frac{dx}{d\tau}+2(x+\mu)\frac{dy}{d\tau}-6\tau^3\left[y\frac{dy}{d\tau}+(x+\mu)\frac{dx}{d\tau}\right]\\
&\qquad\qquad =27\mu\tau^5\left\{(x+\mu)y+2(x+\mu)^2y+\cdots\right\}.
\end{aligned}\right\} \quad (33)$$

The solution (27) may be written in the form

$$\left.\begin{aligned}
&x+\mu=c\tau^2\left[1+\sum_{j=1}^{\infty}a_{2j}\tau^{2j}\right], \qquad y=c\tau^5\left[-1+\sum_{j=1}^{\infty}b_{2j}\tau^{2j}\right],\\
&c=\left[\frac{9(1-\mu)}{2}\right]^{1/3}, \qquad a_2=-b_2=a=\text{arbitrary constant.}
\end{aligned}\right\} \quad (34)$$

The next step is to form general expressions for the terms involved in (33). Nearly all of the terms written are of the second degree in $x+\mu$ and y; those which are of degree higher than the second are all multiplied by the factor μ and will be in general of little importance. They will not be included in the general formula and must be added to it when it is used. Since these terms contain τ^{11} as a factor, they will not contribute much to the solution unless it is carried very far. The general expressions for the terms of the second degree in $x+\mu$ and y are found from (34) to be

$$\tau(x+\mu)\frac{d^2x}{d\tau^2}=2c^2\tau^3\left\{1+\sum_{j=1}^{\infty}(2j^2+3j+2)a_{2j}\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+1)(2k+1)a_{2k}a_{2j-2k}\tau^{2j}\right\},$$

$$\tau y\frac{d^2y}{d\tau^2}=2c^2\tau^3\left\{10\tau^6-\sum_{j=4}^{\infty}(2j^2-3j+11)b_{2j-6}\tau^{2j}+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}(k+2)(2k+5)b_{2k}b_{2j-2k-6}\tau^{2j}\right\},$$

$$-2(x+\mu)\frac{dx}{d\tau}=-4c^2\tau^3\left\{1+\sum_{j=1}^{\infty}(j+2)a_{2j}\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+1)a_{2k}a_{2j-2k}\tau^{2j}\right\},$$

$$-2y\frac{dy}{d\tau}=-2c^2\tau^3\left\{5\tau^6-2\sum_{j=4}^{\infty}(j+2)b_{2j-6}\tau^{2j}+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}(2k+5)b_{2k}b_{2j-2k-6}\tau^{2j}\right\},$$

$$\begin{aligned}-6\tau^3(x+\mu)\frac{dy}{d\tau}=-6c^2\tau^3\Big\{-5\tau^6-\sum_{j=4}^{\infty}\left[5a_{2j-6}-(2j-1)b_{2j-6}\right]\tau^{2j}\\
+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}(2k+5)b_{2k}a_{2j-2k-6}\tau^{2j}\Big\},\end{aligned}$$

$$6\tau^3 y\frac{dx}{d\tau}=12c^2\tau^3\Big\{-\tau^6-\sum_{j=4}^{\infty}\Big[(j-2)a_{2j-6}-b_{2j-6}\Big]\tau^{2j}+\sum_{j=6}^{\infty}\sum_{k=1}^{j-4}(k+1)a_{2k}b_{2j-2k-6}\tau^{2j}\Big\},$$

$$-\frac{27}{2}(1+2\mu)\tau^5(x+\mu)^2=-\frac{27}{2}c^2(1+2\mu)\tau^3\Big\{\tau^6+2\sum_{j=4}^{\infty}a_{2j-6}\tau^{2j}+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}a_{2k}a_{2j-2k-6}\tau^{2j}\Big\},$$

$$-\frac{27}{2}(1-\mu)\tau^5 y^2=-\frac{27}{2}c^2(1-\mu)\tau^3\Big\{\tau^{12}-2\sum_{j=7}^{\infty}b_{2j-12}\tau^{2j}+\sum_{j=8}^{\infty}\sum_{k=1}^{j-7}b_{2k}b_{2j-2k-12}\tau^{2j}\Big\},$$

$$\frac{\tau}{2}\left(\frac{dx}{d\tau}\right)^2=2c^2\tau^3\Big\{1+2\sum_{j=1}^{\infty}(j+1)a_{2j}\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+1)(j-k+1)a_{2k}a_{2j-2k}\tau^{2j}\Big\},$$

$$\frac{\tau}{2}\left(\frac{dy}{d\tau}\right)^2=\frac{1}{2}c^2\tau^3\Big\{25\tau^6-10\sum_{j=4}^{\infty}(2j-1)b_{2j-6}\tau^{2j}+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}(2k+5)(2j-2k-1)b_{2k}b_{2j-2k-6}\tau^{2j}\Big\};$$

$$\tau y\frac{d^2x}{d\tau^2}=2c^2\tau^6\Big\{-1-\sum_{j=1}^{\infty}\Big[(j+1)(2j+1)a_{2j}-b_{2j}\Big]\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+1)(2k+1)a_{2k}b_{2j-2k}\tau^{2j}\Big\},$$

$$-\tau(x+\mu)\frac{d^2y}{d\tau^2}=-2c^2\tau^6\Big\{-10-\sum_{j=1}^{\infty}\Big[10a_{2j}-(j+2)(2j+5)b_{2j}\Big]\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+2)(2k+5)a_{2j-2k}b_{2k}\tau^{2j}\Big\},$$

$$-2y\frac{dx}{d\tau}=-4c^2\tau^6\Big\{-1-\sum_{j=1}^{\infty}\Big[(j+1)a_{2j}-b_{2j}\Big]\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+1)a_{2k}b_{2j-2k}\tau^{2j}\Big\},$$

$$-6\tau^3 y\frac{dy}{d\tau}=-6c^2\tau^6\Big\{5\tau^6-2\sum_{j=4}^{\infty}(j+2)b_{2j-6}\tau^{2j}+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}(2k+5)b_{2k}b_{2j-2k-6}\tau^{2j}\Big\},$$

$$2(x+\mu)\frac{dy}{d\tau}=2c^2\tau^6\Big\{-5-\sum_{j=1}^{\infty}\Big[5a_{2j}-(2j+5)b_{2j}\Big]\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=2}^{j-1}(2k+5)a_{2j-2k}b_{2k}\tau^{2j}\Big\},$$

$$-6\tau^3(x+\mu)\frac{dx}{d\tau}=-12c^2\tau^6\Big\{1+\sum_{j=1}^{\infty}(j+2)a_{2j}\tau^{2j}+\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}(k+1)a_{2k}a_{2j-2k}\tau^{2j}\Big\},$$

$$-27\mu\tau^5(x+\mu)y=-27c^2\mu\tau^6\Big\{-\tau^6-\sum_{j=4}^{\infty}\Big[a_{2j-6}-b_{2j-6}\Big]\tau^{2j}+\sum_{j=5}^{\infty}\sum_{k=1}^{j-4}a_{2k}b_{2j-2k-6}\tau^{2j}\Big\}.$$

On substituting these expressions in (33) and equating the coefficients of τ^{2j+3} and τ^{2j+6} respectively, it is found that

$$
\left.\begin{aligned}
2j(2j+3)a_{2j} = &-2\sum_{k=1}^{j-1}(k+1)(k+j)a_{2k}a_{2j-2k}+3\left[4j-9(1+2\mu)\right]a_{2j-6}\\
&+\left[4j^2+12j-9\right]b_{2j-6}-27(1-\mu)b_{2j-12}\\
&-\frac{1}{2}\sum_{k=1}^{j-4}\left[(2k+5)(2j+2k+3)b_{2k}+12(k+1)a_{2k}\right]b_{2j-2k-6}\\
&+6\sum_{k=4}^{j-4}(2k+5)b_{2k}a_{2j-2k-6}+\frac{27}{2}(1+2\mu)\sum_{k=1}^{j-4}a_{2k}a_{2j-2k-6}\\
&+\frac{27}{2}(1-\mu)\sum_{k=1}^{j-7}b_{2k}b_{2j-2k-12}+\text{quantities coming from}\\
&\qquad\text{terms in (33) of the third and higher degrees,}\\
-2(j+2)(2j+3)b_{2j} = &\,2(j+2)(2j+3)a_{2j}-2\sum_{k=1}^{j-1}(k+1)(2k-1)a_{2k}b_{2j-2k}\\
&+12\sum_{k=1}^{j-1}(k+1)a_{2k}a_{2j-2k}+2\sum_{k=1}^{j-1}(k+1)(2k+5)a_{2j-2k}b_{2k}\\
&-12(j+2)b_{2j-6}+6\sum_{k=1}^{j-4}(2k+5)b_{2k}b_{2j-2k-6}\\
&-27\mu[a_{2j-6}-b_{2j-8}]+27\mu\sum_{k=1}^{j-4}a_{2k}b_{2j-2k-6}+\text{quantities}\\
&\qquad\text{coming from terms in (33) of the third and}\\
&\qquad\text{higher degrees.}
\end{aligned}\right\}\quad(35)
$$

These formulas are to be used when j is 3 or greater, and care must be taken in adding terms coming from the higher powers of $(x+\mu)$ and y in the right members of equations (33).

The results of applying (35) are

$$
\left.\begin{aligned}
&x+\mu=r\cos\tau^3+x_1\mu+x_2\mu^2+\cdots,\\
&-y=r\sin\tau^3-y_1\mu-y_2\mu^2+\cdots,\\
&r=c\tau^2\Big[1+a\tau^2-\frac{3}{7}a^2\tau^4+\frac{23}{63}a^3\tau^6-\frac{1894}{4851}a^4\tau^8+\frac{3293}{7007}a^5\tau^{10}\\
&\qquad-\frac{2,418,092}{3,972,769}a^6\tau^{12}+\frac{55,964,945}{67,540,473}a^7\tau^{14}-\frac{38,481,084,886}{32,937,237,333}a^8\tau^{16}+\cdots\Big],\\
&x_1=c\tau^8\Big[\frac{1}{2}+\Big(\frac{2}{11}a+\frac{9}{22}c\Big)\tau^2+\Big(\frac{45}{1001}a^2+\frac{135}{286}ac+\frac{9}{26}c^2\Big)\tau^4\\
&\qquad\qquad-\Big(\frac{1270}{9009}a^3-\frac{35}{1802}a^2c-\frac{9}{13}ac^2-\frac{3}{10}c^3\Big)\tau^6\cdots\Big],\\
&y_1=c\tau^{11}\Big[-\frac{1}{5}+\frac{2}{55}a\tau^2-\Big(\frac{4392}{35,035}a^2-\frac{380}{901}ac-\frac{27}{182}c^2\Big)\tau^4\cdots\Big],\\
&x_2=c\tau^{14}\Big[\frac{1}{20}+\frac{139}{3740}a\tau^2+\cdots\Big],\qquad y_2=c\tau^{17}\Big[-\frac{1}{800}+\cdots\Big].
\end{aligned}\right\}\quad(36)
$$

229. The Conditions for Existence of Closed Orbits of Ejection.—The series (34) converge for all $|\mu| \leqq \mu_0$ and $|\alpha-\alpha_0| \leqq \rho$ provided $|\tau| \leqq R$, where R is a positive constant depending on μ_0, α_0, and ρ. The coefficients of the various powers of τ are polynomials in α and μ, so far as μ occurs explicitly, and they also involve μ implicitly through c. These coefficients are expansible as power series in $\alpha-\alpha_0$ and μ which converge for all finite values of $|\alpha-\alpha_0|$ and for $|\mu|<1$. Therefore the expressions for $x+\mu$ and y are expansible as power series in $\alpha-\alpha_0=\beta$ and μ, and if $|\beta| \leqq \rho$, $|\mu| \leqq \mu_0$ the series converge for all $|\tau| \leqq R$. They may be written

$$x=p_1(\beta, \mu; \tau), \quad \frac{dx}{d\tau}=p_2(\beta, \mu; \tau), \quad y=p_3(\beta, \mu; \tau), \quad \frac{dy}{d\tau}=p_4(\beta, \mu; \tau), \quad (37)$$

where $p_1, \ldots, p_4$ are power series in β and μ.

Now α_0 will be determined so that when μ is zero the period from ejection to collision shall be $2j\pi$. From (5) and (13) it is found that α_0 satisfying this condition is

$$\alpha_0=-\frac{c}{10}(2j)^{-2/3}. \quad (38)$$

Suppose $T<R$ and $\mu=0$ and that for $\tau=T$ the coördinates of the infinitesimal body are x_0, x_0', y_0, y_0', where the accents denote derivatives with respect to τ. Suppose now $0<\mu<\mu_0$ and let the values of the coördinates at $\tau=T$ be

$$x=x_0+\beta_1, \quad x'=x_0'+\beta_2, \quad y=y_0+\beta_3, \quad y'=y_0'+\beta_4. \quad (39)$$

The conditions that these values of the coördinates shall belong to an orbit of ejection are

$$\left.\begin{aligned} x_0+\beta_1&=p_1(\beta, \mu;\ T), & y_0+\beta_3&=p_3(\beta, \mu;\ T), \\ x_0'+\beta_2&=p_2(\beta, \mu;\ T), & y_0'+\beta_4&=p_4(\beta, \mu;\ T). \end{aligned}\right\} \quad (40)$$

Since the right members of these equations are expansible as converging power series in β and μ, it follows from the definitions of x_0, x_0', y_0, and y_0' that

$$\beta_1=q_1(\beta, \mu), \quad \beta_2=q_2(\beta, \mu), \quad \beta_3=q_3(\beta, \mu), \quad \beta_4=q_4(\beta, \mu), \quad (41)$$

where $q_1, \ldots, q_4$ are power series in β and μ, vanishing with β and μ.

If the infinitesimal body crosses the x-axis perpendicularly at any time the orbit is symmetrical with respect to the x-axis. It follows from the definitions of x_0, x_0', y_0, and y_0' that, when $\mu=0$,

$$x_0'(P/2)=0, \quad y_0(P/2)=0, \quad (42)$$

where P is the period in τ from ejection to collision. It will be shown that analogous conditions can be satisfied when μ is distinct from zero, and therefore that closed orbits of ejection exist in the restricted problem of three bodies.

Suppose μ is distinct from zero and consider the solution with the initial conditions $x_0+\beta_1$, $x_0'+\beta_2$, $y_0+\beta_3$, $y_0'+\beta_4$. In order to leave the period arbitrary an undetermined parameter δ is introduced by the transformation

$$\tau=\tau_1(1+\delta), \tag{43}$$

where τ_1 is the new independent variable. Now if the orbit for $\mu=0$ does not pass through the position of μ the solutions can be expanded as power series in $\beta_1, \ldots, \beta_4$, δ, and μ; and if Q is arbitrarily chosen in advance, the moduli of $\beta_1, \ldots, \beta_4$, δ, and μ can be taken so small that the solutions converge for all $T \gtreqless \tau_1(1+\delta) \leqq Q$. The quantity Q will be taken equal to $(1+\delta)P/2$ where, as before, P is the period from ejection to collision when $\mu=0$.

In order to complete the discussion in regard to the convergence it is necessary to show that none of the orbits in question for $\mu=0$ passes through the position of μ. Suppose $\tau=\tau_0$ is the time at which the infinitesimal body crosses the positive x-axis on which, for $\mu=0$, the body μ lies at the distance unity from the origin. It is necessary to show that for none of the values of ξ_0 defined in (16) is equation (4) satisfied by $\xi=1$ and $t_0-t_1=n\pi$, where t_1 is the time of ejection.

It follows from (4) that the larger $|\xi_0|$ is the shorter is the time required for the infinitesimal body to pass from ejection to the distance unity. Therefore, if for the smallest value of ξ_0 belonging to the problem, viz., $\xi_0=2$, it reaches the distance unity in less than π, then it will always be at a distance greater than unity at $t-t_1=\pi$ and all multiples of π until it reaches the greatest distance ξ_0. Consequently, it can not pass through the point occupied by μ while receding from $1-\mu$; and since the path referred to rotating axes is symmetrical with respect to the x-axis, it can not pass through the position of μ on its return to $1-\mu$. In making the computation it is convenient to let $\xi=\xi_0\rho$. Then, transferring the origin to t_1, equation (4) becomes

$$t-t_1=\frac{\sqrt{2}}{8}[\pi-4\sqrt{\rho-\rho^2}+2\sin^{-1}(-1+2\rho)](\pm\xi_0)^{3/2}.$$

It is found from this equation that if $\xi_0=2$ the value of $t-t_1$ for $\xi=1$ is 0.18π, which is less than π. Therefore none of the orbits in question for $\mu=0$ passes through the position of μ.

If the infinitesimal body is moving in an orbit of ejection and crosses the x-axis perpendicularly at any time, then it follows from the symmetry of its motion that its orbit is also an orbit of collision. Therefore sufficient conditions for a closed orbit of ejection are

$$\frac{dx}{d\tau_1}=y=0 \text{ at } \tau_1=\frac{P}{2}.$$

If the initial conditions are $x_0+\beta_1$, $x_0'+\beta_2$, $y_0+\beta_3$, $y_0'+\beta_4$ and the parameter δ has been introduced by (43), these equations become

$$\frac{dx}{d\tau_1}=P_1(\beta_1,\ldots,\beta_4,\delta,\mu;P/2)=0, \quad y=P_2(\beta_1,\ldots,\beta_4,\delta,\mu;P/2)=0, \tag{44}$$

where P_1 and P_2 are converging power series in $\beta_1, \ldots, \beta_4, \delta$, and μ. It follows from (42) that P_1 and P_2 vanish for $\beta_1 = \cdots = \beta_4 = \delta = \mu = 0$.

If $\beta_1, \ldots, \beta_4$ are determined by (41) the orbit is an orbit of ejection. Therefore, upon substituting the series for these constants in (44), sufficient conditions for the existence of closed orbits of ejection become

$$\frac{dx}{d\tau_1}=Q_1(\beta,\ \delta,\ \mu;\ P/2)=0, \qquad y=Q_2(\beta,\ \delta,\ \mu;\ P/2)=0, \tag{45}$$

where Q_1 and Q_2 are power series in β, δ, and μ, which vanish with $\beta=\delta=\mu=0$. The coördinates can, therefore, be developed as power series in β, δ, and μ and the moduli of these parameters can be taken so small that the series converge for $|\tau| < P$, where P is the period from ejection to collision for $\mu=0$.

230. Proof of the Existence of Closed Orbits of Ejection.—The proof of the existence of closed orbits of ejection resolves itself into the demonstration that equations (45) have solutions when μ is distinct from zero. These equations are not satisfied by $\mu=0$ unless β and δ are both also zero, because, when $\mu=0$, the problem reduces to that of two bodies in which the period in τ from ejection to greatest distance depends upon β, and in which the distance depends upon δ. Therefore equations (45) have one or more solutions for β and δ as power series in μ, vanishing with μ, according as the functional determinant is distinct from zero or is zero for $\beta=\delta=\mu=0$.

Since the functional determinant involves derivatives only with respect to β and δ, the μ may be put equal to zero before forming it. Then the determinant in question is

$$\Delta=\left|\begin{array}{cc} \dfrac{\partial \dot{x}}{\partial \alpha}, & \dfrac{\partial \dot{x}}{\partial \delta} \\[2ex] \dfrac{\partial y}{\partial \alpha}, & \dfrac{\partial y}{\partial \delta} \end{array}\right|_{\substack{\alpha-\alpha_0=0\\ \beta=\delta=0}},$$

where $\dot{x}$ is the derivative of x with respect to τ_1. Before forming the elements of the first column δ may be put equal to zero, and before forming the elements of the second column $\beta=\alpha-\alpha_0$ may be put equal to zero.

It follows from (15) that when $\delta=\mu=0$ and $t-t_1=2j\pi$ the value of y is zero whatever α may be. Therefore $\partial y/\partial \alpha$ is zero and the determinant becomes simply $\Delta=(\partial\dot{x}/\partial\alpha)(\partial y/\partial\delta)$.

In the case under consideration the value of $t-t_1$ for which Δ is formed is

$$t-t_1=\tau^3=\tau_1^3(1+\delta)^3=-\frac{P^3}{8}(1+\delta)^3=j\pi(1+\delta)^3. \tag{46}$$

The second factor of Δ will be computed first. Upon putting $\alpha-\alpha_0=\beta=0$, it is found from the second equation of (15) that

$$\frac{\partial y}{\partial \delta}=\left[-\frac{\partial \xi}{\partial \delta}\sin j\pi(1+\delta)^3-3j\pi(1+\delta)^2\xi\cos j\pi(1+\delta)^3\right]_{\delta=0}=(-1)^{j+1}3j\pi\xi_0\neq 0.$$

Before computing the first factor of Δ the parameter δ may be put equal to zero. Hence it follows, from (46), (15), and (9), that

$$\frac{\partial \dot{x}}{\partial a}=\frac{\partial \xi}{\partial a}=4\tau^3\left[1-\tfrac{9}{7}a\tau^2+\tfrac{46}{21}a^2\tau^4-\tfrac{18{,}940}{4851}a^3\tau^6+-\cdots\right]_{\substack{a=a_0\\ \tau=P/2}}.$$

It was proved in §223 that the signs in this series alternate and that a is negative for those orbits which lie entirely in the finite part of the plane. Therefore $\partial\dot{x}/\partial a$ is distinct from zero for all values of a under consideration.

It follows from this discussion that Δ is distinct from zero for $\beta=a-a_0=\delta=0$, $\tau=j\pi$, and consequently that the sufficient conditions for the existence of closed orbits of ejection can be uniquely satisfied for $|\mu|$ sufficiently small. There is a closed orbit of the type in question for ejection in both the positive and the negative direction for all integral values of j upon which the ξ_0, or a_0, of (16) depends.

In the special case in which the finite masses are equal, a closed orbit of ejection for $j=2$, with ejection in the positive direction,* was discovered from numerical experiments by Burrau in two interesting memoirs.† Since in his problem μ had the large value 0.5, it is not to be expected that the results of this analysis would agree very closely with the results of his computations. Hence the comparison will be made only for the constant of the Jacobian integral. Upon taking into account the difference in his units and those employed here, it is found that his Jacobian constant C_B, equation (5) loc. cit., is related to C of (29) by the equation

$$-2C_B=C=-\tfrac{20}{9}ca+\mu(2+\mu)=\pm\frac{2(1-\mu)}{\xi_0}+\mu(2+\mu).$$

Burrau's computation gave $-2C_B=2.2528$; and for $\mu=0.5$, $\xi_0^{3/2}=2\sqrt{2(1-\mu)}$ it is found that $C=2.38$, and the agreement is fully as close as would be expected. It follows from these numbers that a larger value of the constant $-a$, corresponding to a smaller value of ξ_0, belongs to the undisturbed orbit having the period 2π than to that computed by Burrau. In the undisturbed orbit the greatest distance to which the infinitesimal body recedes is, by (16), $\xi_0=2$; it has this value at $t-t_1=\pi$, and it is then on the negative half of the x-axis. The greatest distance found by Burrau in his computation was 1.9972, or a little less than that in the undisturbed motion.

If the infinitesimal body is ejected toward or from the body μ with a small value of $|\xi_0|$, it will be disturbed so that on its return it will revolve around $1-\mu$ in the positive direction. This can be seen when the motion is considered in fixed axes, for under the conditions postulated the disturbance is positive all the time that the infinitesimal body is going out and returning. If it is ejected farther, it will be accelerated by μ in the negative

*Burrau's orbit of ejection was from the body called μ here, but permuting $1-\mu$ and μ and changing the positive directions of the axes, the statements are correct.

†Recherches numériques concernant des solutions périodiques d'un cas spécial du problème des trois corps, *Astronomische Nachrichten*, vol. 135 (1894), No. 3230; and ibid, vol. 136 (1894), No. 3251.

direction part of the time. While in general the body will not collide with $1-\mu$ on its return, it may possibly do so under special conditions. Indeed, Sir George Darwin has discovered one such orbit by numerical experiment* having the period π. The ejection was from μ in the direction of $1-\mu$, and the body collided with μ going in the same direction.† This orbit is one of a pair which together are the limit of certain periodic orbits, though they are not periodic themselves, either physically or mathematically. The constant C belonging to this orbit in the units employed here is $20/11=1.818$. The values of the masses used by Darwin were $1-\mu=10/11$, $\mu=1/11$. It follows from (16) that $\xi_0=2^{1/3}$ for this period, and from (30) that $C=1.716$. In this case the ξ_0 belonging to the undisturbed orbit is larger than that belonging to Darwin's orbit. The value of ξ_0 is $2^{1/3}=1.26$; the greatest distance in Darwin's orbit, according to his diagram, is 1.3.

231. Conditions at an Arbitrary Point for an Orbit of Ejection.—Since the motion of the infinitesimal body is regular for all finite values of τ and all finite values of the coördinates except those for which it collides with one of the finite masses,‡ it becomes a matter of interest to determine in any special case whether the trajectory is one of ejection or collision for a finite value of t. It is sufficient, as Painlevé conjectured and as Levi-Civita proved,§ that the coördinates and velocities shall satisfy one analytic condition in order that the orbit shall pass through one of the finite masses for a finite value of t. This conclusion will be established here in a different way.

Suppose μ is zero and consider the problem of defining the initial conditions for an orbit of ejection so that it shall pass through the point in question, and so that the components of velocity at the point shall satisfy as many conditions as possible. The velocity in rotating axes at any distance from the finite mass $1-\mu$ is the resultant of the velocity with respect to fixed axes and that due to the rotation of the axes. The velocity with respect to fixed axes at any finite distance can be made any finite quantity by a suitable determination of the constant a, or the equivalent constant ξ_0. Consequently, an arbitrary speed, or one of the components of velocity, with respect to rotating axes at any distance can be secured.

Suppose the speed at a given distance has been assigned; then it is possible to determine the initial direction of ejection so that the orbit of the body will pass through any point having the given distance, for it is possible to do it in fixed axes and the rotation simply changes the direction of ejection by an angle which is proportional to the time required for the body to reach the distance in question. It is clear from this that when $\mu=0$ the conditions of ejection can be so determined that the infinitesimal

*On certain families of periodic orbits, *Monthly Notices of the Royal Astronomical Society*, vol. 70 (1909), p. 134.

†See further remarks on this orbit at the close of §234.

‡Painlevé, *Leçons sur la Théorie Analytique des Equations Différentielles*, p. 583.

§*Acta Mathematica*, vol. 30 (1906), pp. 306–327.

body shall pass through any assigned point with any assigned speed. Of the four quantities required to define an orbit, viz., two coördinates and the speed and direction of motion, three can be taken arbitrarily and the fourth is determined by the condition that the orbit shall be one of ejection. The determination of the fourth quantity is double because the body has the same speed twice, once when it is receding from $1-\mu$, and once when it is returning toward $1-\mu$.

Suppose that, for $\mu=0$, an orbit of ejection passes through the point x_T, y_T with the speed $v_T=\sqrt{x_T'^2+y_T'^2}$ at $(t-t_1)^{1/3}=T$. If ξ_T' represents the speed with respect to fixed axes, then, since the component of velocity due to the rotation of the axes equals numerically the distance of the point from the origin, the relation between v_T and ξ_T' is

$$v_T^2=1+\xi_T'^2. \tag{47}$$

Equation (2) determines ξ_0, the greatest distance to which the body recedes, and (13) gives the constant a_0. Equation (4) gives the value of T, and the direction of ejection is T degrees in the negative direction from the line joining $1-\mu$ and the point $(x_T,\ y_T)$. Let the angle of ejection be θ_0.

Now suppose that μ is distinct from zero, but small. Let the initial values of a and θ be $a_0+\beta$ and $\theta_0+\gamma$. Let a new independent variable τ_1 and a parameter δ be introduced by (43). Then the solution can be written in the form

$$x=p_1(\beta,\gamma,\delta,\mu;\tau_1),\qquad y=p_3(\beta,\gamma,\delta,\mu;\tau_1),$$
$$\frac{dx}{d\tau_1}=p_2(\beta,\gamma,\delta,\mu;\tau_1),\qquad \frac{dy}{d\tau_1}=p_4(\beta,\gamma,\delta,\mu;\tau_1),$$

where $p_1,\ldots,p_4$ are power series in β, γ, δ, and μ. The moduli of these parameters can be taken so small that the series converge for $0\gtreqless\tau_1\leqq T$.

The conditions that the body shall pass through the point $(x_T,\ y_T)$ with the velocity v_T at $\tau_1=T$ are

$$p_1(\beta,\gamma,\delta,\mu;T)-x_T=0,\quad p_3(\beta,\gamma,\delta,\mu;T)-y_T=0,\quad \sqrt{p_2^2+p_4^2}-v_T=0. \tag{48}$$

Since these equations are satisfied by $\beta=\gamma=\delta=\mu=0$, they can be written as power series in β, γ, δ, and μ, vanishing with $\beta=\gamma=\delta=\mu=0$, of the form

$$P_1(\beta,\gamma,\delta,\mu;T)=0,\quad P_2(\beta,\gamma,\delta,\mu;T)=0,\quad P_3(\beta,\gamma,\delta,\mu;T)=0. \tag{49}$$

Equations (49) are not satisfied by $\mu=0$ unless also $\beta=\gamma=\delta=0$. Therefore they have solutions for β, γ, and δ in terms of μ which vanish for $\mu=0$. If the determinant of the linear terms in β, γ, and δ is distinct from zero the solution is unique. In treating the problem it is convenient to use equations derived from (49) rather than these equations themselves. Let φ represent the angle between the positive end of the x-axis and the line from the origin to the point $(x_T,\ y_T)$. Then let Q_1, Q_2, and Q_3 be defined by

$$Q_1=P_1\cos\varphi+P_2\sin\varphi,\qquad Q_2=-P_1\sin\varphi+P_2\cos\varphi,\qquad Q_3=P_3. \tag{50}$$

This transformation is equivalent to rotating the axes so that (x_T, y_T) lies on the positive half of the new x-axis. The solutions of

$$Q_1(\beta, \gamma, \delta, \mu; T)=0, \quad Q_2(\beta, \gamma, \delta, \mu; T)=0, \quad Q_3(\beta, \gamma, \delta, \mu; T)=0 \quad (51)$$

are identical with those of (49), for the two sets of functions are linearly related with non-vanishing determinant.

The determinant of the terms of the Q_i which are linear in β, γ, and δ is

$$\Delta = \left| \begin{array}{ccc} \frac{\partial Q_1}{\partial \beta}, & \frac{\partial Q_1}{\partial \gamma}, & \frac{\partial Q_1}{\partial \delta} \\ \frac{\partial Q_2}{\partial \beta}, & \frac{\partial Q_2}{\partial \gamma}, & \frac{\partial Q_2}{\partial \delta} \\ \frac{\partial Q_3}{\partial \beta}, & \frac{\partial Q_3}{\partial \gamma}, & \frac{\partial Q_3}{\partial \delta} \end{array} \right|_{\beta=\gamma=\delta=\mu=0.} \quad (52)$$

Before forming this determinant μ may be put equal to zero, and before computing the elements of each column the parameters with respect to which the derivations are taken in the other columns may be put equal to zero. When $\gamma=\delta=\mu=0$ the value of Q_2 is zero for all values of β; therefore $\partial Q_2/\partial\beta=0$. Since, for $\mu=0$, the distance of the infinitesimal body from the origin is independent of γ it follows that Q_1 is an even function of γ; therefore $\partial Q_1/\partial\gamma=0$ for $\beta=\gamma=\delta=\mu=0$. Also, since, for $\beta=\delta=\mu=0$, the velocity is independent of γ, it follows that $\partial Q_3/\partial\gamma=0$. Hence the determinant reduces to

$$\Delta = \frac{\partial Q_2}{\partial \gamma} \left| \begin{array}{cc} \frac{\partial Q_1}{\partial \beta}, & \frac{\partial Q_1}{\partial \delta} \\ \frac{\partial Q_3}{\partial \beta}, & \frac{\partial Q_3}{\partial \delta} \end{array} \right|_{\beta=\gamma=\delta=\mu=0.} \quad (53)$$

When $\mu=0$ the values of x and y, which are the coördinates referred to rotating axes, are

$$x=\xi\cos[\theta_0+\gamma-(t-t_1)], \qquad y=\xi\sin[\theta_0+\gamma-(t-t_1)],$$

where ξ has the value given in (9). Therefore P_1 and P_2 become

$$P_1=\xi(a_0+\beta,\ \delta;\ T^3)\cos[\theta_0+\gamma-T^3]-\xi(a_0+0,\ 0,\ T^3)\cos(\theta_0+0-T^3),$$

$$P_2=\xi(a_0+\beta,\ \delta;\ T_3)\sin[\theta_0+\gamma-T^3]-\xi_0(a_0+0,\ 0,\ T^3)\sin(\theta_0+0-T^3).$$

If $\beta=\delta=0$, the first terms of the expansions of these expressions as power series in γ are

$$P_1=-\xi(a_0+0,0;\ T^3)\sin(\theta_0-T^3)\gamma+\cdots,$$

$$P_2=+\xi(a_0+0,0;\ T^3)\cos(\theta_0-T^3)\gamma+\cdots.$$

Under the restrictions which have been imposed, $\varphi=\theta_0-T^3$ and Q_2 becomes

$$Q_2=\xi(a_0+0,\ 0;\ T^3)\gamma+\cdots. \quad (54)$$

Since $\xi(a_0+0,\ 0;\ T^3)$ is distinct from zero, $\partial Q_2/\partial\gamma$ is also distinct from zero.

Now suppose $\gamma=\mu=0$. Then $\varphi=\theta_0-T^3$ and Q_1 becomes

$$Q_1=\xi(\alpha_0+\beta,\ \delta;\ T^3)-\xi(\alpha_0+0,\ 0;\ T^3). \tag{55}$$

Therefore $\partial Q_1/\partial\beta=\partial\xi/\partial\beta$, $\partial Q_1/\partial\delta=\partial\xi/\partial\delta$, the first of which is positive by the properties of ξ which were derived in §223. The second one of these partial derivatives is positive or negative according as, for $\mu=\beta=\delta=0$, the infinitesimal body is receding from, or approaching toward, the origin at $t-t_1=T^3$. If ξ has its greatest value for $t-t_1=T^3$, then $\partial Q_1/\partial\delta$, which is the derivative of ξ with respect to τ, is zero.

It follows from (47), (48), and (50) that, for $\gamma=\mu=0$,

$$Q_3=\sqrt{1+[\xi'(\alpha_0+\beta,\delta;\ T)]^2}-\sqrt{1+[\xi'(\alpha_0+0,0;\ T)]^2}. \tag{56}$$

Therefore the partial derivatives of Q_3 which appear in (52) are

$$\frac{\partial Q_3}{\partial\beta}=\frac{\dfrac{\partial\xi'(\alpha_0+\beta,0;\ T)}{\partial\beta}}{\sqrt{1+[\xi'(\alpha_0+0,\ 0;\ T)]^2}},\qquad \frac{\partial Q_3}{\partial\delta}=\frac{\dfrac{\partial\xi'(\alpha_0+0,\ \delta;\ T)}{\partial\delta}}{\sqrt{1+[\xi'(\alpha_0+0,\ 0;\ T)]^2}}. \tag{57}$$

From equation (9) it is found that

$$\xi'=2c(1+\delta)^2\tau_1\Big[(1+2a(1+\delta)^2\tau_1^2-\tfrac{9}{7}a^2(1+\delta)^4\tau_1^4+\ \cdots\ \Big],$$

where the signs of the coefficients alternate. Therefore the partial derivatives in question are

$$\frac{\partial\xi'}{\partial\beta}=+4\tau_1^3-\frac{36}{7}\,\alpha_0\tau_1^5+\cdots,\qquad \frac{\partial\xi'}{\partial\delta}=4c\tau_1+16\alpha_0\tau_1^3-\frac{108}{7}\alpha_0^2\tau_1^5+\cdots, \tag{58}$$

Since α_0 is negative, the first of these expressions is positive; since the velocity decreases or increases with increasing time (according as the infinitesimal body is receding from or approaching toward the origin) the second of these expressions is negative if the body is receding from, and positive if it is approaching toward, the origin. If ξ' is zero for $t-t_1=T^3$, ξ is an even function of δ because the motion is symmetrical with respect to T^3 as initial time. In this case the second of (58) is zero. Therefore if the infinitesimal body for $\mu=\beta=\delta=0$ is not at its greatest distance at $t-t_1=T^3$, the form of Δ is

$$\Delta=\begin{vmatrix}+ & \pm\\ + & \mp\end{vmatrix}, \tag{59}$$

where the upper sign is to be used if it is receding from, and the lower if it is approaching toward, the origin. Since $\partial Q_2/\partial\gamma$ is distinct from zero, Δ is distinct from zero. Therefore in this case equations (48) are uniquely solvable for β, γ, and δ as power series in μ, vanishing with μ. This means

that if an arbitrary point in the xy-plane, and a velocity greater than that of this point with respect to fixed axes, be selected, then there exist two orbits of ejection passing through this point such that, for μ sufficiently small, the infinitesimal body will pass the point with the given velocity. The direction with which the body passes the point depends upon the initial conditions, of which it is a regular analytic function. This is Painlevé's theorem for the restricted problem of three bodies.

If the velocity chosen equals that of the arbitrary point with respect to fixed axes, so that, for $\mu=0$, the point in question is at the greatest distance to which the infinitesimal body recedes, then the determinant Δ is zero and the solution is multiple. The reason for it is that the two solutions which were distinct in the other case have united, and the solution has become double.

232. Closed Orbits of Ejection for Large Values of μ.—It was proved in §230 that closed orbits of ejection exist provided $|\mu|$ is sufficiently small. The question of their existence for large values of μ will now be considered.

If $\tau=\tau_1(1+\delta)$, the solutions of (23) may be written

$$x=f_1(a_0+\beta,\ \delta,\ \mu;\ \tau_1),\qquad y=f_2(a_3+\beta\ \delta,\ \mu;\ \tau_1),\tag{60}$$

and the conditions for a closed orbit of ejection with the period 2π in τ_1 are

$$\frac{dx(\pi)}{d\tau_1}=f_1'(a_0+\beta,\ \delta,\ \mu;\ \pi)=0,\qquad y(\pi)=f_2(a_0+\beta,\ \delta,\ \mu;\ \pi)=0.\tag{61}$$

It has been shown that, for $|\mu|$ sufficiently small, equations (61) can be solved for β and δ as converging power series in μ, vanishing with μ, and that when these results are substituted in (60) the latter become power series in μ which converge for $0\overline{\gtrless}\tau\leqq\pi$ provided $|\mu|$ is sufficiently small.

Suppose the series for the closed orbit converge for $\mu=\mu_1$, and that the values of $a_0+\beta$ and δ for this value of μ are a_1 and δ_1. The solutions of (23) are expansible as power series in τ for values of a, δ, and μ in the vicinity of a_1, δ_1, and μ_1. If $|a-a_1|<r_1$, $|\delta-\delta_1|<r_2$ $|\mu-\mu_1|<r_3$ the series converge if $\tau_1<T$, where T is any arbitrary quantity less than the period from ejection to collision. The result will have the form of (27) where τ is replaced by $\tau_1(1+\delta)$. Each term of (27) can be expanded as a power series in $a-a_1$, $\delta-\delta_1$, $\mu-\mu_1$ which will converge provided $|\mu-\mu_1|<1-\mu_1$. The solution may be written in the form

$$x=p_1(a-a_1,\ \delta-\delta_1,\ \mu-\mu_1;\ \tau_1),\qquad y=p_2(a-a_1,\ \delta-\delta_1,\ \mu-\mu_1;\ \tau_1),\tag{62}$$

where p_1 and p_2 are power series in $a-a_1$, $\delta-\delta_1$, and $\mu-\mu_1$. Suppose the values of x, y, and their derivatives at $\tau_1=T$ for $a-a_1=\delta=\delta_1=\mu-\mu_1=0$ are x_T, y_T, x_T', and y_T'. Let their values for an arbitrary set of values of $a-a_1$, $\delta-\delta_1$, and $\mu-\mu_1$, satisfying the inequalities which insure convergency,

be $x_T+\beta_1$, $y_T+\beta_2$, $x'_T+\beta_3$, and $y'_T+\beta_4$. Then $\beta_1, \ldots, \beta_4$ are expansible as power series in $\alpha-\alpha_1$, $\delta-\delta_1$, $\mu-\mu_1$ of the form

$$\beta_i=q_i(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1;\ T) \qquad (i=1, \ldots, 4), \tag{63}$$

where the q_i vanish for $\alpha-\alpha_1=\delta-\delta_1=\mu-\mu_1=0$.

Now consider a solution with the initial conditions $x_T+\beta_1$, $y_T+\beta_2$, $x'_T+\beta_3$, $y'_T+\beta_4$. Suppose for $\beta_1=\beta_2=\beta_3=\beta_4=0$ the infinitesimal body does not pass through the position of μ for $T \lesseqgtr \tau_1 \leqq \pi$. Therefore the solutions can be expanded as power series in $\beta_1, \ldots, \beta_4$ which will converge for $T \lesseqgtr \tau_1(1+\delta) \lesseqgtr \pi$ provided the moduli of $\beta_1, \ldots, \beta_4$ are sufficiently small. At $\tau_1=\pi$ the expressions for the coördinates become, making use of (63),

$$\left.\begin{aligned}
x&=P_1(\beta_1, \ldots, \beta_4;\ \pi)=Q_1(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1),\\
y&=P_2(\beta_1, \ldots, \beta_4;\ \pi)=Q_2(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1),\\
x'&=P_3(\beta_1, \ldots, \beta_4;\ \pi)=Q_3(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1),\\
y'&=P_4(\beta_1, \ldots, \beta_1;\ \pi)=Q_4(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1).
\end{aligned}\right\} \tag{64}$$

Conditions that the orbit shall be closed with the period $2\pi/(1+\delta)$ are

$$Q_2(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1)=0, \qquad Q_3(\alpha-\alpha_1,\ \delta-\delta_1,\ \mu-\mu_1)=0. \tag{65}$$

It has been seen that in general the solution of these equations for $\alpha-\alpha_1$ and $\delta-\delta_1$ in terms of $\mu-\mu_1$, vanishing with $\mu-\mu_1$, is unique. This is always true unless the solution becomes multiple. If the multiplicity is odd, there is one real solution for both positive and negative values of $\mu-\mu_1$. There are three solutions altogether for $|\mu|$ sufficiently small because ξ_0, defined in (16), has three values for which the conditions of a closed orbit of ejection can be satisfied, but only one of them is real. Consequently the real solution can not disappear by uniting with one of the others unless they first unite and become real. Then, if two of the real solutions should unite and become complex, there would be one real one left. That is, there is one real closed orbit of ejection from $1-\mu$ for all values of μ from 0 to 1, excluding the value unity. The argument has been made for the period 2π, but it is entirely similar for any multiple of 2π.

It has been tacitly assumed in the argument that none of the orbits of ejection under consideration passes through the position of μ for any value of μ; for it was only under this condition that the convergence of the series was assured. It has been proved that the closed orbits of ejection do not pass through μ for μ sufficiently small. Since the coördinates in these orbits are regular analytic functions of μ, it follows that if any one of them passes through the position of μ for any value of μ, then for values near this one it will pass near μ. The motion of the infinitesimal body in the vicinity of a finite body when referred to rotating axes is always in the

retrograde direction, and the orbits in question are always symmetrical with respect to the x-axis.

Consider the motion with respect to μ in a closed orbit of ejection from $1-\mu$. Whether the ejection is toward or from μ the motion with respect to μ in those parts of the orbit which are near to it is direct instead of retrograde. Therefore, the orbit can not pass near μ without first developing folds so that a line from μ in certain directions will intersect it three times. It is extremely improbable, though not absolutely certain, that this sort of development could take place. It is probable that the real orbits of ejection which exist for μ sufficiently small continue in the analytic sense for all values of μ from 0 to unity, and that they do not pass near μ.

For $\mu=0$ the orbits in which the ejection is toward μ are symmetrically opposite to those in which the ejection is away from μ, and these conditions

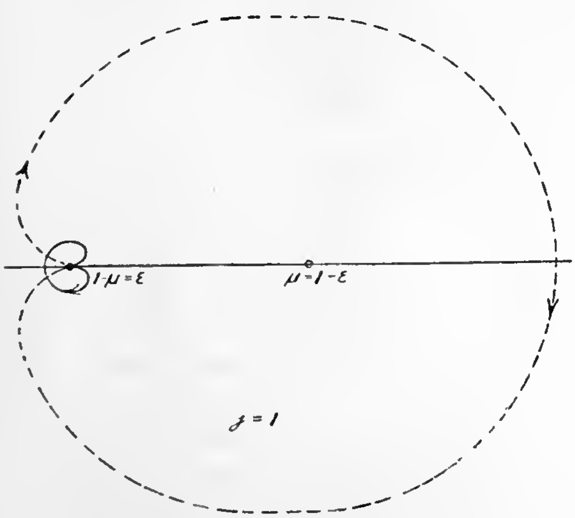


FIG. 18.

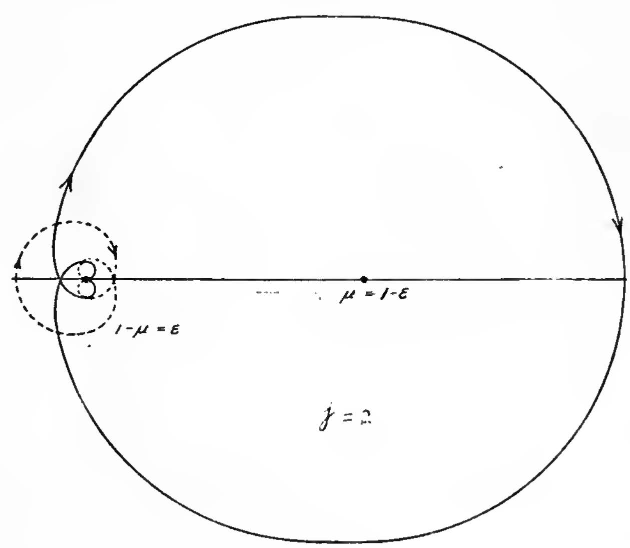


FIG. 19.

are approximately realized when μ is small. When μ increases, the loops which surround $1-\mu$ diminish in size and preserve their approximate forms, while those which surround μ approach circles whose radii are $j^{2/3}$, where $2j\pi$ is the period, and whose centers are at μ. For $\mu=1-\epsilon$, where ϵ is very small, the orbits have the form shown in Figs. 18 and 19. There are, of course, closed orbits of ejection from both of the finite bodies.

233. Periodic Orbits Related to Closed Orbits of Ejection.—There are periodic orbits passing near one of the finite bodies of which the closed orbits of ejection are the limits. Suppose $\mu=0$ and consider the motion of the infinitesimal body with respect to the finite body $1-\mu$. Let the infinitesimal body cross the x-axis perpendicularly at $t=0$ near the body $1-\mu$, and let the initial velocity be determined so that the period is 2π, or a multiple of 2π. Then the motion with respect to the rotating axes is periodic,

the orbit is symmetrical with respect to the x-axis, and crosses it perpendicularly at the half period. The limits of these orbits, as the nearest approach to $1-\mu$ becomes zero, are the closed orbits of ejection.

The orbits under consideration exist for initial motion near the finite body in both the positive and the retrograde directions, but in both cases the motion in the remote parts of the orbits when referred to rotating axes is in the retrograde direction. Therefore, those in which the motion in the vicinity of the finite body is direct have loops, while the others do not. The character of the two classes of orbits for periods 2π and 4π are shown in Figs. 20 and 21. There are, of course, orbits of a similar character which are symmetrically opposite with respect to the y-axis.

Suppose the initial conditions for one of the periodic orbits in question when $\mu=0$, are $x(0)=a$, $x'(0)=0$, $y(0)=0$, $y'(0)=b$, and represent the coördinates for this solution by x_0, x_0', y_0, and y_0'. The distance a is small and the orbit does not pass through the point (1, 0) occupied by μ.

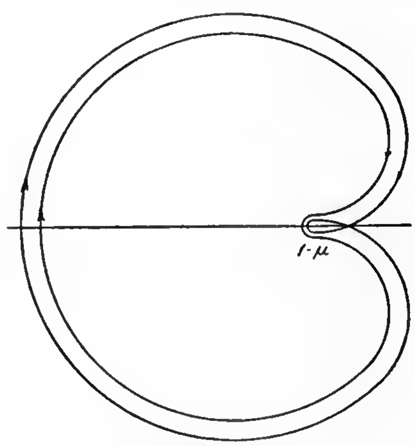


Fig. 20.

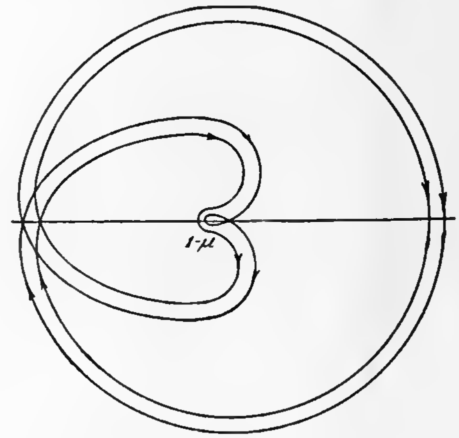


Fig. 21.

Now suppose μ is distinct from zero and that the initial conditions are $x(0)=a+\alpha$, $x'(0)=0$, $y(0)=0$, $y'(0)=b+\beta$. Let the variable τ be introduced by $t=\tau(1+\delta)$, where δ is an undetermined parameter. Then the solutions of the differential equations of motion, which are regular functions of the coördinates and δ in the vicinity of $x=x_0$, $x'=x_0'$, $y=y_0$, $y'=y_0'$, $\delta=0$ for $0 \gtreqless \tau \leqq j\pi$ (j an integer), can be written in the form

$$\left.\begin{aligned} x&=x_0+\xi\ =x_0(\tau)+p_1(\alpha,\beta,\delta,\mu;\tau), & y&=y_0+\eta\ =y_0(\tau)+p_3(\alpha,\beta,\delta,\mu;\tau),\\ x'&=x_0'+\xi'=x_0'(\tau)+p_2(\alpha,\beta,\delta,\mu;\tau), & y'&=y_0'+\eta'=y_0'(\tau)+p_4(\alpha,\beta,\delta,\mu;\tau),\end{aligned}\right\} \quad (66)$$

where $p_1, \ldots, p_4$ are power series in α, β, and δ, vanishing with α, β, and δ. Moreover, the moduli of α, β, and δ can be taken so small that the series converge for $0 \leqq \tau \gtreqless j\pi$, where j is any integer.

Sufficient conditions that the orbit for μ distinct from zero shall be periodic with the period $2j\pi$ are

$$p_2(\alpha,\beta,\delta,\mu;j\pi)=0, \qquad p_3(\alpha,\beta,\delta,\mu;j\pi)=0. \qquad (67)$$

The problem is to show that α, β, and δ can be determined so that these equations shall be satisfied for an arbitrary μ sufficiently small. In the problem for $\mu=0$, either a or b can be taken arbitrarily when the other is determined in terms of j except for sign; one sign belongs to the direct and the other to the retrograde orbit. It will be supposed that a is the arbitrary. Therefore it may be supposed that it absorbs the undetermined α and leaves only two parameters in (67) besides the arbitrary μ.

Equations (67) can be solved uniquely for β and δ as power series in μ, vanishing with μ, provided

$$\Delta=\left|\begin{matrix}\frac{\partial p_2}{\partial\beta}, & \frac{\partial p_2}{\partial\delta}\\ \frac{\partial p_3}{\partial\beta}, & \frac{\partial p_3}{\partial\delta}\end{matrix}\right|_{\beta-\delta=\mu=0} \tag{68}$$

is distinct from zero. Before this determinant is formed μ can be put equal to zero, and therefore Δ depends only upon the two-body problem. Before the second column is formed β can be put equal to zero. When $\mu=\beta=0$ the period in t is $2j\pi$; hence at the half period the infinitesimal body is on the x-axis and the value of x is an even function of $t-j\pi$. Now the parameter δ serves only to vary the period in t (keeping it fixed in τ), and is therefore equivalent to varying t from the half period. When t is near $j\pi$, δ can be determined so that $t-j\pi=j\pi(1+\delta)-j\pi$ consistently with the definition of τ. Therefore $p_2(0,\ 0,\ \delta,\ \mu;\ j\pi)$ is an even function of δ, and consequently $\partial p_2/\partial\delta=0$ for $\beta=\delta=\mu=0$. Hence the determinant δ becomes

$$\Delta=\frac{\partial p_2}{\partial\beta}\,\frac{\partial p_3}{\partial\delta}. \tag{69}$$

The second factor is distinct from zero because, except for a constant factor, it is the derivative of y with respect to t at $\tau=j\pi$, and this derivative is distinct from zero.

In considering the first factor of (69) the parameter δ can be put equal to zero. If ξ and η represent the coördinates referred to fixed axes, the expression for x becomes

$$x=\xi\cos t+\eta\sin t.$$

Therefore the value of x' at $t=j\pi$ is

$$x'=(-1)^j[\xi'+\eta].$$

The problem is reduced to finding whether or not the expression

$$(-1)^j\frac{\partial x'}{\partial\beta}=\frac{\partial\xi'}{\partial\beta}+\frac{\partial\eta}{\partial\beta} \tag{70}$$

is zero under the assumed conditions.

The expressions for ξ, ξ', and η, as given by the two-body problem, are

$$\xi=\bar{a}\,[\cos E-e],\qquad \xi'=-\bar{a}\sin E\frac{dE}{dt}=-\frac{\omega\bar{a}\sin E}{1-e\cos E},\qquad \eta=\bar{a}\sqrt{1-e^2}\sin E,$$

where E is the eccentric anomaly, $\bar{a}$ is the major semi-axis of the orbit, and ω is the mean angular motion in the orbit. Since $\sin E=0$ and $\cos E=-1$ for $t=j\pi$ and $\beta=0$, it follows that

$$\frac{\partial\xi'}{\partial\beta}=-\frac{\omega\bar{a}}{1+e}\frac{\partial E}{\partial\beta},\qquad \frac{\partial\eta}{\partial\beta}=\bar{a}\sqrt{1-e^2}\,\frac{\partial E}{\partial\beta}.$$

Therefore (70) is not zero unless $\partial E/\partial\beta$ is zero. But it is found from Kepler's equation that

$$\frac{\partial E}{\partial\beta}=\frac{j\pi}{1-e}\frac{\partial\omega}{\partial\beta}.$$

It follows from the properties of the two-body problem that at $t=0$

$$\bar{a}(1-e)=a,\qquad \bar{a}=\omega^{-2/3},\qquad (b+\beta)^2=\frac{1+e}{\bar{a}(1-e)}. \tag{71}$$

From these equations it is found that

$$\frac{2}{1-e}=a[\omega^{2/3}+(b+\beta)^2],\qquad \frac{\partial\omega}{\partial\beta}=-3b\omega^{1/3}\neq 0.$$

Therefore neither factor of the right member of Δ is zero, and consequently the solution of (67) for β and δ as power series in μ, vanishing with μ, is unique. When the results of the solution of (67) are substituted in (66), the expressions for x, x', y, and y' become power series in μ alone (α having been taken equal to zero) and they are periodic with the period $2j\pi$.

When $\mu=0$ the limits, as a approaches zero, of the periodic orbits which are being considered are the closed orbits of ejection. There are two families, depending upon the sign of b, which have the same limit. Now when μ is distinct from zero the expressions for the coördinates are expansible as power series in μ, the parts independent of μ are the periodic orbits for $\mu=0$, and the series converge, for $|\mu|$ sufficiently small, while t runs through at least half a period. Therefore the coördinates for any value of t are continuous functions of μ. Since the solutions were developed as power series in α they are continuous functions of a for any t and μ if a is distinct from zero. But in the variables of Levi-Civita it is not necessary to make any restrictions on the initial conditions. The coördinates are in all cases uniformly continuous functions of the initial conditions for all μ, and therefore the limits of the periodic orbits under discussion as a becomes zero are the closed orbits of ejection, and this holds not only for μ equal to zero but also for all μ sufficiently small.

234. Periodic Orbits having Many Near Approaches.—The orbits considered in the preceding article are characterized by the fact that, at least for small values of μ, after the infinitesimal body leaves the point nearest $1-\mu$ it continually recedes until the mid-period, which is a multiple of π, and then returns symmetrically. Orbits will now be considered in which the infinitesimal body recedes from and returns toward μ many times before they re-enter.

Suppose μ is zero and that the infinitesimal body is started near $1-\mu$ on, and perpendicularly to, the line joining $1-\mu$ and μ; and suppose the initial conditions are such that its period is commensurable with 2π without being a multiple of 2π. Let the period be

$$P=2\pi\frac{p}{q}, \tag{72}$$

where p and q are relatively prime integers. Then the motion with respect to the rotating axes is periodic with the period

$$T=Pq=2\pi p. \tag{73}$$

In this period the infinitesimal body runs through q of its periods with respect to fixed axes and the movable axes make p rotations.

Now suppose that μ is distinct from zero and that the initial conditions are given slight variations, but of such a character that the infinitesimal body is started at right angles to the line joining the finite bodies. A new independent variable τ and a parameter δ are introduced by the relation $t=\tau(1+\delta)$. The solutions can be developed as power series in μ, δ, and the increments to the initial conditions, and the moduli of these quantities can be taken so small that these series converge for $\tau \leqq T/2$. Then the conditions that the solution shall be periodic are that the orbit shall cross the x-axis perpendicularly at $\tau=T/2$. These conditions have the form (67) and all of the properties of (67) which were used in proving the existence and character of their solution. Therefore, the periodic orbits which are in question exist.

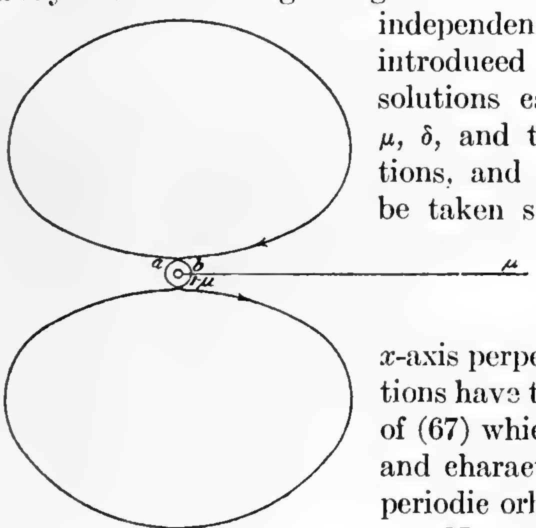


FIG. 22.

Now suppose that the initial distance from $1-\mu$, which was arbitrary, is made to approach zero as a limit. During this approach to zero the distances to the other near apses vary, but there is no apparent reason why all these apsidal distances should vanish at the same time. In fact, from the lack of symmetry it is doubtful whether any two of them are simultaneously zero.

The simplest orbits of the type under consideration are those for which $p=1$, $q=2$. Their general form for retrograde motion in the vicinity of the finite body $1-\mu$ is shown in Fig. 22. If the distance from $1-\mu$ to a becomes zero, the orbit of ejection discovered by Darwin is obtained. The question whether the distance from $1-\mu$ to b becomes zero is one that is hard to answer. Certainly it can not be answered affirmatively with complete rigor by numerical experiments, though the existence of certain classes of periodic orbits can be proved in this way. If, for perpendicular projection from a given point on the x-axis with a certain speed, the next crossing of the x-axis is at an angle which is greater than $\pi/2$; and if, for a perpendicular projection from the same point with a different speed, the next crossing is at an angle less than $\pi/2$, then, from the analytic continuity, it can be inferred that there is an intermediate speed at which the crossing will be perpendicular. But in the present case these conditions are not present, and all that can be said is that when the distance from $1-\mu$ to a vanishes, the distance from $1-\mu$ to b is small, and the approach to $1-\mu$ is almost exactly along the x-axis. This is, of course, to be expected from the nature of these orbits.

CHAPTER XVI.

SYNTHESIS OF PERIODIC ORBITS IN THE RESTRICTED PROBLEM OF THREE BODIES

235. Statement of Problem.—In the problem of two bodies there are circular orbits whose dimensions range from infinitely great to infinitely small. They form a continuous series geometrically and their coördinates are continuous functions of the various parameters by which they may be defined. There are orbits in which the direction of motion is forward, and others in which it is retrograde. The two series are identical only when the orbits are infinitely great and when they are infinitely small.

In the restricted problem of three bodies* the orbits which are analogous to the circular orbits in the problem of two bodies are those which revolve around one or both of the finite bodies and which re-enter, when referred to rotating axes, after one synodical revolution. Those inclosing but a single finite body were treated in Chapter XII, and it was shown there that the deviations from uniform circular motion are due to the attraction of the second finite mass. Those orbits which revolve around both finite masses, and which are analogous to circular orbits, were treated in Chapter XIII, and it was shown there that the deviations from uniform circular motion are due to the fact that the finite masses are separated by a finite distance.

The problem of the present chapter is to trace, so far as possible, a continuous series of orbits from those inclosing both finite masses and having infinitely great dimensions to those revolving around the two finite masses separately in orbits of infinitesimal dimensions. There are no difficulties for very great or for very small orbits, but since in some way the infinitely great are eventually divided into two series which become infinitely small, it is clear that there is a region in which the resemblance to the two-body problem is very remote. The difficulties arise in following the orbits through these critical forms.

The method of treatment is that of analytic continuation of the solutions with respect to the parameters upon which they depend. The differential equations which define the motion are, when referred to rotating axes and in canonical units,

$$\left.\begin{aligned} &\frac{d^2x}{dt^2}-2\frac{dy}{dt}=x-\frac{(1-\mu)(x+\mu)}{r_1^3}-\mu\frac{(x-1+\mu)}{r_2^3},\\ &\frac{d^2y}{dt^2}+2\frac{dx}{dt}=y-(1-\mu)\frac{y}{r_1^3}-\mu\frac{y}{r_2^3},\\ &r_1^2=(x+\mu)^2+y^2, \qquad r_2^2=(x-1+\mu)^2+y^2. \end{aligned}\right\} \qquad (1)$$

* The restricted problem of three bodies is that in which there are two finite bodies and one infinitesimal, the finite bodies revolving in circles.

These equations admit Jacobi's integral

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=x^2+y^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}-C. \tag{2}$$

The general solutions of (1) can be written in the form

$$x=f_1(\alpha_1, \ldots, \alpha_4, \mu;\ t), \qquad y=f_2(\alpha_1, \ldots, \alpha_4, \mu;\ t), \tag{3}$$

where $\alpha_1, \ldots, \alpha_4$ are the initial values of x, y and their first derivatives. The conditions that the solutions (3) shall be periodic eliminate three of the four α_j, and the periodic solutions have the form

$$x=F_1(\alpha, \mu;\ nt), \qquad y=F_2(\alpha, \mu;\ nt), \tag{4}$$

where α is one of the α_j, or a function of them (for example, the Jacobian constant C), and n is a constant depending on α and μ and so associated with t that the period is $2\pi/n$. The problem under consideration is to follow the solutions as α, μ, or n varies through its possible range of values. It will be found convenient in the discussion to use sometimes one and sometimes another of these parameters as independent. The correspondence between them is not one-to-one, so that a series of orbits may branch at a certain point with respect to one of them and not with respect to another. The functions in question are highly transcendental and it has not been found possible to follow them with complete rigor through branch-points and infinities by direct processes. But the fact that the orbits may have a branching with respect to one parameter and not with respect to another makes it possible sometimes to establish the existence of critical forms indirectly.

There are orbits whose coördinates are complex and whose periods are real. With varying values of the parameter α they may become real. This does not arise in the problem of two bodies. It makes it necessary to include in the discussion certain orbits which are complex for the values of the parameters in terms of which they are most conveniently expressed.

Sir George Darwin discovered several classes of periodic orbits by numerical experiments.* Burrau found the limiting form of the orbits of one family of oscillating satellites by the same process.† Many other families have been discovered by computations carried out in connection with this work. All these examples illustrate the theories to be set forth here and place them on a solid basis at points where they fall short of complete rigor because of the difficulty of following orbits by analytical processes through critical forms.

Before taking up the direct synthesis, some of the properties of the periodic orbits which have been previously given will be enumerated and some additional ones will be derived.

236. Periodic Satellite and Planetary Orbits.—It was shown in Chapters XII and XIII that there are periodic orbits encircling each of the finite bodies separately, and others encircling both of them together, which are closed

* Acta Mathematica, vol. 21 (1897).

† Astronomische Nachrichten, vol. 135 (1894), No. 3230; and *ibid.*, vol. 136 (1894), No. 3251.

after one synodical revolution. The direction of motion of the finite bodies is considered as being forward, and the opposite is considered as being retrograde. For small values of the parameter in terms of which the solutions are developed, corresponding to very small and very large orbits respectively, there are three classes in which the motion is direct and three in which it is retrograde. Only one class of the direct and one of the retrograde orbits is real for small values of the parameters. Darwin's computations show that at least in some cases the complex orbits may become real.

One of the most important properties of the orbits in question is that they all cross the line joining the finite bodies perpendicularly at every half period. If the parameters upon which the periodic orbits depend are varied, these properties persist, for the solutions are expansible in integral or fractional powers of the parameters, and the property in question is a consequence of the character of their coefficients. Therefore, the whole series of orbits from infinite to infinitesimal dimensions will possess this property, unless indeed they branch into two series which are symmetrical with respect to the line joining the finite bodies.

If the coördinates of a real periodic orbit are analytic in a parameter, then the orbit can not disappear without becoming identical with another periodic orbit;* and a complex orbit can not become real without becoming identical with another complex orbit, the two becoming real as they become identical. Real orbits disappear and appear in pairs with the variation of the parameters in terms of which their coordinates are analytic functions.

237. The Non-Existence of Isolated Periodic Orbits.—Appeal will be made to the numerical computations in establishing the existence of periodic orbits near certain critical forms. In order that this procedure may be justified, it is necessary to prove that the orbits which they have shown to exist are not isolated examples which exist only for special values of the masses and the other parameter on which they depend.

Suppose that for $\mu=\mu_0$ equations (1) admit the periodic solution

$$x=F_1\,(\mu_0;\ n_0t), \qquad y=F_2(\mu_0;\ n_0t), \tag{5}$$

where the period is $2\pi/n_0$. The initial conditions are

$$x(0)=x_0, \quad x'(0)=x'_0, \quad y(0)=y_0, \quad y'(0)=y'_0. \tag{6}$$

The initial conditions determine the value of the constant of the Jacobian integral

$$C_0=-(x'^2_0+y'^2_0)+x_0^2+y_0^2+\frac{2(1-\mu_0)}{r_1^{(0)}}+\frac{2\mu_0}{r_2^{(0)}}. \tag{7}$$

The orbit in question will not pass through one of the finite bodies, for then it would not be strictly periodic, as was explained in §222. It will not have any infinite branches, for then it could not have a finite period.

* Les Méthodes Nouvelles de la Mécanique Céleste, vol. I, p. 83.

There are two problems to be considered: (A) To determine whether or not periodic orbits exist for $\mu=\mu_0+\lambda$, $C=C_0$, which reduce to (5) for $\lambda=0$; and (B) to determine whether or not periodic orbits exist for $\mu=\mu_0$, $C=C_0+\gamma$, which reduce to (5) for $\gamma=0$. If periodic orbits exist when the parameters are varied separately, they also exist when they are varied simultaneously.

(A) Let $\mu=\mu_0+\lambda$ and take as initial conditions

$$x(0)=x_0+\alpha,\qquad x'(0)=x'_0+\alpha',\qquad y(0)=y_0+\beta,\qquad y'(0)=y'_0+\beta'. \quad (8)$$

Since the periodic orbit (5) does not pass through a singular point of the differential equations, the solutions can be developed as series of the form

$$\left.\begin{aligned} x &=F_1(\mu_0;\ n_0t)+\alpha\ +p_1(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ t),\\ x' &=F_1'(\mu_0;\ n_0t)+\alpha'+p_2(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ t),\\ y &=F_2(\mu_0;\ n_0t)+\beta\ +p_3(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ t),\\ y' &=F_2'(\mu_0;\ n_0t)+\beta'\ +p_4(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ t),\end{aligned}\right\} \quad (9)$$

where p_1 . . . , p_4 are power series in α, α', β, β', and λ which vanish identically with these parameters and are zero for $t=0$. Moreover, if any finite T is taken in advance the moduli of these parameters can be taken so small that p_1, . . . , p_4 converge for all $0\gtreqless t\leqq T$.

The integral (2) can be written

$$F(p_1,\ .\ .\ .\ ,\ p_4,\ \alpha,\ \alpha',\ \beta,\ \beta',\ \lambda)-F(0,\ .\ .\ .\ ,\ 0,\ \alpha,\ \alpha',\ \beta,\ \beta',\ \lambda)=0, \quad (10)$$

where F is a power series in p_1, . . . , p_4, α, α', β, β', and λ. Equation (10) is identically satisfied by $p_1=\ .\ .\ .\ =p_4=0$.

Sufficient conditions that the solution (9) shall be periodic with the period $P=2\pi/n_0$ are

$$\left.\begin{aligned} &p_1(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ P)=0, \qquad && p_3(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ P)=0,\\ &p_2(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ P)=0, \qquad && p_4(\alpha,\ \alpha',\ \beta,\ \beta',\ \lambda;\ P)=0.\end{aligned}\right\} \quad (11)$$

It follows from the form of (2) that unless x_0' is zero, equation (10) can be solved for p_2 as a power series in p_1, p_3, p_4, α, α', β, β', and λ, vanishing identically for $p_1=p_3=p_4=0$. If x_0' were zero and y_0' were not zero the solution would be made for p_4 instead of for p_2. If x_0' and y_0' were both zero the origin of time would be shifted so that at least one of them would be distinct from zero. This is always possible unless they are identically zero. But they are identically zero only when the infinitesimal body is at an equilibrium point, and it has been shown in Chapters V and IX that these solutions are not isolated. Consequently, it may be supposed that (10) is solved for p_4 in the form

$$p_4=\varphi(p_1,\ p_2,\ p_3;\ \alpha,\ \alpha',\ \beta,\ \beta',\ \lambda),\qquad \varphi(0,\ 0,\ 0;\ \alpha,\ \alpha',\ \beta,\ \beta',\ \lambda)\equiv 0. \quad (12)$$

Therefore, if p_1, p_2, and p_3 are periodic with the period P, then by virtue of this relation p_4 is also periodic with the period P. Hence the fourth equation of (11) is redundant and may be suppressed.

Now consider the solution of the first three equations of (11) for α, α', and β in terms of λ. The parameter β' is superfluous and may be taken equal to zero. This amounts to a definite determination of the initial time. Since (5) is a real periodic solution the coefficients of p_1, p_2, and p_3 are real. Equations (11) are not satisfied by $\lambda=0$ with α, α', and β arbitrary, for then all orbits would be periodic with the same period when $\mu=\mu_0$. Similarly, they are not satisfied by $\lambda=0$ unless all three of the parameters α, α', and β are zero. They are not satisfied by $\alpha=\alpha'=\beta=0$ and λ arbitrary, for then fixed initial conditions would give a periodic orbit for all distribution of mass between the finite bodies. This is certainly not true for μ near zero. Therefore, the three equations can be solved for α, α', and β as power series in integral or fractional powers of λ. If the powers are integral the solution will be unique and real for both positive and negative values of λ. If the powers are odd fractional, there will be a single real solution for both positive and negative values of λ. If the powers are even fractional, there will be two real solutions for λ either positive or negative, depending on the signs of the coefficients of certain terms, and all the solutions will be complex for λ negative or positive, depending upon the signs of the same coefficients. Hence in all cases there are real periodic solutions for small values of λ which reduce to (5) for $\lambda=0$.

(B) Let $C=C_0+\gamma$ and take the initial conditions (8). Also let

$$t=(1+\delta)\tau, \tag{13}$$

where δ is an arbitrary parameter and τ a new independent variable. The solutions in this case have the form

$$\left.\begin{aligned} x &=F_1(\mu_0;\ n_0\tau)+\alpha+p_1(\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \delta;\ \tau),\\ x' &=F_1'(\mu_0;\ n_0\tau)+\alpha'+p_2(\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \delta;\ \tau),\\ y &=F_2(\mu_0;\ n_0\tau)+\beta+p_3(\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \delta;\ \tau),\\ y' &=F_2'(\mu_0;\ n_0\tau)+\beta'+p_4(\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \delta;\ \tau), \end{aligned}\right\} \tag{14}$$

where $p_1, \ldots, p_4$ are power series in α, α', β, β', γ, and δ. The parameter γ is determined in terms of α, α', β, β', and δ by equation (2) at $\tau=0$. Therefore it may be omitted from $p_1, \ldots, p_4$. Or, if y_0' is not zero, equation (2) determines β' as a power series in α, α', β, γ, and δ, vanishing with these quantities. It will be supposed that β' is eliminated. It follows from the integral that the last equation corresponding to (11) is redundant, and it will be suppressed. The constant α' is superfluous and may be taken equal to zero. Then the conditions that the solution (14) shall be periodic with the period P in τ are

$$p_1(\alpha,\ \beta,\ \gamma,\ \delta;\ P)=0,\quad p_2(\alpha,\ \beta,\ \gamma,\ \delta;\ P)=0,\quad p_3(\alpha,\ \beta,\ \gamma,\ \delta;\ P)=0. \tag{15}$$

Consider the solution of equations (15) for α, β, and δ in terms of γ. They are not satisfied unless all four of these parameters are zero, and consequently solutions for α, β, and δ as power series in integral or fractional powers of γ exist, and the circumstances under which they are real are strictly analogous to those of Case (A).

238. The Persistence of Double Orbits with Changing Mass-Ratio of the Finite Bodies.—In Chapter XI, p. 359, it was stated that Darwin's computations show that the two orbits which are complex for small values of m, or for large values of the Jacobian constant C, unite and become real for a certain value of C. In this computation the ratio of the masses of the finite bodies was 10 to 1. The question naturally arises whether a corresponding double periodic orbit exists for other ratios of the finite masses.

The conditions for a double periodic orbit will first be developed. Suppose

$$x = f_1(t), \qquad x' = f_1'(t), \qquad y = f_2(t), \qquad y' = f_2'(t), \tag{16}$$

is a periodic solution of equations (1) for $\mu = \mu_0$, having the period P. All orbits except those around the equilateral triangular points, which will be given special consideration in §242, are symmetrical with respect to the x-axis. In them the origin of time can be so chosen that the initial conditions are

$$x(0) = x_0, \qquad x'(0) = 0, \qquad y(0) = 0, \qquad y'(0) = y'_0, \tag{17}$$

and from (2), $C = C_0$.

Now consider a solution with the initial conditions

$$x(0) = x_0 + \alpha, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = y'_0 + \beta, \quad C = C_0 + \gamma. \tag{18}$$

The constant β can be expressed in terms of α and γ by means of (2), and it will be supposed that β is eliminated by this relation. Let a new independent variable τ be defined by

$$t = (1 + \delta)\tau, \tag{19}$$

where δ is a parameter as yet undetermined. Then the solution of (1) can be developed in the form

$$\begin{aligned} x &= p_1(\alpha, \gamma, \delta;\ \tau), & x' &= p_2(\alpha, \gamma, \delta;\ \tau), \\ y &= p_3(\alpha, \gamma, \delta;\ \tau), & y' &= p_4(\alpha, \gamma, \delta;\ \tau), \end{aligned} \tag{20}$$

where $p_1, \ldots, p_4$ are power series in α, γ, and δ.

The conditions that (20) shall be a periodic solution with the period P in τ are

$$p_2(\alpha, \gamma, \delta;\ P/2) = 0, \qquad p_3(\alpha, \gamma, \delta;\ P/2) = 0. \tag{21}$$

It will now be shown that δ can be eliminated by means of the second of these equations. Suppose equations (1) with t as the independent variable and initial conditions (17) are integrated as power series in δ. The terms independent of δ will be periodic with the period P. Non-periodic terms will enter only when the terms in the right members at some stage of the integration contain terms with the period P. Then t times periodic terms will appear in the solution, and terms multiplied by t^2 and higher powers of t will not enter until later stages of the integration. Terms of this character

will actually arise, for otherwise the solution would be periodic for all δ; that is, the coördinates would be constants with respect to t. The expression for $y=p_3$ is an odd function of t with the initial conditions (18). Therefore the term in t is multiplied by an even function, that is, a cosine term which does not vanish at $t=P/2$. Therefore p_3 carries δ to the first degree and this parameter can be eliminated, giving

$$P(\alpha, \gamma)=0, \tag{22}$$

where P is a power series in α and γ vanishing for $\alpha=\gamma=0$.

Suppose γ is taken as the independent parameter and that (22) is solved for α in terms of γ. A necessary and sufficient condition that (16) be a double periodic solution with respect to the Jacobian constant C is that

$$\frac{\partial P}{\partial \alpha}=P_1(\alpha, \gamma)=0 \tag{23}$$

for $\alpha=\gamma=0$. Suppose this condition is satisfied.

Now suppose $\mu=\mu_0+\lambda$ and consider the question of the existence of a double periodic solution for this value of μ. The double periodic solution will exist provided the equation corresponding to (23) is satisfied. Suppose the initial value of x is

$$x(0)=x_0+\sigma+\alpha, \tag{24}$$

where σ is a parameter which remains to be determined. The steps corresponding to those leading up to (23) can be taken in this case, and the equation corresponding to (23) becomes

$$P_2(\sigma, \lambda)=0 \tag{25}$$

for $\alpha=\gamma=0$, where P_2 is a power series in σ and λ vanishing for $\sigma=\lambda=0$. The series P_2 contains terms in σ alone; otherwise, not only would every orbit whose initial conditions were

$$x(0)=x_0+\sigma, \qquad x'(0)=0, \qquad y(0)=0, \qquad y'(0)=y'_0, \tag{26}$$

where $x(0)$ is arbitrary, be a periodic orbit, but it would be a double periodic orbit. Therefore (25) can be solved for σ as a power series in λ, or in some fractional power of λ, vanishing with λ, the multiplicity of the solution being equal to the degree of the term of lowest degree in σ alone. Hence, in the analytic sense, if there is a double periodic orbit for $\lambda=0$ (that is, for $\mu=\mu_0$), then there is a double periodic orbit for every value of $\mu=\mu_0+\lambda$.

If the solution of (25) has the form

$$\sigma=\lambda p(\lambda), \tag{27}$$

where $p(\lambda)$ is a power series in λ, then there is a single double solution of the series for every $|\lambda|$ sufficiently small. But if the solution is in $\lambda^{\frac{1}{2}}$, then there are two real double solutions when λ has one sign and none when it has the other. That is, for $\lambda=0$ two real double periodic orbits unite and disappear

by becoming complex. And in general, *double periodic solutions appear or disappear in pairs, which become identical for certain values of μ*, a result analogous to Poincaré's theorem respecting the appearance and disappearance of real periodic orbits.

Now consider the real periodic satellite orbits which are re-entrant after a single synodical revolution. Darwin's computation shows that for the mass ratio of 10 to 1 there is one, and but one, real double periodic orbit in which the motion is direct. Hence, there is no other double periodic orbit with which it could unite to disappear for any value of μ.

The same result also is a consequence of the analysis of Chapter XI, where it was shown that for $\mu=0$, and therefore for all μ, there are but three real and complex orbits of the type in question. From the fact that there are only three periodic orbits of the type in question it follows that there can not be more than one direct double periodic orbit; and from its existence for $\mu={}^1/_{11}$, $1-\mu={}^{10}/_{11}$, it follows that there is one direct double periodic orbit for all values of μ from zero to unity. The orbits of inferior planets differ from those of satellites only in the ratio of the masses. Therefore, there are also double periodic orbits of inferior planets in which the motion is in the forward direction.

When one of the masses which revolve in circles becomes zero, those orbits around the other which are complex for small periods are complex for all periods from zero to infinity. In this case there are no double orbits except those of infinite and infinitesimal dimensions. The question arises as to the character of the double orbits for very small values of the second finite mass. Consider the totality of real circular orbits around one of the finite bodies when the mass of the other one is zero. As the second mass becomes finite, the periodic orbits are continuous deformations of the circular orbits with the exception of that circular orbit whose period is 2π. It passes through the point where the second mass becomes finite, and the force function for this orbit has a discontinuity. This means that the orbit itself has a discontinuity. It is conjectured that the complex orbits have corresponding discontinuities; that when the second finite mass is very small there are three real orbits about the larger mass in which the motion is in the forward direction and which have a mean distance near unity and a period near 2π; that there are three corresponding real orbits of small dimensions about the smaller mass; and, finally, that there are three similar orbits of the nature of superior planets. For increasing values of the Jacobian constant two of the three in each case unite and form the double orbits. Consequently, if this conjecture is correct, and if the three types of double orbits are followed as one of the finite bodies approaches zero as a limit, the one around the larger finite mass and the one around both finite masses approach the unit circle, and the one around the smaller finite mass approaches zero dimensions.

The question of direct double periodic orbits was put to numerical test for $\mu=\frac{1}{2}$. In all, 53 orbits were computed for various values of C, starting with $C=3.58$ and ending with $C=3.086$. For $C=3.58$ there were two direct periodic orbits which were geometrically much alike, and which intersected the x-axis near $-.785$. For smaller values of C the corresponding periodic orbits were more nearly identical. For $C=3.086$ they were sensibly identical. The computation for this value of C gave the results set forth in the following table (Fig. 23):

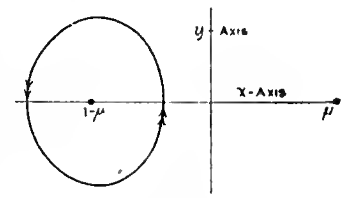


FIG. 23.

$\mu=\frac{1}{2}$, $C=3.086$, Double periodic orbit.					$\mu=\frac{1}{2}$, $C=3.086$, Double periodic orbit.				
t	x	y	x'	y'	t	x	y	x'	y'
0	−.7740	0	0	−1.193	.40	−.5643	−.3322	.756	− .317
.05	−.7693	−.0592	.187	−1.168	.50	−.4893	−.3514	.738	− .072
.10	−.7556	−.1161	.357	−1.097	.60	−.4176	−.3470	.695	+ .159
.15	−.7341	−.1684	.498	− .990	.70	−.3510	−.3197	.633	.385
.20	−.7064	−.2148	.605	− .862	.80	−.2917	−.2698	.547	.612
.25	−.6741	−.2545	.679	− .724	.90	−.2429	−.1974	.420	.833
.30	−.6389	−.2872	.725	− .585	1.00	−.2097	−.1048	.235	1.009
.35	−.6020	−.3131	.749	− .448	1.10	−.1978	+.0007	−.003	1.079

There are also three retrograde periodic orbits, only one of which is real for small values of the parameters in terms of which their coördinates are developed. The question arises as to whether the retrograde complex periodic orbits unite and become real. In order to test the question by numerical experiment, 20 orbits were computed. For $\mu=\frac{1}{2}$ and $C=3.75$, five orbits were computed, starting with various values of x_0 and determining y_0' so that C should be 3.75. The correspondence between x_0 and the angle φ at which the orbit crossed the x-axis after a half revolution is given in the accompanying table:

$\mu=\frac{1}{2}$, $C=3.75$, Motion retrograde.					
x_0	−.780,	−.730,	−.700,	−.650,	−.600
φ	84°,	85°,	86°,	91°30′,	99°

This shows, especially when taken in conjunction with more extensive computations for other values of C, that there is only one retrograde periodic orbit about each of the finite bodies separately for $C=3.75$. Since only a few retrograde periodic orbits have heretofore been given, the coördinates for the approximately periodic orbit defined by $\mu=\frac{1}{2}$, $C=3.75$, $x_0=-.650$ will be given for enough values of t to show its geometrical characteristics (Fig. 24).

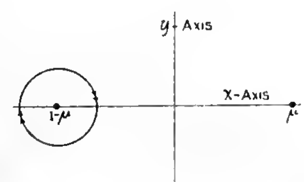


FIG. 24.

$\mu=\frac{1}{2}$, $C=3.75$, Retrograde periodic orbit.				
t	x	y	x'	y'
0	−.6500	0	0	2.052
.025	−.6420	.0504	.637	1.943
.050	−.6189	.0955	1.194	1.634
.075	−.5835	.1308	1.611	1.173
.100	−.5398	.1534	1.854	.625
.125	−.4922	.1618	1.920	.048
.150	−.4451	.1561	1.821	− .501
.175	−.4024	.1374	1.582	− .982
.200	−.3670	.1078	1.233	−1.367
.225	−.3414	.0699	.807	−1.637
.250	−.3270	.0266	.336	−1.781
.275	−.3246	−.0184	− .151	−1.800

Eight retrograde orbits were computed for $\mu=\frac{1}{2}$, $C=3.214$. In this case also the existence of but one periodic orbit was indicated, as is shown

$\mu=\frac{1}{2}$, $C-3.214$, Motion retrograde.								
x_0	−1.0000,	− .9000,	− .8000,	− .7700,	− .719,	− .685,	− .650,	− .625
φ	near collision	80°,	83°21′,	84°56′,	87°41′,	91°15′,	96°	100°

by the accompanying table of correspondences between x_0 and φ.

The retrograde orbit defined by $\mu=\frac{1}{2}$, $C=3.214$, $x_0=-.685$, is nearly periodic. Its coördinates for various values of t are given in the following table (Fig. 25):

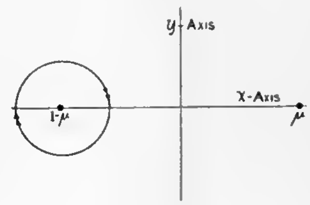


Fig. 25.

$\mu=\frac{1}{2}$, $C=3.214$, Retrograde periodic orbit.				
t	x	y	x'	y'
0	−.6850	0	0	1.872
.025	−.6794	.0461	.446	1.815
.050	−.6630	.0896	.867	1.650
.075	−.6368	.1278	1.221	1.389
.100	−.6026	.1586	1.503	1.053
.125	−.5624	.1802	1.694	.664
.150	−.5187	.1916	1.787	.247
.175	−.4739	.1925	1.782	− .174

$\mu=\frac{1}{2}$, $C=3.214$, Retrograde periodic orbit.				
t	x	y	x'	y'
.200	−.4314	.1831	1.684	− .577
.225	−.3904	.1640	1.502	− .942
.250	−.3559	.1364	1.248	−1.253
.275	−.3285	.1019	.936	−1.497
.300	−.3094	.0622	.583	−1.664
.325	−.2995	.0194	.207	−1.747
.350	−.2991	−.0244	− .174	−1.743

Five retrograde orbits were computed for $\mu=\frac{1}{2}$, $C=2.95$. In this case also the existence of only one periodic orbit was indicated. The correspondence between x and φ is given in the following table:

$\mu=\frac{1}{2}$, $C-2.95$, Motion retrograde.					
x_0	−1.0000,	−.9000,	−.8000,	−.719,	−.625
φ	81°,	79°10′,	82°20′,	89°0′,	103°43′

The orbit defined by $\mu=\frac{1}{2}$, $C=2.50$, $x=-.719$ is nearly periodic. Its coördinates for various values of t are given in the following table (Fig. 26, page 496):

$\mu=\frac{1}{2}$, $C=2.95$, Retrograde periodic orbit.					$\mu=\frac{1}{2}$, $C=2.95$, Retrograde periodic orbit.				
t	x	y	x'	y'	t	x	y	x'	y'
0	−.7190	0	0	1.719	.200	−.5007	.2146	1.716	− .054
.025	−.7149	.0424	.334	1.675	.225	−.4581	.2088	1.678	− .408
.050	−.7025	.0832	.655	1.577	.250	−.4174	.1943	1.570	− .752
.075	−.6824	.1208	.952	1.416	.275	−.3805	.1715	1.392	−1.072
.100	−.6553	.1536	1.212	1.100	.300	−.3486	.1411	1.149	−1.352
.125	−.6222	.1803	1.427	.934	.325	−.3235	.1043	.849	−1.578
.150	−.5844	.2000	1.587	.629	.350	−.3065	.0627	.504	−1.735
.175	−.5434	.2116	1.685	.296	.375	−.2987	.0182	.129	−1.811

Two orbits were also computed for $C=2.75$. The results were similar to those for $C=2.95$. All the results indicate that there is only one real retrograde periodic orbit about each of the finite bodies.

239. Cusps on Periodic Orbits.—The orbits of ejection in a certain sense have cusps at the point of collision with a finite body. But they have been treated in Chapter XIV and require no further comments here.

The coördinates of the infinitesimal body can be expressed as power series in $t-t_1$, if for $t=t_1$ it is at any point for which the differential equations are regular. A necessary condition that the orbit shall have a cusp at $t=t_1$ is that the expressions for both x and y shall have no linear terms in $t-t_1$. It follows that if the orbit has a cusp at $t=t_1$, then $x'(t_1)=y'(t_1)=0$. That is, the body is on a curve of zero relative velocity at $t=t_1$. Suppose its coördinates at $t=t_1$ are x_1 and y_1. Now let

$$x=x_1+\xi, \qquad y=y_1+\eta; \tag{28}$$

then equations (1) become

$$\left.\begin{aligned}
&\frac{d^2\xi}{dt^2}-2\frac{d\eta}{dt}=A_0+A_1\xi+A_2\eta+\ \cdots,\\
&\frac{d^2\eta}{dt^2}+2\frac{d\xi}{dt}=B_0+B_1\xi+B_2\eta+\ \cdots,\\
&A_0=x_1-\frac{(1-\mu)\,(x_1+\mu)}{r_1^3}-\mu\frac{(x_1-1+\mu)}{r_2^3},\\
&B_0=y_1-\frac{(1-\mu)y_1}{r_1^3}-\frac{\mu y_1}{r_2^3},\\
&A_1=1+(1-\mu)\frac{[2(x_1+\mu)^2-y_1^2]}{r_1^5}+\mu\frac{2(x_1-1+\mu)^2-y_1^2]}{r_2^5},\\
&B_1=A_2=3(1-\mu)\frac{(x_1+\mu)y_1}{r_1^5}+3\mu\frac{(x_1-1+\mu)y_1}{r_2^5},\\
&B_2=1+(1-\mu)\frac{[-(x_1+\mu)^2+2y_1^2]}{r_1^5}+\mu\frac{[-(x_1-1+\mu)^2+2y_1^2]}{r_2^5}.
\end{aligned}\right\} \tag{29}$$

The solution of equations (29) as power series in $t-t_1$ with the initial conditions $\xi=\eta=\xi'=\eta'=0$ is

$$\left.\begin{aligned}\xi&=\frac{A_0}{2}(t-t_1)^2+\frac{B_0}{3}(t-t_1)^3+\frac{[-4A_0+A_0A_1+B_0B_2]}{24}(t-t_1)^4+\ldots,\\ \eta&=\frac{B_0}{2}(t-t_1)^2-\frac{A_0}{3}(t-t_1)^3+\frac{[-4B_0+A_0B_1+B_0B_2]}{24}(t-t_1)^4+\ldots,\end{aligned}\right\}\quad(30)$$

The direction cosines of the normal to the curve of zero relative velocity at the point (x_1, y_1) are proportional to A_0 and B_0. Therefore the tangent to the cusp is perpendicular to the curve of zero relative velocity at the point (x_1, y_1).

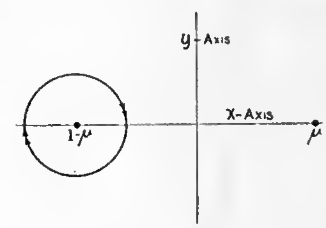


FIG. 26.

Now take a new set of axes (u, v) with origin at (x_1, y_1), with u having the direction of the tangent and v perpendicular to it. If the positive ends of the new axes are chosen so that the cosine and sine of the angle from the positive end of the ξ-axis counted counter-clockwise to the positive end of the u-axis are proportional to A_0 and B_0, and if the positive end of the v-axis is 90° forward counter-clockwise from the positive end of the u-axis, then the equations of transformation are

$$\left.\begin{aligned}u&=A_0\xi+B_0\eta=\tfrac{1}{2}[A_0^2+B_0^2](t-t_1)^2+\\ &\qquad\frac{[-4(A_0^2+B_0^2)+2A_0B_0A_2+A_0^2A_1+B_0B_2]}{24}(t-t_1)^4+\ldots,\\ v&=B_0\xi+A_0\eta=-\tfrac{1}{3}[A_0^2+B_0^2](t-t_1)^2+\\ &\qquad\frac{[A_0B_0(B_2-A_1)+(A_0^2-B_0^2)A_2]}{24}(t-t_1)^4+\ldots.\end{aligned}\right\}\quad(31)$$

The value of u is positive for both positive and negative values of $(t-t_1)$, because the coefficient of $(t-t_1)^2$ can vanish only at an equilibrium point. Therefore the positive end of the u-axis extends from the point (x_1, y_1) into the region of real velocities. The value of v is positive for small negative values of $t-t_1$, and negative for small positive values of $t-t_1$. Therefore, *if motion along the curves of zero relative velocity is taken as positive when the region of real velocity is on the left, the motion in the cusp orbits, in the neighborhood of the cusps, is in the positive direction; that is, the infinitesimal body crosses the tangent to the cusp in the positive direction.* In Fig. 27, C is a curve of zero velocity, P is a cusp, O is the orbit near the cusp, and T is the tangent to the orbit at the cusp.

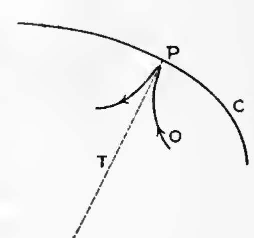


FIG. 27.

Now suppose the orbit having the cusp is a periodic orbit. If it has no other double points than at the cusps, and if it is inside of the curves of zero relative velocity, then it revolves around one of the finite bodies in the positive direction. If it is outside of the curves of zero relative velocity, it revolves with respect to the rotating axes in the retrograde direction. In-

deed, all periodic orbits of superior planets revolve in the retrograde direction with respect to the rotating axes. But the orbits with cusps, if they have no other double points, revolve in the forward direction with respect to fixed axes, because at the cusps they have precisely the forward motion of the rotating axes.

240. Periodic Orbits Having Loops Which Are Related to Cusps.—Suppose the orbit defined by the initial conditions

$$x(0)=x_0, \qquad x'(0)=0, \qquad y(0)=0, \qquad y'(0)=y_0', \qquad C=C_0, \tag{32}$$

is periodic with the period P. Therefore

$$x'(P/2)=0, \qquad y(P/2)=0. \tag{33}$$

Suppose it has a cusp at $t=t_1$, or

$$x(t_1)=x_1, \qquad x'(t_1)=0 \qquad y(t_1)=y_1, \qquad y'(t_1)=0. \tag{34}$$

If the initial conditions are varied in such a way that the orbit remains periodic, its character in the vicinity of the cusp will be changed. The nature of these changes will now be considered. Suppose the initial conditions are

$$x(0)=x_0+\alpha, \quad x'(0)=0, \quad y(0)=0, \quad y'(0)=y_0'+\beta, \quad C=C_0+\gamma; \tag{35}$$

and that

$$t=(1+\delta)\tau, \tag{36}$$

where τ is a new independent variable and δ is an undetermined parameter.

The solutions can be expanded as power series in α, β, and δ. The conditions that they shall be periodic are

$$x'[(1+\delta)P/2]=p_1(\alpha,\ \beta,\ \delta)=0, \qquad y[(1+\delta)P/2]=p_2(\alpha,\ \beta,\ \delta)=0, \tag{37}$$

where p_1 and p_2 are power series in α, β, and δ, vanishing identically with α, β, and δ. Now consider the solution of (37) for β and δ in terms of α, vanishing with α. The solution is always possible either in integral or fractional powers unless the equations are identically satisfied by $\beta=\delta=0$. But this means that all orbits in which the infinitesimal body crosses the x-axis near x_0 with fixed velocity y_0' are periodic, and that they all have the same period. Since the results are analytic in α, the orbits may be continued with respect to α, in the analytic sense, until the place of crossing, x_0, is small. But then the methods of Chapter XIV apply and it is known that the period depends upon x_0. Therefore equations (37) are not identically satisfied by $\beta=\delta=0$, and they can be solved for β and δ in terms of α. The solutions will have the form

$$\beta=\alpha^{1/p}P_1(\alpha^{1/p}), \qquad \delta=\alpha^{1/p}P_2(\alpha^{1/p}), \tag{38}$$

where p is unity if the Jacobian of p_1 and p_2 with respect to β and δ is distinct from zero for $\alpha=\beta=\delta=0$. If p is not unity, it is some other positive integer. In general it will be unity.

In the computations of Darwin γ was taken as the parameter which defines the orbits. The change can be made here because (2) is uniquely solvable for α as a power series in β, γ, and δ unless

$$x_0-\frac{(1-\mu)(x_0+\mu)}{r_1^3}-\frac{\mu(x_0-1+\mu)}{r_2^3}=0, \qquad y_0=0.$$

But these equalities are satisfied only at one of the collinear solution points. The orbits in the vicinity of these points will be omitted from this discussion because they belong to quite another category.

If α is eliminated by means of (2), and β and δ by means of (38), the solutions will be expressed as power series in $\gamma^{1/p}$, and the values of the coördinates at $\tau=t_1$ are

$$\left.\begin{aligned} x(\tau)_{\tau=t_1}&=x_1+\gamma^{1/p}\theta_1, & x'(\tau)_{\tau=t_1}&=0+\gamma^{1/p}\theta_2,\\ y(\tau)_{\tau=t_1}&=y_1+\gamma^{1/p}\theta_3, & y'(\tau)_{\tau=t_1}&=0+\gamma^{1/p}\theta_4,\end{aligned}\right\} \tag{39}$$

where $\theta_1, \ldots, \theta_4$ are power series in $\gamma^{1/p}$.

The expressions for the coördinates in the vicinity of $t=t_1$, satisfying the relations (39), can be expanded as power series in $t-t_1$. The solution is found from equations (29) to be

$$\left.\begin{aligned} x-x_1=\xi=a_1(\tau-t_1)+a_2(\tau-t_1)^2+a_3(\tau-t_1)^3+\cdots,\\ y-y_1=\eta=b_1(\tau-t_1)+b_2(\tau-t_1)^2+b_3(\tau-t_1)^3+\cdots,\end{aligned}\right\} \tag{40}$$

where

$$\left.\begin{aligned} a_1&=\gamma^{1/p}\theta_2, \qquad b_1=\gamma^{1/p}\theta_4,\\ a_2&=\ (1+\delta)b_1+\tfrac{1}{2}(1+\delta)^2A_0,\\ b_2&=-(1+\delta)a_1+\tfrac{1}{2}(1+\delta)^2B_0,\\ a_3&=\tfrac{2}{3}(1+\delta)b_2+\tfrac{1}{6}(1+\delta)^2[A_1a_1+A_2b_1],\\ b_3&=-\tfrac{2}{3}(1+\delta)a_2+\tfrac{1}{6}(1+\delta)^2[B_1a_1+B_2b_1],\\ 1+\delta&=1+\gamma^{1/p}P(\sigma^{1/p}),\end{aligned}\right\} \tag{41}$$

where A_0, A_1, A_2, B_0, B_1, B_2 are given in (29) and P is a power series in $\gamma^{1/p}$

The transformation (31) gives

$$\left.\begin{aligned} u=\gamma^{1/p}(\theta_2A_0+\theta_4B_0)(\tau-t_1)+(a_2A_0+b_2B_0)(\tau-t_1)^2\\ +(a_3A_0+b_3B_0)(\tau-t_1)^3+\cdots,\\ v=\gamma^{1/p}(\theta_4A_0-\theta_2B_0)(\tau-t_1)+(b_2A_0-a_2B_0)(\tau-t_1)^2\\ +(b_3A_0-a_3B_0)(\tau-t_1)^3+\cdots.\end{aligned}\right\} \tag{42}$$

It follows from (41) that for $\gamma=0$

$$u=\tfrac{1}{2}(A_0^2+B_0^2)(\tau-t_1)^2+\cdots, \qquad v=-\tfrac{1}{3}(A_0^2+B_0^2)(\tau-t_1)^3+\cdots, \tag{43}$$

The equation $v=0$ determines the points at which the periodic orbit crosses the u-axis. It follows from the second of (43) that $(t-t_1)=0$ is a triple but not a quadruple solution for $\gamma=0$. Therefore there are three solutions of $v=0$ for $t-t_1$ as power series in integral or fractional powers of $\gamma^{1/p}$, vanishing with γ. One of them is simply

$$\tau-t_1=0 \tag{44}$$

The others depend upon the values of the coefficients of the right member of the second equation of (42). Unless $\theta_4 A_0 - \theta_2 B_0 = K = 0$ for $\gamma = 0$ the two remaining solutions have the form

$$\tau - t_1 = +\frac{\sqrt{3K}}{\sqrt{A_0^2 + B_0^2}}\gamma^{1/2p} \qquad \tau - t_1 = -\frac{\sqrt{3K}}{\sqrt{A_0^2 + B_0^2}}\gamma^{1/2p} \tag{45}$$

If $K = 0$, which will be exceptionally if at all, the corresponding solutions exist but may be in integral powers of $\gamma^{1/p}$.

It has been remarked that p will in general be unity. When it is odd the periodic orbit with the cusp at (x_1, y_1) is a multiple orbit. If p is even, two orbits which are real when γ has one sign unite for $\gamma = 0$ and disappear by becoming imaginary when γ has the other sign. When p is odd there is a single real orbit for γ both positive and negative. It is clear that only exceptionally, if at all, will a double periodic orbit have a cusp. If it were so in any particular case the value of μ could be changed, when it would no longer be true. Therefore it will be supposed that p is unity.

It follows from (45) that the second and third intersections of the curve with the u-axis are real for γ positive or negative according as K is positive or negative, and that they are not real when γ has the other sign. When the second and third intersections of the curve are real the curve consists of a small loop as is indicated in Fig. 28; and when they are complex the curve has a point near which the curvature is sharp, as is indicated in Fig. 29. When there are three intersections of the curve with the u-axis, one occurs before t_1 and one after t_1.

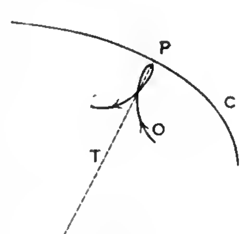


FIG. 28.

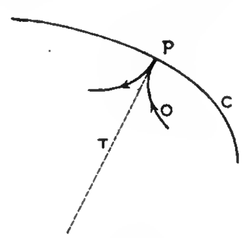


FIG. 29.

It follows from this discussion that *if a periodic orbit for a certain value* C_0 *of the Jacobian constant has a cusp, then for a slightly larger* (or smaller) *value it has a point, near a curve of zero relative velocity, in the vicinity of which there is very sharp curvature; for diminishing* (increasing) *values of* C *the point of sharp curvature approaches the cusp form on the corresponding curve of zero velocity, which it reaches for* $C = C_0$*; and for still further diminishing* (increasing) *values of* C *it has a small loop near a curve of zero velocity.* In Darwin's computations examples of periodic orbits with cusps were found. It follows, of course, from the symmetry of the periodic orbits with respect to the x-axis that if there is a cusp at $x = x_1$, $y = y_1$, then there is also a cusp at $x = x_1$, $y = -y_1$.

241. The Persistence of Cusps with Changing Mass-Ratio of the Finite Bodies.—Suppose for $\mu = \mu_0$ equations (1) have a periodic solution satisfying the initial conditions

$$x(0) = x_0, \qquad x'(0) = 0, \qquad y(0) = 0, \qquad y'(0) = y'_0, \tag{46}$$

and the cusp conditions at $t = t_1$

$$x(t_1)=x_1, \qquad x'(t_1)=0, \qquad y(t_1)=y_1, \qquad y'(t_1)=0. \tag{47}$$

If P represents the period of the solution, the expressions for x' and y satisfy the equations

$$x'(P/2)=0, \qquad y(P/2)=0. \tag{48}$$

Now suppose $\mu=\mu_0+\lambda$ and consider the question of the existence of a periodic orbit having a cusp for this value of μ. Let

$$t=(1+\delta)\tau, \quad x(0)=x_0+\alpha, \quad x'(0)=0, \quad y(0)=0, \quad y'(0)=y'_0+\beta. \tag{49}$$

The solution can be expanded as power series in α, β, δ, and λ. The conditions that it shall be periodic in τ with the period P are

$$x'(\tau)_{\tau=P/2}=p_1(\alpha, \beta, \delta, \lambda)=0, \qquad y(\tau)_{\tau=P/2}=p_2(\alpha, \beta, \delta, \lambda)=0, \tag{50}$$

where p_1 and p_2 are power series in α, β, δ, and λ, vanishing with α, β, δ, and λ. Unless the initial conditions (46) define a double periodic orbit these equations can be solved uniquely for β and δ as power series in α and λ, vanishing with α and λ. The results will have the form

$$\beta=q_1(\alpha, \lambda), \qquad \delta=q_2(\alpha, \lambda), \tag{51}$$

where q_1 and q_2 are power series in α and λ, vanishing for $\alpha=\lambda=0$.

Suppose β and δ are eliminated from the solutions by means of equations (51). The results will be expanded as power series in α and λ. The values of the coördinates at $\tau=t_1$ will be

$$\left.\begin{array}{ll} x(\tau)_{\tau=t_1}=x_1+P_1(\alpha, \lambda), & x'(\tau)_{\tau=t_1}=0+P_1'(\alpha, \lambda), \\ y(\tau)_{\tau=t_1}=y_1+P_2(\alpha, \lambda), & y'(\tau)_{\tau=t_1}=0+P_2'(\alpha, \lambda), \end{array}\right\} \tag{52}$$

where P_1, P_1', P_2, P_2' are power series in α and λ which vanish with α and λ.

The values of the coördinates near $\tau=t_1$ can be expanded as power series in $\tau-t_1$ satisfying equations (52). The results are

$$\left.\begin{array}{l} x(\tau)=x_1+P_1(\alpha, \lambda)+\ a(\tau-t_1)^2+\ \dots, \\ x'(\tau)=0\ +P_1'(\alpha, \lambda)+2a(\tau-t_1)\ +\ \dots, \\ y(\tau)=y_1+P_2(\alpha, \lambda)+\ b(\tau-t_1)^2+\ \dots, \\ y'(\tau)=0\ +P_2'(\alpha, \lambda)+2b(\tau-t_1)\ +\ \dots. \end{array}\right\} \tag{53}$$

The conditions that the orbit shall have a cusp at $\tau=t_2$ are

$$0=P_1'(\alpha, \lambda)+2a(t_2-t_1)+\dots, \qquad 0=P_2'(\alpha, \lambda)+2b(t_2-t_1)+\dots. \tag{54}$$

These equations are not satisfied by $\lambda=0$, because a and b contain terms which depend upon x_1 and y_1 alone. Neither are they satisfied by $\alpha=0$, $t_2-t_1=0$, unless orbits crossing the x-axis at $x=x_0$ are periodic for all values of μ and have a cusp for the same $t=t_1$. But it is known that the points at which the periodic orbits cross the x-axis depend upon the value of μ. Therefore equations (54) can be solved for t_2-t_1 and α in integral or fractional powers of λ. In general the solution will be in integral powers of λ. If the

solution is in integral or odd fractional powers of λ, it is real for both positive and negative values of λ. If the solution is in even fractional powers of λ, there are two real solutions when λ has one sign and only complex solutions when it has the other.

It follows from this discussion that if a real cusp exists for any value of μ, it will exist for all other values of μ unless two real cusps become identical and disappear by becoming complex. Since an orbit is uniquely defined by the conditions for a cusp, as well as by any other initial conditions, cusps disappear by becoming complex only when two orbits become identical.

Darwin's computations showed that in the case of one of the orbits which was complex for large values of the Jacobian constant ("satellites of Class C") there were periodic orbits without loops near the cusp form, and others for smaller values of the Jacobian constant having loops. It follows from the results of §240 that between the two orbits there exists one having two cusps which are symmetrically situated with respect to the x-axis; and it follows from the discussion of this article that the orbits with cusps exist for all values of μ unless a cusp develops on another orbit which later becomes identical with this.

242. Some Properties of the Periodic Oscillating Satellites near the Equilateral Triangular Points.—In Chapter IX, Dr. Buck has treated the periodic oscillating satellites which are near the equilateral triangular points, using in a general way the methods of Chapter V. It will be necessary for the purposes of the latter part of this chapter to develop a few additional properties of these orbits; and the most important of them can not be established by the methods of Chapter V, but follow from the methods of Chapter VI.

The differential equations will be transformed by letting $\mu=\mu_0+\lambda$. For motion in the vicinity of the equilateral triangular points they are

$$\left.\begin{aligned} &\frac{d^2x}{dt^2}-2\frac{dy}{dt}-\frac{3}{4}x-\frac{3\sqrt{3}}{4}(1-2\mu_0)y=X,\\ &\frac{d^2y}{dt^2}+2\frac{dx}{dt}-\frac{3\sqrt{3}}{4}(1-2\mu_0)x-\frac{9}{4}y=Y,\end{aligned}\right\} \qquad (55)$$

where X and Y are of the second and higher degrees in x, y, and λ.

In this article the character of the small oscillations will be discussed. In treating them X and Y may be provisionally put equal to zero. The characteristic equation on which the nature of the solutions depends is

$$\left\{\begin{matrix} S^2-\frac{3}{4}, & -2S-\frac{3\sqrt{3}}{4}(1-2\mu_0) \\ 2S-\frac{3\sqrt{3}}{4}(1-2\mu_0), & S^2-\frac{9}{4}\end{matrix}\right\} \equiv S^4+S^2+\frac{27}{4}\mu_0(1-\mu_0)=0. \qquad (56)$$

The roots of this equation are all purely imaginary or complex in conjugate pairs according as $1-27\mu_0(1-\mu_0)$ is or is not greater than zero. When $1-27\mu_0(1-\mu_0)$ is zero there are two pairs of equal purely imaginary solutions

of (56). It will be supposed for the present that μ_0 has such a value that the roots of (56) are pure imaginaries and distinct, and that μ has such a value that $\lambda=\mu-\mu_0$ is very small.

Let the roots of (56) be $+\sigma\sqrt{-1}$, $-\sigma\sqrt{-1}$, $+\rho\sqrt{-1}$, $-\rho\sqrt{-1}$, where σ and ρ are real and $\sigma<\rho$. There are two periodic solutions of the linear terms of (55), one with the period $2\pi/\sigma$ and the other with the period $2\pi/\rho$. If $x=0$, $y=c_1>0$ at $t=0$, the solution having the period $2\pi/\sigma$ is

$$x=\frac{c_1}{b_1}\sin\sigma t, \qquad y=\frac{a_1c_1}{b_1}\sin\sigma t+c_1\cos\sigma t, \tag{57}$$

where

$$b_1=\frac{2\sigma}{\sigma^2+\frac{9}{4}}, \qquad a_1=\frac{-3\sqrt{3}}{4}\,\frac{(1-2\mu_0)}{\sigma^2+\frac{9}{4}}<0. \tag{58}$$

The curve described by the infinitesimal body is an ellipse whose equation is

$$\frac{a_1^2+b_1^2}{c_1^2}x^2-\frac{2a_1}{c_1^2}xy+\frac{y^2}{c_1^2}=1;$$

and if θ represents the angle between the positive end of the x-axis and the major axis of the ellipse, it is easily found to be defined, except as to quadrant, by the equation

$$\tan 2\theta=\frac{2a_1}{1-a_1^2-b_1^2}=\frac{-3\sqrt{3}(1-2\mu_0)[\sigma^2+\frac{9}{4}]}{2[\sigma^4+\frac{1}{2}\sigma^2+\frac{81}{16}-\frac{27}{16}(1-2\mu_0)^2]}<0. \tag{59}$$

The direction of motion in the orbit is found to be retrograde from

$$x'(0)=\frac{c_1\sigma}{b_1}>0, \qquad y'(0)=\frac{a_1c_1\sigma}{b_1}<0.$$

There are equations for the period $2\pi/\rho$ which differ from (57), (58), and (59) only in that σ is replaced by ρ, and the subscript 1 on a, b, and c is replaced by 2. The motion in these orbits is also in the retrograde direction.

The linear terms of equations (55) admit the integral

$$x'^2+y'^2=\tfrac{3}{4}x^2+\frac{3\sqrt{3}}{2}(1-2\mu_0)xy+\tfrac{9}{4}y^2-C. \tag{60}$$

Let the value of C for the orbits having the period $2\pi/\sigma$ and $2\pi/\rho$ be C_σ and C_ρ. They are found from equations (57) to have the values

$$\left.\begin{aligned}C_\sigma&=c_1^2\left[\tfrac{9}{4}-\frac{\sigma^2}{b_1^2}(1+a_1^2)\right]=\frac{c_1^2}{4}\left[9-(\sigma^2+\tfrac{9}{4})^2-\frac{27}{16}(1-2\mu_0)^2\right],\\ C_\rho&=c_2^2\left[\tfrac{9}{4}-\frac{\rho^2}{b_2^2}(1+a_2^2)\right]=\frac{c_2^2}{4}\left[9-(\rho^2+\tfrac{9}{4})^2-\frac{27}{16}(1-2\mu_0)^2\right].\end{aligned}\right\} \tag{61}$$

It follows from (56) that the values of σ^2 and ρ^2 are

$$\sigma^2=\frac{1-\sqrt{1-27\mu_0(1-\mu_0)}}{2}, \qquad \rho^2=\frac{1+\sqrt{1-27\mu_0(1-\mu_0)}}{2}. \tag{62}$$

Since μ_0 is small σ^2 is small and the limit of σ^2 as μ_0 approaches zero is zero. On the other hand ρ^2 is near unity, and its limit as μ_0 approaches zero is 1.

Therefore C_σ is positive and C_ρ is negative, at least for small values of μ_0. For the Lagrangian equilateral triangular solution $x\equiv y\equiv 0$ the value of $C_\sigma = C_\rho$ is zero. Hence the value of the Jacobian constant is greater for the orbits whose period is $2\pi/\sigma$, and less for those whose period is $2\pi/\rho$, than it is for the Lagrangian equilateral triangular point solution.

Now consider the curves of zero relative velocity. They are known to be real only if the value of C is greater than that which belongs to the equilateral triangular point solution. Therefore they are real only for the solution with the period $2\pi/\sigma$. Their equation is

$$C_\sigma = \tfrac{3}{4}x^2 + \frac{3\sqrt{3}}{2}(1-2\mu_0)xy + \tfrac{9}{4}y^2. \tag{63}$$

This is the equation of an ellipse, the direction of whose major axis is given by

$$\tan 2\varphi = -\sqrt{3}(1-2\mu_0). \tag{64}$$

The limit of the right member of this expression for $\mu_0 = 0$ is $-\sqrt{3}$.

Since σ^2 is small when μ_0 is small, the approximate value of the right member of equation (59) is $-\sqrt{3}(1-2\mu_0)$. Therefore the orbit whose period is $2\pi/\sigma$ has its axes, for small μ_0, nearly coincident with the axes of the corrresponding curves of zero relative velocity. The x-axis is in the line joining the finite body μ with the equilateral triangular point, and the other is of course at right angles to it.

Let the coördinates in the orbit whose period is $2\pi/\sigma$ referred to its axes be ξ and η; its equation is then

$$\left.\begin{aligned} &A_1\xi^2 + B_1\eta^2 = 1,\\ &A_1 = \frac{1}{c_1^2}[(a_1 \cos\theta - \sin\theta)^2 + b_1^2 \cos^2\theta],\\ &B_1 = \frac{1}{c_1^2}[(a_1 \sin\theta + \cos\theta)^2 + b_1^2 \sin^2\theta]. \end{aligned}\right\} \tag{65}$$

The corresponding equations for the curves of zero relative velocity are

$$\left.\begin{aligned} &A\xi^2 + B\eta^2 = 1,\\ &A = \frac{1}{C_\sigma}[\tfrac{3}{4}\cos^2\varphi + \frac{3\sqrt{3}}{2}(1-2\mu_0)\sin\varphi\cos\varphi + \tfrac{9}{4}\sin^2\varphi],\\ &B = \frac{1}{C_\sigma}[\tfrac{3}{4}\sin^2\varphi - \frac{3\sqrt{3}}{2}(1-2\mu_0)\sin\varphi\cos\varphi + \tfrac{9}{4}\cos^2\varphi]. \end{aligned}\right\} \tag{66}$$

It follows from (58) that when μ_0 is small the approximate values of a_1 and b_1 are $\sqrt{3}/3$ and zero respectively. Since the limit of θ for $\mu_0 = 0$ is $-30°$, it is found from (65) that

$$\lim_{\mu_0=0}\frac{A_1}{B_1} = \lim_{\mu_0=0}\frac{\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 3\sin^2\theta}{3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + \sin^2\theta} = 0. \tag{67}$$

The limit of the ellipse for $\mu_0 = 0$ is a straight line through the origin and the position of μ, and for small values of μ_0 the eccentricity is near unity.

The approximate value of φ is also $-30°$ when μ_0 is small. Hence it is found from (66) that the ratio A/B also has the same value as A_1/B_1. Therefore at the limit the orbit with period $2\pi/\sigma$ and the corresponding curve of zero relative velocity have not only the same orientation, but they have also the same eccentricity.

It follows from (61), (65), and (66) that at the limit $\mu_0=0$

$$\frac{A}{A_1}=\tfrac{9}{4}\frac{c_1^2}{C\sigma}=4.$$

The ratio of the dimensions of the periodic orbit whose period is $2\pi/\sigma$ to that of the curve of zero relative velocity corresponding to the same value of C is equal to the square root of this number, or 2. This is actually the limit of the ratio of the linear dimensions of the orbits to the curves of zero relative velocity as μ_0 approaches zero.

The discussion so far has pertained to the linear terms alone of the differential equations. The results are the first terms of the series for the periodic solutions which can be shown to exist by the methods of Chapter VI. Consequently, for small values of the parameter λ they give close approximations to the periodic orbits and the corresponding curves of zero relative velocity. The period $2\pi/\sigma$ is very long for small values of μ_0.

Now consider the periodic orbits whose period is $2\pi/\rho$. The approximate value of ρ for small values of μ_0 is unity, and from the equations corresponding to (58) it is found that $b_2=\frac{3}{13}$, $a_2=-3\frac{\sqrt{3}}{13}$. Therefore, for these orbits also

$$\tan 2\theta=-\sqrt{3}$$

when μ_0 has the limit zero. It is found from the equations corresponding to (65) that

$$\lim_{\mu_0=0}\frac{A_2}{B_2}=\tfrac{1}{4}.$$

Therefore in these orbits the length of one axis is twice that of the other. The limit, for $\mu_0=0$, of the eccentricity of the orbits whose period is $2\pi/\sigma$ is unity and the limit of the periods is infinity; the corresponding limits for the orbits whose period is $2\pi/\rho$ are $\sqrt{3}/2$ and 2π.

243. The Analytic Continuity of the Orbits about the Equilateral Triangular Points.—The periodic solutions as developed by the methods of Chapter VI are power series in $\pm\lambda^{\frac{1}{2}}$, and they involve μ_0. The coefficients of the power series are continuous functions of μ_0. The orbits are real when λ has one sign and complex when it has the other. As λ passes through zero from one sign to the other, two real solutions for $\pm\lambda^{\frac{1}{2}}$ unite and disappear by becoming complex. They do not belong to the physical problem except when $\lambda=\mu-\mu_0$, but since the orbits exist for every value of μ_0 distinct from zero it is easy to get an understanding of the situation from the behavior of

the more general solutions when λ does not equal $\mu-\mu_0$. Or, since the coefficients are continuous functions of μ_0, this parameter can be considered as varying with λ so that their sum is μ. That is, the solutions and the period can be expressed in terms of μ and λ by replacing μ_0 by $\mu-\lambda$.

The two real orbits which unite and disappear for $\lambda=0$ are not geometrically distinct. This appears to be an exception to the theorem that real orbits appear or disappear only in pairs. It arises because in the analysis adopted the conditions that the orbit shall be periodic give a double determination of the same orbit. The two determinations coincide when the orbits shrink to zero dimensions for $\lambda=0$. Such a situation can arise only at the five equilibrium points. Moreover, the matter is quite different when the solutions are developed by the method of Chapter V. When the parameter ϵ' passes through zero the orbits do not disappear, but the same series is obtained for both positive and negative values of ϵ', the origin of time belonging to a different place on the orbit. In the symmetrical orbits around the equilibrium points which are on the x-axis the origin of time is displaced by half a period. In the non-symmetrical orbits about the equilateral triangular points the origin is shifted from one point where the arbitrary initial condition, e. g., $x'(0)=0$, is satisfied to the other point in the orbit where it is also satisfied.

The Jacobian integral exists when the right members of the differential equations are not limited to their linear terms. Hence, in place of (60), the right member is an infinite series in x and y. When the expressions for x and y as series in $\lambda^{1/2}$ are substituted the constant C becomes a power series in $\lambda^{1/2}$, the term of the lowest degree in λ being of the first degree. Consequently the series can be solved for $\lambda^{1/2}$ as a power series in $\pm C^{1/2}$, and the result substituted for the solution in powers of $\lambda^{1/2}$ will give x and y expressed as power series in $C^{1/2}$, which converge for $|C|$ sufficiently small. As C goes through zero the orbits whose period is $2\pi/\sigma$ change from real to complex, and those whose period is $2\pi/\rho$ change from complex to real. There is a branch on each of the series at $C=0$, but the two series are distinct.

When μ_0 satisfies the equation

$$1-27\mu_0(1-\mu_0)=0$$

the values of σ and ρ are equal to $\sqrt{2}/2$. In this case four solutions branch at $\lambda=0$.

244. The Existence of Periodic Orbits about the Equilateral Triangular Points for Large Values of μ.—The orbits heretofore discussed have been for values of μ such that $1-27\mu(1-\mu)$ is positive. If it is zero, there is a double solution of zero dimensions. Suppose now that μ has such a value that this function of μ is very little less than zero, and take μ_0 so that $1-27\mu_0(1-\mu_0)$ is a little greater than zero. Then there are real periodic orbits with the periods $2\pi/\sigma$ and $2\pi/\rho$ for λ sufficiently small. When λ

is less than $\mu-\mu_0$ these orbits do not belong to the physical problem. The analytic continuation of the solutions with respect to the parameter λ can be made until $\lambda=\mu-\mu_0$ unless they have some singularity for a real positive value of λ. It is very improbable that there is an infinity in the solutions for a real positive value of λ because an infinity implies either an infinite branch of the orbit or one passing through one of the finite bodies. The orbit could not acquire an infinite branch without winding infinitely many times, in the rotating plane, about the finite bodies. Even if it should pass through one of the finite bodies its continuity would be maintained, as was seen in the case of other orbits of ejection in Chapter XV. A branch-point would imply the existence of other real orbits which could become identical with the ones under consideration. It also seems very improbable that there is such a branch-point for small variations in λ. Therefore it will be assumed, as being probable, that the periodic solutions can be continued to the ones belonging to the physical problem for $\lambda=\mu-\mu_0$. This applies both to those whose periods are $2\pi/\sigma$ and to those whose periods are $2\pi/\rho$. The possibility of their having acquired loops about one or both of the finite bodies by having passed through ejectional forms must, however, be admitted. This circumstance makes numerical verification difficult.

Now suppose the value of μ_0 approaches 0.0385 . . . as a limit, the value of μ_0 which satisfies $1-27\mu_0(1-\mu_0)=0$, and that the analytic continuation can be made with respect to λ for all μ_0. The expressions for the coördinates in the two classes of orbits are the same except that σ and ρ are interchanged. As μ_0 approaches 0.0385 . . . σ and ρ approach equality, and the corresponding orbits approach identity for $\lambda=\mu-\mu_0$. A difficulty in attempting complete rigor arises from the fact that a certain determinant which is distinct from zero in the proof of the existence of the solutions approaches zero as μ_0 approaches 0.0385 . . . . But if it is admitted that the analytic continuation with respect to λ can be made starting with any μ_0, it follows that even if μ is a little larger than 0.0385 . . . there is a double periodic orbit, and it surrounds a small real curve of zero relative velocity in the vicinity of one of the equilateral triangular solution points. As it is decreased toward the limit 0.0385 . . . , the dimensions of this double periodic orbit diminish toward zero as a limit. There is in this analysis a double determination of a double periodic orbit, just as of a single periodic orbit, and the two determinations coincide when it has zero dimensions. Consequently it can disappear at zero dimensiens without uniting with another double periodic orbit.

If μ increases the double periodic orbit persists, according to the principles of §238, unless it becomes identical with another double periodic orbit. If there were another double periodic orbit with which it could unite it would envelop neither of the finite masses and would have two distinct branches which are symmetrical with respect to the x-axis. It is improbable in the extreme that there is another such double periodic orbit, which would mean

the existence of four single periodic orbits of the type under consideration. There are only two periodic orbits which shrink on the equilateral triangular points, and others of the type could arise only from orbits, which originally had loops about a finite body, passing through an ejectional form.

The existence of a double periodic orbit for all values of μ implies the existence or two single periodic orbits which branch from it for values of the parameters which define the orbit, for example the Jacobian constant C, its linear dimension, or its period. It should be added, of course, that the two series of orbits may branch at the double orbit when considered with respect to one parameter, and form a continuous series when considered with respect to another.

245. Numerical Periodic Orbits about Equilateral Triangular Points.—In accordance with the principles of §244, two periodic orbits about the equilateral triangular points should exist for all values of μ from 0 to $\frac{1}{2}$, and for all values of C near that belonging to the equilateral triangular equilibrium points. The only way they could cease to exist for at least some value of C would be for all of them to pass through an ejectional form for every C. These orbits have an axis of symmetry only when $\mu=\frac{1}{2}$. It is very difficult to establish by numerical processes the existence of a periodic orbit when it has no axis of symmetry because, for a given initial point, there are two arbitrary components of velocity, and interpolations must be made from a two-parameter family. Therefore the computations were restricted to the case $\mu=\frac{1}{2}$. It follows from the differential equations that in this case the orbits in question have the line $x=0$ as a line of symmetry. Since $\mu=\frac{1}{2}$ is far from the values $(0 \lesseqqgtr \mu \leqq 0.0385 \ldots)$ for which the existence of the orbits in question was established by direct processes, those found by computation can not be expected to have much geometrical resemblance to those found by analysis.

Since the surfaces of zero relative velocity expand with increasing C and unite on the x-axis, it follows that either the periodic orbits about the equilateral triangular points unite in pairs and disappear with increasing values of C, or they pass through the collinear equilibrium points with infinite periods. Therefore it seemed best to start computations for values of C not much greater than that belonging to the equilibrium point, viz, 3.

In attempting to discover periodic orbits about the equilateral triangular points 40 orbits were computed. In 17 of these C was taken equal to 3.03; in 16 it was taken equal to 3.20; in the remaining 7 it was taken equal to 3.284. The initial values of the coördinates and components of velocity were $x_0=0$, y_0 arbitrarily chosen, $y_0'=0$, x_0' determined so as to give the adopted value of C. The computation was continued until x became again equal to zero, and the approach to periodicity was determined by the approximation of y_0' to zero.

$C = 3.03$, Period $= 2 \times 4.388 = 8.776$ (Fig. 30).

t	x	y	x'	y'	t	x	y	x'	y'
0.0	0	.1200	−1.060	0	1.2	−1.0287	− .5841	− .708	− .070
0.05	− .0533	.1216	−1.076	.067	1.4	−1.1768	− .5687	− .762	+ .222
0.10	− .1083	.1266	−1.127	.131	1.6	−1.3285	− .4967	− .745	.495
0.15	− .1667	.1345	−1.219	.185	1.8	−1.4701	− .3726	− .660	.740
0.20	− .2311	.1446	−1.366	.214	2.0	−1.5890	− .2032	− .521	.947
0.25	− .3047	.1549	−1.593	.182	2.2	−1.6755	+ .0030	− .338	1.107
0.30	− .3921	.1600	−1.919	− .022	2.4	−1.7221	.2361	− .124	1.216
0.35	− .4966	.1454	−2.217	− .661	2.6	−1.7238	.4856	+ .108	1.271
0.40	− .6014	.0865	−1.798	−1.661	2.8	−1.6786	.7409	.344	1.274
0.45	− .6685	− .0060	− .927	−1.897	3.0	−1.5865	.9917	.575	1.226
0.50	− .7019	− .0957	− .479	−1.674	3.2	−1.4497	1.2285	.790	1.134
0.6	− .7353	− .2397	− .273	−1.233	3.4	−1.2720	1.4429	.982	1.003
0.7	− .7638	− .3475	− .313	− .941	3.6	−1.0586	1.6274	1.146	.838
0.8	− .7995	− .4305	− .403	− .725	3.8	− .8158	1.7762	1.277	.646
0.9	− .8445	− .4937	− .498	− .543	4.0	− .5503	1.8846	1.371	.435
1.0	− .8986	− .5396	− .583	− .378	4.2	− .2699	1.9495	1.427	.212
1.1	− .9605	− .5695	− .654	− .221	4.4	+ .0177	1.9689	1.443	− .018

$C = 3.03$, Period $= 2 \times 5.95 = 11.90$ (Fig. 31).

t	x	y	x'	y'	t	x	y	x'	y'
0	0	.7428	.074	0	2.3	− .7940	− .5196	− .323	− .605
.1	.0072	.7406	.070	− .044	2.4	− .8320	− .5766	− .434	− .500
.2	.0137	.7340	.058	− .087	2.5	− .8804	− .6198	− .532	− .365
.3	.0185	.7232	.038	− .130	2.6	− .9380	− .6496	− .616	− .231
.4	.0210	.7082	.010	− .170	2.7	−1.0030	− .6660	− .684	− .097
.5	.0202	.6892	− .026	− .209	2.8	−1.0742	− .6689	− .735	+ .039
.6	.0154	.6664	− .070	− .245	3.0	−1.2272	− .6340	− .784	.312
.7	.0060	.6402	− .121	− .278	3.2	−1.3828	− .5459	− .762	.570
.8	− .0090	.6109	− .180	− .308	3.4	−1.5275	− .4076	− .675	.808
.9	− .0303	.5788	− .247	− .332	3.6	−1.6490	− .2249	− .532	1.018
1.0	− .0588	.5446	− .323	− .352	3.8	−1.7222	− .0054	− .343	1.174
1.1	− .0954	.5086	− .411	− .367	4.0	−1.7841	+ .2412	− .121	1.284
1.2	− .1414	.4713	− .514	− .378	4.2	−1.8095	.5047	+ .120	1.342
1.3	− .1988	.4329	− .638	− .390	4.4	−1.7358	.7742	.368	1.344
1.4	− .2702	.3925	− .793	− .416	4.6	−1.6376	1.0390	.612	1.296
1.5	− .3587	.3482	− .986	− .489	4.8	−1.4920	1.2892	.842	1.199
1.6	− .4683	.2906	−1.204	− .696	5.0	−1.3024	1.5157	1.049	1.060
1.7	− .5951	.1988	−1.269	−1.199	5.2	−1.0744	1.7107	1.227	.884
1.8	− .6994	.0492	− .710	−1.700	5.4	− .8140	1.8674	1.371	.679
1.9	− .7384	− .1158	− .157	−1.522	5.6	− .5288	1.9806	1.475	.450
2.0	− .7459	− .2519	− .042	−1.209	5.8	− .2268	2.0464	1.538	.205
2.1	− .7522	− .3604	− .100	− .972	6.0	+ .0834	2.0623	1.557	− .024
2.2	− .7674	− .4483	− .207	− .792					

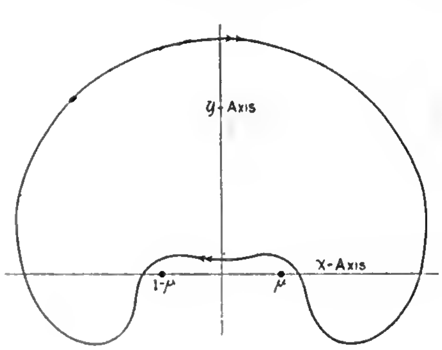


FIG. 30.

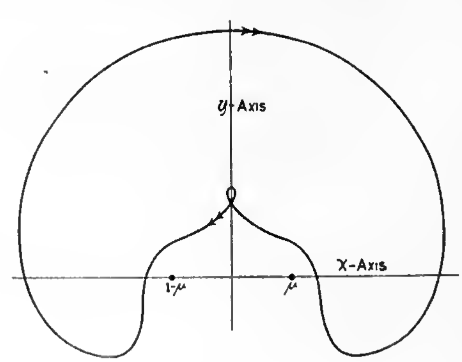


FIG. 31.

It was found that for $C=3.03$ there are two periodic orbits about the equilateral triangular points differing considerably in dimensions and periods; for $C=3.20$ there are also two periodic orbits which differ less

$C=3.20$, Period $=2\times4.68=9.36$ (Fig. 32).

t	x	y	x'	y'	t	x	y	x'	y'
0	0	.2030	− .893	0	1.6	−1.2324	− .4760	− .653	.273
.05	− .0449	.2038	− .904	.034	1.8	−1.3606	− .3976	− .621	.507
.10	− .0908	.2064	− .936	.066	2.0	−1.4772	− .2750	− .537	.714
.15	− .1390	.2104	− .993	.091	2.2	−1.5723	− .1146	− .408	.884
.20	− .1907	.2152	−1.076	.104	2.4	−1.6381	+ .0758	− .245	1.014
.25	− .2472	.2202	−1.191	.093	2.6	−1.6689	.2878	− .060	1.098
.30	− .3104	.2238	−1.339	.039	2.8	−1.6612	.5121	+ .138	1.138
.40	− .4618	.2119	−1.683	− .365	3.0	−1.6136	.7400	.337	1.134
.5	− .6288	.1309	−1.474	−1.268	3.2	−1.5268	.9632	.528	1.091
.6	− .7350	− .0142	− .693	−1.470	3.4	−1.4031	1.1741	.706	1.013
.7	− .7848	− .1484	− .372	−1.199	3.6	−1.2458	1.3663	.865	.904
.8	− .8181	− .2548	− .320	− .940	3.8	−1.0588	1.5343	1.001	.772
.9	− .8515	− .3380	− .355	− .731	4.0	− .8470	1.6736	1.113	.619
1.0	− .8901	− .4022	− .418	− .556	4.2	− .6154	1.7810	1.198	.452
1.1	− .9352	− .4498	− .484	− .399	4.4	− .3696	1.8537	1.256	.274
1.2	− .9864	− .4823	− .542	− .254	4.6	− .1150	1.8901	1.284	.090
1.4	−1.1036	− .5055	− .626	+ .018	4.8	+ .1426	1.8892	1.288	− .096

$C=3.20$, Period $=2\times5.72=11.44$ (Fig. 33).

t	x	y	x'	y'	t	x	y	x'	y'
0	0	.5554	− .186	0	2.2	−1.0165	− .5708	− .554	− .129
.05	− .0093	.5549	− .187	− .020	2.4	−1.1359	− .5722	− .631	+ .114
.10	− .0188	.5534	− .191	− .040	2.6	−1.2650	− .5256	− .650	.350
.15	− .0285	.5509	− .199	− .060	2.8	−1.3920	− .4334	− .611	.569
.20	− .0387	.5474	− .209	− .080	3.0	−1.5059	− .2997	− .520	.763
.25	− .0494	.5430	− .222	− .098	3.2	−1.5970	− .1307	− .385	.921
.30	− .0609	.5376	− .238	− .117	3.4	−1.6577	+ .0660	− .218	1.039
.4	− .0867	.5241	− .280	− .152	3.6	−1.6827	.2818	− .029	1.111
.5	− .1173	.5073	− .335	− .184	3.8	−1.6690	.5076	+ .168	1.139
.6	− .1542	.4874	− .406	− .214	4.0	−1.6158	.7345	.363	1.123
.7	− .1990	.4645	− .494	− .244	4.2	−1.5243	.9544	.549	1.069
.8	− .2537	.4385	− .603	− .278	4.4	−1.3974	1.1600	.718	.982
.9	− .3205	.4082	− .737	− .331	4.6	−1.2388	1.3452	.865	.867
1.0	− .4019	.3708	− .893	− .428	4.8	−1.0531	1.5052	.988	.730
1.1	− .4989	.3196	−1.050	− .618	5.0	− .8453	1.6361	1.085	.577
1.2	− .6079	.2419	−1.098	− .963	5.2	− .6208	1.7354	1.155	.414
1.4	− .7697	− .0152	− .411	−1.400	5.4	− .3851	1.8013	1.198	.244
1.6	− .8120	− .2561	− .137	− .990	5.6	− .1434	1.8329	1.215	.072
1.8	− .8495	− .4181	− .259	− .805	5.8	+ .0989	1.8301	1.205	− .100
2.0	− .9180	− .5202	− .423	− .649					

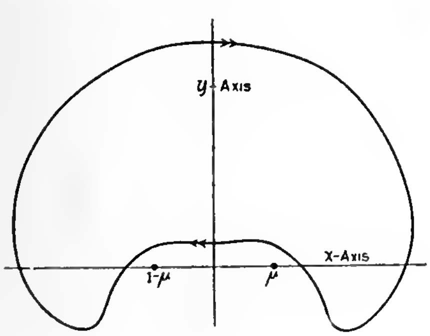


Fig. 32.

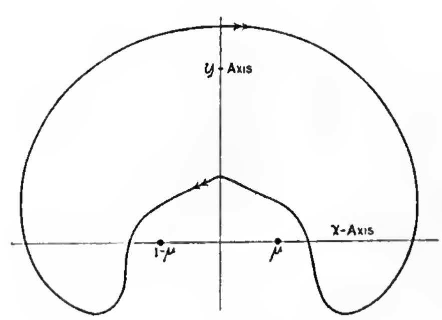


Fig. 33.

in dimensions and periods; and for $C=3.3284$ there is a very close approach to a periodic orbit, though one was not actually found. The computations indicate that the two series of periodic orbits unite and disappear for some value of C slightly smaller than 3.3284. But there is an orbit so nearly periodic for $C=3.3284$ that it is included as being very nearly that double

$C=3.3284$, Approximate Double Periodic Orbit, Period $=2\times 6.14=12.28$ (Fig. 34, page 511).

t	x	y	x'	y'	t	x	y	x'	y'
0	0	.3500	− .567	.000	2.2	−1.3027	− .3950	− .497	.434
.05	− .0284	.3499	− .571	− .003	2.4	−1.3953	− .2911	− .422	.600
.10	− .0572	.3496	− .583	− .006	2.6	−1.4691	− .1504	− .310	.731
.15	− .0868	.3492	− .603	− .010	2.8	−1.5178	+ .0051	− .173	.821
.20	− .1176	.3486	− .632	− .016	3.0	−1.5375	.1748	− .023	.869
.25	− .1501	.3476	− .670	− .024	3.2	−1.5270	.3498	+ .128	.876
.30	− .1848	.3461	− .717	− .036	3.4	−1.4869	.5226	.270	.846
.4	− .2624	.3405	− .840	− .081	3.6	−1.4201	.6862	.395	.785
.5	− .3542	.3281	−1.000	− .180	3.8	−1.3307	.8350	.495	.701
.6	− .4628	.3009	−1.168	− .390	4.0	−1.2239	.9655	.569	.603
.7	− .5842	.2444	−1.226	− .768	4.2	−1.1050	1.0759	.615	.500
.8	− .6972	.1460	− .979	−1.174	4.4	− .9795	1.1658	.631	.400
.9	− .7742	.0206	− .568	−1.273	4.6	− .9517	1.2364	.638	.308
1.0	− .8166	− .1004	− .315	−1.126	4.8	− .7254	1.2900	.623	.229
1.1	− .8428	− .2032	− .232	− .931	5.0	− .6033	1.3292	.597	.166
1.2	− .8658	− .2871	− .237	− .752	5.2	− .4869	1.3575	.567	.120
1.3	− .8915	− .3542	− .280	− .595	5.4	− .3763	1.3782	.540	.089
1.4	− .9223	− .4066	− .335	− .455	5.6	− .2705	1.3939	.520	.070
1.5	− .9586	− .4455	− .390	− .326	5.8	− .1675	1.4068	.511	.060
1.6	−1.0000	− .4720	− .439	− .204	6.0	− .0651	1.4182	.515	.055
1.8	−1.0954	− .4895	− .507	+ .026	6.2	+ .0397	1.4288	.536	.051
2.0	−1.1996	− .4627	− .527	.240					

periodic orbit at which the two single periodic orbits unite and disappear. It is believed that in all cases the computations covered so wide a range of initial conditions that no periodic orbits of the type in question escaped detection.

The results shown in the preceding tables (omitting intermediate steps) were obtained by the computations, the origin of coördinates being at the center of gravity of the finite bodies.

246. Closed Orbits of Ejection for Large Values of μ.—It was shown in Chapter XV that for small values of μ there exist closed orbits of ejection from $1-\mu$ for projections both toward and from $1-\mu$ and that their periods reduce to $2j\pi$ $(j=1, 2, \ldots)$ for $\mu=0$. It was also shown in §234 that these orbits can be continued, in the analytic sense, to any value of μ unless two of them disappear by becoming identical and vanishing. In order to confirm this conclusion and to get an idea of the form of these orbits for large values of μ, 63 orbits of ejection were computed. It was also desired to discover orbits which are orbits of ejection from one finite mass and of collision with the other.

The computations were all started by means of the series (36) of §228. After the values of x, y, x', and y' had been determined for a few small values of t, the computations were continued by the ordinary processes.

In all cases the infinitesimal body was ejected from the finite body $1-\mu$ in the positive or negative x-direction.

$\mu = \frac{1}{2}$, $C=2.242$, Closed Orbit of Ejection (Fig. 35).

t	x	y	x'	y'	t	x	y	x'	y'
0	− .5000	0	+ ∞	0	1.4	.5001	−1.0542	− .254	− .831
.10	− .2343	− .0266	1.653	− .432	1.6	.4241	−1.2173	− .514	− .793
.15	− .1592	− .0512	1.378	− .549	1.8	.2964	−1.3678	− .759	− .703
.20	− .0950	− .0811	1.202	− .644	2.0	.1221	−1.4949	− .979	− .560
.25	− .0380	− .1153	1.081	− .722	2.2	− .0929	−1.5884	−1.163	− .367
.30	+ .0138	− .1530	.997	− .784	2.4	− .3401	−1.6388	−1.300	− .131
.35	.0622	− .1934	.940	− .829	2.6	− .6091	−1.6385	−1.380	+ .139
.40	.1081	− .2356	.901	− .857	2.8	− .8877	−1.5820	−1.396	.429
.5	.1954	− .3221	.845	− .864	3.0	−1.1628	−1.4666	−1.343	.724
.6	.2767	− .4070	.776	− .832	3.2	−1.4203	−1.2931	−1.220	1.008
.7	.3496	− .4884	.676	− .799	3.4	−1.6464	−1.0654	−1.030	1.263
.8	.4112	− .5674	.554	− .784	3.6	−1.8284	− .7910	− .781	1.472
.9	.4600	− .6458	.421	− .787	3.8	−1.9559	− .4804	− .486	1.621
1.0	.4954	− .7250	.286	− .799	4.0	−2.0209	− .1474	− .162	1.699
1.2	.5025	− .8878	.014	− .827	4.2	−2.0201	+ .1938	+ .170	1.699

$\mu = \frac{1}{2}$, $C=2.840$, Closed Orbit of Ejection (Fig. 36).

t	x	y	x'	y'	t	x	y	x'	y'
0	− .5000	0	+ ∞	0	3.0	.3901	− .9546	− .019	− .174
.10	− .2445	− .0257	1.514	− .408	3.2	.3838	− .9930	− .045	− .210
.15	− .1769	− .0485	1.214	− .506	3.4	.3710	−1.0390	− .086	− .250
.20	− .1215	− .0758	1.016	− .580	3.6	.3483	−1.0930	− .144	− .290
.25	− .0744	− .1062	.874	− .636	3.8	.3121	−1.1546	− .220	− .325
.30	− .0335	− .1391	.769	− .677	4.0	.2591	−1.2222	− .312	− .348
.35	+ .0029	− .1737	.690	− .702	4.2	.1861	−1.2928	− .419	− .357
.40	.0358	− .2092	.630	− .715	4.4	.0907	−1.3621	− .536	− .335
.5	.0943	− .2806	.546	− .706	4.6	− .0284	−1.4251	− .656	− .290
.6	.1459	− .3492	.486	− .663	4.8	− .1717	−1.4761	− .775	− .214
.7	.1918	− .4126	.432	− .604	5.0	− .3377	−1.5087	− .882	− .106
.8	.2322	− .4698	.376	− .540	5.2	− .5232	−1.5165	− .969	+ .033
.9	.2670	− .5208	.321	− .481	5.4	− .7234	−1.4938	−1.028	.108
1.0	.2964	− .5662	.267	− .427	5.6	− .9319	−1.4359	−1.051	.384
1.2	.3402	− .6424	.174	− .338	5.8	−1.1409	−1.3395	−1.032	.581
1.4	.3674	− .7028	.102	− .269	6.0	−1.3417	−1.2033	− .968	.780
1.6	.3826	− .7509	.053	− .214	6.2	−1.5249	−1.0281	− .857	.969
1.8	.3898	− .7894	.022	− .173	6.4	−1.6816	− .8172	− .702	1.135
2.0	.3924	− .8208	.006	− .143	6.6	−1.8032	− .5764	− .510	1.268
2.2	.3928	− .8474	.002	− .125	6.8	−1.8834	− .3130	− .288	1.358
2.4	.3929	− .8716	.000	− .119	7.0	−1.9164	− .0366	− .038	1.397
2.6	.3927	− .8959	− .001	− .126	7.2	−1.8976	+ .2420	+ .227	1.379
2.8	.3923	− .9228	− .005	− .145					

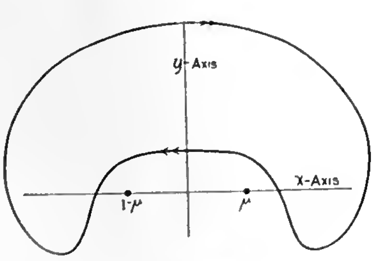


Fig. 34.

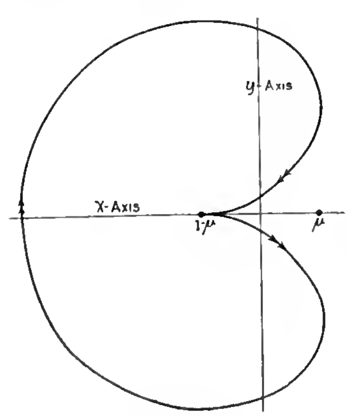


Fig. 35.

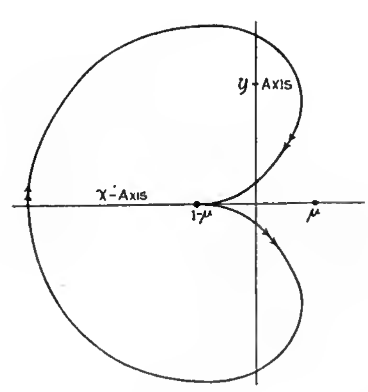


Fig. 36.

Any orbit which intersects the x-axis perpendicularly is symmetrical with respect to the x-axis. Hence, it follows that if one of these orbits of ejection intersects the x-axis perpendicularly, then it is a closed orbit of ejection of the type treated in Chapter XV.

Computations were first made for $\mu = \frac{1}{2}$ to discover orbits of the type characterized by $j=1$ with ejection toward μ and shown in Fig. 15. It was proved in Chapter XV that such an orbit exists for small values of μ and that its period is approximately 2π. Such an orbit was found for $\mu = \frac{1}{2}$, but its period was about 8. Another orbit, also of a similar type, was discovered whose period was about 14. One of these orbits is undoubtedly the limit, for decreasing values of C, of the oscillating satellite about the collinear equilibrium point, as Burrau's calculations have indicated. The value of C corresponding to the equilibrium point for $\mu = \frac{1}{2}$ is about 3.46, and the values of C for these orbits are 2.24 and 2.84. The question arises regarding the origin of the other orbit of this type. It is probably the limiting form of a periodic orbit about $1-\mu$ consisting of a double loop and having a double point on the x-axis. Such orbits were treated by Poincaré in *Les Méthodes Nouvelles de la Mécanique Céleste,* Chapter XXXI. The analytic continuation of the former beyond the ejectional form for decreasing values of C is also a periodic orbit with two loops. For greater or smaller values of C the latter will have also the character of an oscillating satellite, but it can not reduce to the equilibrium point because there is only one orbit of this type. The results for $\mu = \frac{1}{2}$ are given in the tables of page 511.

For $\mu = \frac{4}{5}$ similar results were found. Since orbits of this type have not been computed heretofore, the results for the four orbits will be given for enough values of t to exhibit their properties.

The corresponding results for $\mu = \frac{4}{5}$, with ejections from $1-\mu = \frac{1}{5}$, are given in the following tables:

$\mu = 4/5$, $C = 2.696$, Closed Orbit of Ejection (Fig. 37, page 513)

t	x	y	x'	y'	t	x	y	x'	y'
0	− .8000	0	+ ∞	0	2.0	.5307	−1.0537	− .130	− .605
.10	− .6021	− .0198	1.252	− .324	2.2	.4863	−1.1762	− .314	− .616
.15	− .5447	− .0383	1.062	− .413	2.4	.4051	−1.2978	− .497	− .593
.20	− .4947	− .0609	.945	− .489	2.6	.2892	−1.4112	− .672	− .535
.25	− .4495	− .0870	.870	− .554	2.8	.1384	−1.5094	− .834	− .440
.30	− .4073	− .1162	.822	− .610	3.0	− .0429	−1.5851	− .975	− .311
.35	− .3669	− .1478	.796	− .657	3.2	− .2498	−1.6316	−1.090	− .150
.40	− .3273	− .1816	.788	− .694	3.4	− .4766	−1.6430	−1.172	+ .040
.5	− .2477	− .2535	.811	− .735	3.6	− .7158	−1.6142	−1.214	.250
.6	− .1642	− .3270	.861	− .725	3.8	− .9591	−1.5419	−1.212	.474
.7	− .0758	− .3969	.904	− .668	4.0	−1.1972	−1.4244	−1.162	.702
.8	+ .0154	− .4596	.912	− .585	4.2	−1.4205	−1.2617	−1.063	.924
.9	.1052	− .5140	.878	− .506	4.4	−1.6189	−1.0561	− .915	1.128
1.0	.1899	− .5615	.810	− .449	4.6	−1.7829	− .8125	− .720	1.302
1.2	.3343	− .6464	.628	− .415	4.8	−1.9041	− .5384	− .487	1.433
1.4	.4404	− .7321	.434	− .450	5.0	−1.9758	− .2429	− .226	1.513
1.6	.5081	− .8282	.244	− .511	5.2	−1.9936	+ .0627	+ .049	1.533
1.8	.5382	− .9361	.056	− .567					

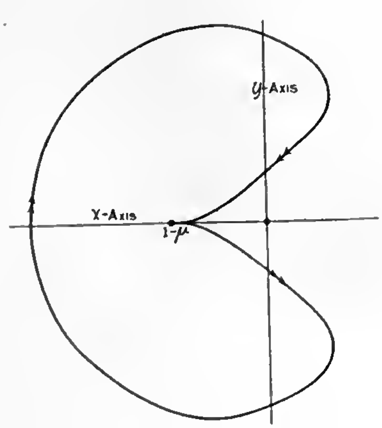
$\mu = 4/5$, $C = 2.965$, Closed Orbit of Ejection (Fig. 38).

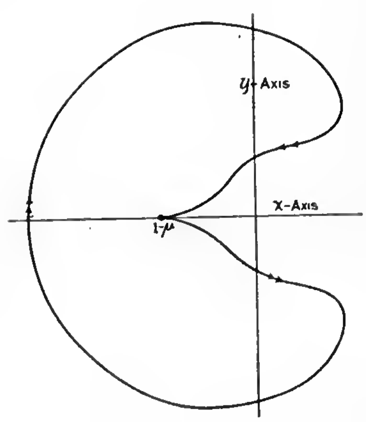
t	x	y	x'	y'	t	x	y	x'	y'
0	− .8000	0	+ ∞	0	2.8	.7358	− .9116	− .028	− .382
.10	− .6092	− .0192	1.168	− .308	3.0	.7212	− .9919	− .119	− .419
.15	− .5554	− .0319	.963	− .388	3.2	.6880	−1.0784	− .214	− .444
.20	− .5107	− .0577	.834	− .452	3.4	.6354	−1.1686	− .313	− .456
.25	− .4713	− .0816	.746	− .504	3.6	.5626	−1.2596	− .415	− .452
.30	− .4356	− .1079	.687	− .546	3.8	.4692	−1.3482	− .519	− .432
.35	− .4023	− .1361	.649	− .581	4.0	.3551	−1.4311	− .622	− .393
.40	− .3704	− .1658	.627	− .607	4.2	.2206	−1.5041	− .722	− .334
.5	− .3083	− .2282	.622	− .636	4.4	.0668	−1.5633	− .816	− .255
.6	− .2448	− .2919	.652	− .633	4.6	− .1049	−1.6047	− .899	− .155
.7	− .1773	− .3538	.697	− .599	4.8	− .2917	−1.6241	− .968	− .036
.8	− .1054	− .4109	.740	− .538	5.0	− .4907	−1.6178	−1.019	+ .102
.9	− .0299	− .4609	.769	− .460	5.2	− .6977	−1.5823	−1.047	.255
1.0	+ .0475	− .5027	.776	− .376	5.4	− .9077	−1.5148	−1.048	.421
1.2	.1991	− .5627	.729	− .232	5.6	−1.1145	−1.4134	−1.016	.593
1.4	.3356	− .5997	.633	− .148	5.8	−1.3116	−1.2774	− .951	.766
1.6	.4513	− .6262	.524	− .125	6.0	−1.4922	−1.1075	− .850	.932
1.8	.5454	− .6524	.419	− .141	6.2	−1.6489	− .9057	− .713	1.083
2.0	.6193	− .6844	.322	− .181	6.4	−1.7748	− .6762	− .541	1.208
2.2	.6746	− .7256	.232	− .232	6.6	−1.8634	− .4249	− .342	1.298
2.4	.7123	− .7774	.145	− .286	6.8	−1.9103	− .1599	− .125	1.346
2.6	.7326	− .8397	.059	− .336	7.0	−1.9129	+ .1102	+ .099	1.348

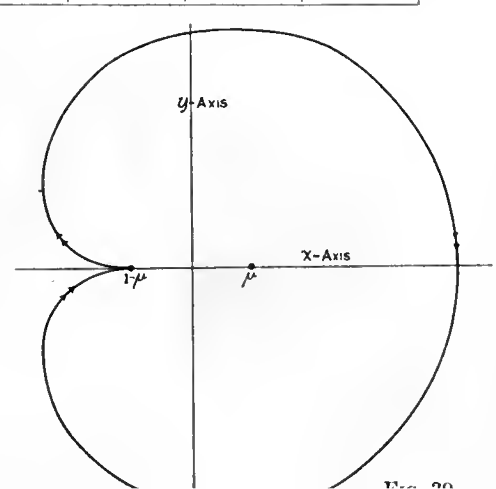
Computations were also made in which the ejection was in the negative direction from $1-\mu$. One periodic orbit of the type characterized by $j=1$, Fig. 15, was discovered, and its coördinates are given in the following table:

$\mu = 1/2$, $C = 1.8224$, Closed Orbit of Ejection (Fig. 39).

t	x	y	x'	y'	t	x	y	x'	y'
0	− .5000	0	− ∞	0	1.3	−1.0232	1.4027	.902	1.236
.10	− .7723	.0273	−1.736	.448	1.4	− .9251	1.5226	1.059	1.158
.15	− .8516	.0530	−1.457	.577	1.5	− .8118	1.6337	1.206	1.062
.20	− .9193	.0847	−1.264	.687	1.6	− .6843	1.7344	1.342	.949
.25	− .9784	.1215	−1.111	.784	1.7	− .5439	1.8230	1.465	.821
.30	−1.0306	.1628	− .979	.870	1.8	− .3919	1.8981	1.573	.679
.35	−1.0765	.2082	− .861	.947	2.0	− .0595	2.0025	1.741	.358
.4	−1.1168	.2573	− .751	1.015	2.2	+ .2993	2.0387	1.835	.000
.5	−1.1816	.3649	− .547	1.132	2.4	.6691	2.0011	1.848	− .376
.6	−1.2265	.4827	− .353	1.222	2.6	1.0331	1.8879	1.778	− .753
.7	−1.2524	.6084	− .164	1.288	2.8	1.3747	1.7009	1.625	−1.113
.8	−1.2593	.7395	+ .023	1.332	3.0	1.6777	1.4453	1.393	−1.436
.9	−1.2478	.8740	.207	1.354	3.2	1.9271	1.1303	1.091	−1.704
1.0	−1.2180	1.0096	.388	1.355	3.4	2.1103	.7685	.733	−1.901
1.1	−1.1703	1.1443	.565	1.335	3.6	2.2179	.3755	.338	−2.015
1.2	−1.1052	1.2760	.737	1.295	3.8	2.2445	− .0315	− .070	−2.039







If μ were zero and the infinitesimal body were ejected from $1-\mu$ either toward or from μ in such a way that its period would be π, the orbits described in rotating axes would consist of two parts symmetrical with respect to the x-axis, as shown in Figs. 40, a, and 41, a. These curves are the limits separating two types of periodic orbits (for $\mu=0$) in the rotating plane, as is shown in Figs. 40, b, and 41, c. As μ increases a dissymmetry develops with respect to the line through $1-\mu$ perpendicular to the x-axis. Suppose the orbits are followed as μ increases in such a way that they shall remain orbits of ejection in one way or the other. Then orbits of the type Fig. 40, a, will go into types having some of the characteristics of both types a and c of Fig. 41. That they partake of the characteristics of type c instead of those of type b was proved by computations for both $\mu=\frac{1}{2}$ and $\mu=\frac{4}{5}$.

The two following tables give the results for ejection from $1-\mu$ toward μ, with μ having the values $\frac{1}{2}$ and $\frac{4}{5}$. The first orbit lacks a little of being closed, but an exactly closed orbit exists between this one and the one which was computed for $C=3.478$.

$\mu=\frac{1}{2}$, $C=3.489$, Closed Orbit of Ejection (Fig. 42).

t	x	y	x'	y'	t	x	y	x'	y'
0	$-.5000$	0	$+\infty$	0	.7	$-.0225$	$-.2933$	$-.038$	$-.229$
.1	$-.2559$	$-.0244$	1.353	$-.380$	.8	$-.0290$	$-.3090$	$-.088$	$-.089$
.15	$-.1972$	$-.0455$	1.021	$-.456$	.9	$-.0393$	$-.3108$	$-.114$	.054
.20	$-.1521$	$-.0697$	.793	$-.506$	1.0	$-.0511$	$-.2980$	$-.119$	.203
.25	$-.1169$	$-.0958$	.621	$-.533$	1.1	$-.0625$	$-.2703$	$-.107$	.353
.30	$-.0894$	$-.1227$	.486	$-.543$	1.2	$-.0721$	$-.2276$	$-.083$	.499
.35	$-.0680$	$-.1498$	.375	$-.537$	1.3	$-.0790$	$-.1708$	$-.055$	.634
.40	$-.0516$	$-.1762$	.283	$-.518$	1.4	$-.0833$	$-.1018$	$-.033$	.741
.5	$-.0308$	$-.2248$	.141	$-.448$	1.5	$-.0860$	$-.0244$	$-.022$	.796
.6	$-.0222$	$-.2647$	.037	$-.346$	1.55	$-.0871$	$+.0154$	$-.022$	.794

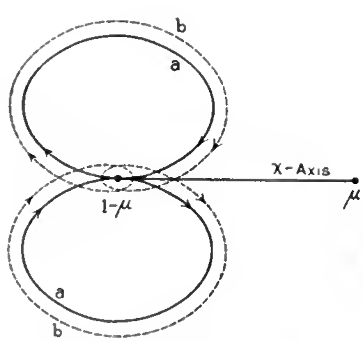


FIG. 40.

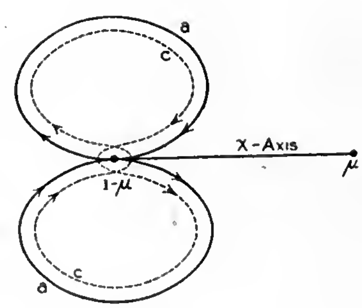


FIG. 41.

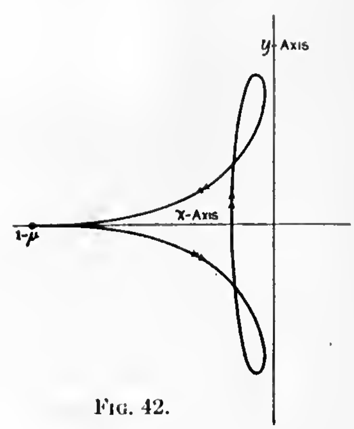


FIG. 42.

Six orbits of a similar type were computed for $\mu=\frac{4}{5}$. The following approximately closed orbit was obtained:

$\mu = 4/5$, $C = 3.5927$, Closed Orbit of Ejection (Fig. 43).

t	x	y	x'	y'	t	x	y	x'	y'
0	−.8000	0	+∞	0	.7	−.4727	−.1973	−.036	−.097
.1	−.6232	−.0177	.957	−.272	.8	−.4775	−.1957	−.056	+.010
.15	−.5920	−.0327	.709	−.323	.9	−.4834	−.1892	−.060	.120
.20	−.5511	−.0497	.538	−.353	1.0	−.4892	−.1716	−.052	.230
.25	−.5274	−.0677	.412	−.367	1.1	−.4937	−.1433	−.038	.335
.30	−.5094	−.0862	.313	−.368	1.2	−.4977	−.1051	−.094	.427
.35	−.4958	−.1044	.232	−.358	1.3	−.4987	−.0588	−.016	.494
.4	−.4859	−.1218	.167	−.339	1.4	−.5003	−.0074	−.191	.525
.5	−.4744	−.1529	.069	−.278	1.425	−.5008	+.0057	−.218	.525
.6	−.4719	−.1767	.004	−.195					

Computations were made for $\mu = \frac{1}{2}$ in which the ejections were from $1-\mu$ in the direction opposite to μ. One closed orbit corresponding to $j = \frac{1}{2}$ was found, the results for which are contained in the following table. Corresponding computations were not made for $\mu = \frac{4}{5}$.

$\mu = 1/2$, $C = 2.73$, Closed Orbit of Ejection (Fig. 44).

t	x	y	x'	y'	t	x	y	x'	y'
0	− .5000	0	− ∞	0	1.4	− .7741	.8052	.566	.162
.1	− .7570	.0254	−1.509	.415	1.5	− .7181	.8159	.552	.052
.15	− .8249	.0490	−1.215	.513	1.6	− .6644	.8156	.518	− .057
.20	− .8800	.0766	− .996	.592	1.7	− .6151	.8046	.465	− .162
.25	− .9254	.1078	− .823	.654	1.8	− .5719	.7833	.395	− .262
.30	− .9628	.1418	− .677	.703	1.9	− .5367	.7523	.307	− .357
.35	− .9934	.1799	− .549	.740	2.0	− .5111	.7121	.203	− .447
.40	−1.0180	.2156	− .434	.767	2.1	− .4967	.6632	.083	− .532
.5	−1.0510	.2940	− .232	.795	2.2	− .4951	.6059	− .052	− .615
.6	−1.0653	.3736	− .058	.793	2.3	− .5076	.5401	− .201	− .704
.7	−1.0635	.4517	+ .093	.766	2.4	− .5356	.4645	− .361	− .813
.8	−1.0476	.5260	.222	.717	2.5	− .5797	.3757	− .579	− .973
.9	−1.0198	.5945	.331	.651	2.6	− .6374	.2660	− .619	−1.244
1.0	− .9821	.6556	.419	.570	2.7	− .6944	.1214	− .445	−1.659
1.1	− .9366	.7081	.487	.478	2.725	− .7040	.0787	− .316	−1.750
1.2	− .8854	.7510	.534	.377	2.750	− .7100	.0342	− .155	−1.808
1.3	− .8306	.7835	.561	.271	2.775	− .7116	−.0112	+ .026	−1.800

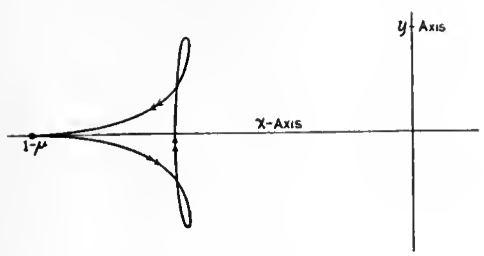


Fig. 43.

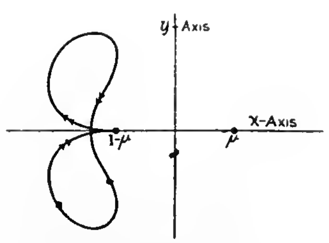


Fig. 44.

247. Orbits of Ejection from $1-\mu$ and Collision with μ.—The motion of the infinitesimal body from ejection may be followed by means of the series (36) of §228. The motion of the infinitesimal body into a collision with the same or the other finite mass can not be followed by numerical processes. But a collision with the second finite mass can be established in

certain cases by making use of properties of symmetry. Any orbit which, for $\mu=\frac{1}{2}$, intersects the y-axis perpendicularly is symmetrical with respect to the y-axis. Hence it follows that if, for $\mu=\frac{1}{2}$, an orbit of ejection from $1-\mu$ intersects the y-axis perpendicularly, then it has a symmetrical collision with the second finite mass.

After 14 computations had been made, an orbit of ejection from $1-\mu$ and collision with μ ($\mu=\frac{1}{2}$) was discovered in which the ejection was toward μ and in which the collision took place without the infinitesimal having encircled, in the rotating plane, either $1-\mu$ or μ. The following table gives the results at a considerable number of intervals from the time of ejection of the infinitesimal body from $1-\mu$ until it crossed the line $x=0$:

$\mu=\frac{1}{2}$, $C=3.4174$, Orbit of Ejection and Collision (Fig. 45, page 517).

t	x	y	x'	y'	t	x	y	x'	y'
0	−.5000	0	+∞	0	.55	−.0082	−.2549	.138	−.440
.10	−.2546	−.0246	1.371	−.383	.60	−.0005	−.2751	.094	−.391
.15	−.1949	−.0459	1.042	−.462	.65	+.0012	−.2933	.058	−.338
.20	−.1487	−.0704	.818	−.514	.70	.0033	−.3088	.028	−.281
.25	−.1122	−.0970	.650	−.545	.75	.0040	−.3214	.004	−.222
.30	−.0831	−.1247	.518	−.559	.80	.0038	−.3310	− .014	−.161
.35	−.0600	−.1527	.412	−.557	.85	.0028	−.3375	− .026	−.099
.40	−.0417	−.1803	.324	−.542	.90	.0013	−.3409	− .032	−.037
.45	−.0274	−.2068	.251	−.516	.95	−.0003	−.3411	− .032	+.027
.50	−.0164	−.2318	.190	−.482					

Another somewhat similar, but larger, orbit of ejection from $1-\mu$ and collision with μ was found, after a number of computations, in which the ejection was toward μ. Part of the results of the computation are given in the following table:

$\mu=\frac{1}{2}$, $C=2.739$, Orbit of Ejection and Collision.

t	x	y	x'	y'	t	x	y	x'	y'
0	−.5000	0	+∞	0	1.4	.3941	− .7683	.032	−.379
.10	−.2428	− .0258	1.537	−.412	1.5	.3947	− .8051	− .018	−.357
.15	−.1739	− .0490	1.242	−.514	1.6	.3906	− .8398	− .063	−.336
.20	−.1169	− .0767	1.048	−.592	1.7	.3822	− .8723	− .105	−.315
.25	−.0682	− .1079	.910	−.652	1.8	.3698	− .9028	− .142	−.295
.30	−.0254	− .1417	.808	−.696	1.9	.3539	− .9313	− .176	−.274
.35	+.0131	− .1773	.733	−.726	2.0	.3348	− .9577	− .206	−.253
.40	.0483	− .2140	.677	−.742	2.1	.3129	− .9819	− .233	−.232
.45	.0810	− .2512	.635	−.745	2.2	.2884	−1.0041	− .257	−.211
.50	.1119	− .2883	.600	−.737	2.3	.2616	−1.0241	− .278	−.189
.55	.1412	− .3248	.570	−.720	2.4	.2328	−1.0419	− .296	−.167
.6	.1690	− .3602	.542	−.697	2.5	.2024	−1.0575	− .312	−.145
.7	.2202	− .4273	.482	−.642	2.6	.1705	−1.0709	− .326	−.122
.8	.2651	− .4886	.416	−.586	2.7	.1373	−1.0820	− .337	−.100
.9	.3033	− .5446	.347	−.536	2.8	.1031	−1.0908	− .346	−.077
1.0	.3345	− .5960	.277	−.493	2.9	.0682	−1.0973	− .353	−.053
1.1	.3588	− .6435	.210	−.458	3.0	.0326	−1.1014	− .357	−.030
1.2	.3766	− .6878	.146	−.428	3.1	−.0032	−1.1032	− .360	−.006
1.3	.3882	− .7293	.087	−.402					

This orbit has a loop about the equilateral triangular point. It follows that there are two families of periodic orbits of the types shown in Fig. 46. Ten orbits were computed in an attempt to find one of them. The difficulties of making the calculations when the infinitesimal body was near one of the finite bodies were so great that wide departures from the orbits of ejection had to be attempted. Indications of such periodic orbits were obtained, but none was actually found.

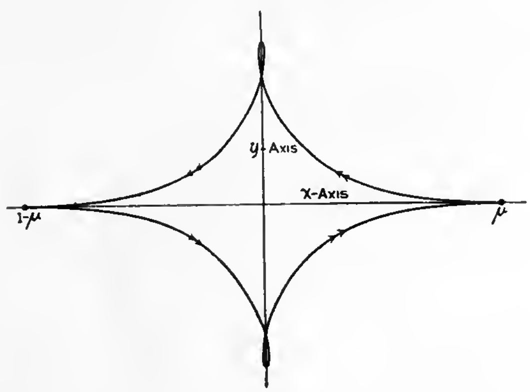


FIG. 45.

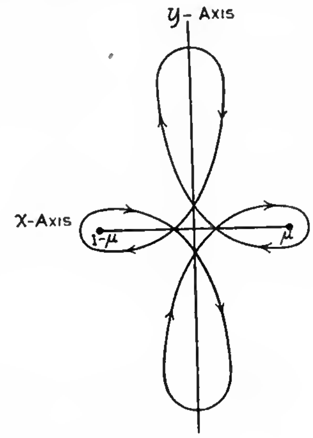


FIG. 46.

248. Proof of the Existence of an Infinite Number of Closed Orbits of Ejection and of Orbits of Ejection and Collision when $\mu=\frac{1}{2}$.—It was proved in Chapter XV that when μ is sufficiently small there are infinitely many closed orbits of ejection, and reasons were given for believing that these orbits persist for all values of μ. The question of the existence of orbits of ejection and collision was not considered.

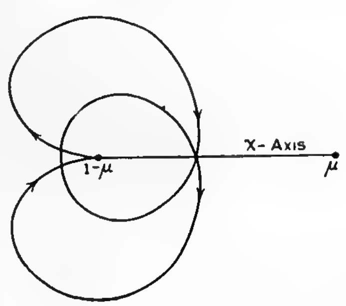


FIG. 47.

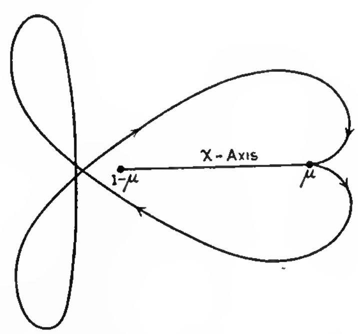


FIG. 48.

It will now be shown that there are infinitely many closed orbits of ejection, and of ejection and collision, for $\mu=\frac{1}{2}$. The differential equations of motion in fixed rectangular axes with the origin at the center of gravity of the system are

$$\frac{d^2x}{dt^2}=-\frac{\frac{1}{2}(x-x_1)}{r_1^3}-\frac{\frac{1}{2}(x-x_2)}{r_2^3},\qquad \frac{d^2y}{dt^2}=-\frac{\frac{1}{2}(y-y_1)}{r_1^3}-\frac{\frac{1}{2}(y-y_2)}{r_2^3},\tag{68}$$

where, if the finite bodies are on the x-axis at $t=0$,

$$x_1=-\tfrac{1}{2}\cos t,\qquad x_2=+\tfrac{1}{2}\cos t,\qquad y_1=-\tfrac{1}{2}\sin t,\qquad y_2=+\tfrac{1}{2}\sin t.\tag{69}$$

Now let $x=r\cos\theta$, and $y=r\sin\theta$, after which equations (68) become

$$\left.\begin{aligned}\frac{d^2r}{dt^2}&=r\left(\frac{d\theta}{dt}\right)^2-\tfrac{1}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]r-\tfrac{1}{4}\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right]\cos(\theta-t),\\ \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)&=\tfrac{1}{2}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)r\sin(\theta-t),\end{aligned}\right\}\tag{70}$$

where

$$r^2=x^2+y^2,\qquad r_1^2=r^2+\tfrac{1}{4}+r\cos(\theta-t),\qquad r_2^2=r^2+\tfrac{1}{4}-r\cos(\theta-t).\tag{71}$$

Suppose it is known that, at $t=T$,

$$r\geqq 2,\quad \frac{dr}{dt}>0.\tag{72}$$

Since $\tfrac{1}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]r<\frac{r}{(r-\frac{1}{2})^3}$ and $\tfrac{1}{4}\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right]<\tfrac{1}{2}\frac{1}{(r-\frac{1}{2})^3}$ and $r\left(\frac{d\theta}{dt}\right)^2$ is always positive, it follows from the first of (70) that

$$\frac{d^2r}{dt^2}>\frac{-(r+\frac{1}{2})}{(r-\frac{1}{2})^3}.\tag{73}$$

The integral of this inequality gives

$$\left(\frac{dr}{dt}\right)^2>\frac{1}{r-\frac{1}{2}}+\frac{\frac{1}{2}}{(r-\frac{1}{2})^2}+K=F(r).\tag{74}$$

Suppose $K>0$ and that $\frac{dr}{dt}>0$ at $t=T$. Then $F(r)$ will always exceed K in value and r will become infinite with t.

Computations were made in which the infinitesimal body was ejected from $1-\mu$ both toward and from μ with initial conditions corresponding to $K>0$, and they were both followed until $r>2$ with $\frac{dr}{dt}$ positive. Hence in both cases the infinitesimal body would recede to infinity. Moreover, it follows from the second equation of (70) that, when referred to rotating axes, the infinitesimal body revolves infinitely many times about the finite bodies, and its distance from the origin continually increases.

Now consider, for example, a closed orbit of ejection from $1-\mu$, for $\mu=\frac{1}{2}$, in which the infinitesimal body makes at least one circuit about the finite body μ. Its orbit therefore crosses the x-axis perpendicularly exactly once, but it does not cross the y-axis perpendicularly. Moreover, it follows from the symmetry of the orbit with respect to the x-axis, that if it crosses the y-axis in the first half of the orbit at an angle $\pi/2+\alpha$, then in the second half it crosses the y-axis at the angle $\pi/2-\alpha$. One or the other of these angles is less than $\pi/2$. Suppose it is the latter. Now suppose the initial conditions of ejection are changed so as to increase K. The orbit will cease to be a closed orbit of ejection and will tend toward one which winds out to infinity with continually increasing r. The angles at which the orbit crosses the axes are continuous functions of the parameter defining the initial conditions, as K for example. Hence the intersection with the y-axis which was at the

angle $\pi/2-\alpha$ and less than $\pi/2$ for the closed orbit of ejection will be exactly perpendicular for a certain value of K. The orbit will be therefore an orbit of ejection from $1-\mu$ and collision with μ.

Now consider an orbit of ejection from $1-\mu$ and collision with μ, crossing the x-axis at least once. From the symmetry of these orbits with respect to the y-axis it follows that they cross the x-axis an even number of times and that if such an orbit crosses the x-axis once at an angle $\pi/2+\beta$, where β is a positive quantity, then it also crosses it at an angle of $\pi/2-\beta$. If the initial conditions are so changed as to increase the constant K, the orbit ceases to be an orbit of ejection and collision, the angle corresponding to $\pi/2-\beta$ eventually becomes greater than $\pi/2$, and therefore, since it is a continuous function of K, there is at least one value of K for which it is exactly $\pi/2$. Such an orbit is a closed orbit of ejection.

It follows from this discussion that if, for any value of K, there is a closed orbit of ejection, then for some larger value of K, corresponding to a smaller value of the Jacobian constant C, there is an orbit of ejection and collision; and that if, for any value of K, there is an orbit of ejection and collision, then for some larger value of K, corresponding to some smaller value of the Jacobian constant C, there is a closed orbit of ejection. Hence, for $1-\mu=\frac{1}{2}$, there are infinitely many closed orbits of ejection and collision. They are all distinct because they have distinct values of K. And since it has been shown that, when $K>0$ for an orbit of ejection, r increases continuously to infinity, it follows that the infinite sets of values of K corresponding to these classes of orbits are bounded. The orbits may be characterized by the number of times they cross the y-axis. For ejections in both the positive and the negative direction there are closed orbits of ejection from each of the finite bodies, and also orbits of ejection from one and collision with the other, which cross the y-axis $2(2j+1)$ times, $j=0, 1, 2, \ldots$

249. On the Evolution of Periodic Orbits about Equilibrium Points.—The evolution of the periodic orbits about the equilibrium points (a) and (c) which are on the x-axis and not between $1-\mu$ and μ was traced, for decreasing values of C, by Burrau's computations from small ovals to the ejectional form. For $\mu=\frac{1}{2}$ they are shown in Fig. 35, and for $\mu=\frac{4}{5}$ in Fig. 37. Beyond these forms they have loops about $1-\mu$.

The periodic orbits about the equilibrium point (b) between $1-\mu$ and μ undergo corresponding evolutions. In the case $\mu=\frac{1}{2}$ as the orbit, for decreasing values of C, becomes an orbit of ejection from one body it becomes an orbit of collision with the other. This limiting form is shown in Fig. 45. It intersects the y-axis six times, twice perpendicularly. Beyond this form it has loops about the finite bodies, and the motion in these loops is in the retrograde direction. With decreasing values of C these loops probably enlarge, the loop about each body eventually becoming an orbit of collision with the other body. In this case the orbit of

ejection and collision intersects the y-axis six times, two of the intersections being perpendicular.

As C decreases, the ejectional and collisional form passes into a loop about the second body, which in turn expands and becomes an ejectional and collisional form with respect to the first body. In this manner the loops of the periodic orbit, with decreasing values of C, pass through ejectional and collisional forms, first with one finite body and then with the other, in a never-ending series, the ejectional and collisional forms being those shown to exist, for $\mu=\frac{1}{2}$, in §248, in which the ejections from each body are in the direction away from the other. They are characterized by the fact that they cross the y-axis $2(2j+1)$ times, $j=0, 1, 2, \ldots$

If the finite masses are unequal the evolution of the periodic orbits is in a general way similar, except that the ejectional forms for the two masses do not occur for the same values of C.

Periodic orbits about the equilateral triangle equilibrium points have been shown in Figs. 30 and 34. With decreasing values of C they probably increase in size and pass through ejectional forms, but ejectional forms in which the direction of ejection is not along the x-axis. Consequently they can not be discovered by numerical processes. If this conjecture is correct, for still smaller values of C they possess loops about the finite bodies, and for still smaller values of C the loop about each of the finite bodies may pass through an ejectional form with the other. There is, however, no evidence to guide conjectures.

250. On the Evolution of Direct Periodic Satellite Orbits.—There are three direct periodic satellite orbits, two of which are complex for large values of the Jacobian constant, but all of which are real for smaller values of C. Suppose the orbits about the finite body μ are under consideration. With decreasing value of C they can pass through ejectional forms. In fact, Darwin's computations showed that two of them were approaching such forms, one by approaching μ from the positive direction and the other by approaching it from the negative direction.

The motion of the infinitesimal body when it is near collision is nearly the same as it would be if the mass of the second body were zero. Consequently its properties can be inferred from a consideration of the motion in the neighborhood of an ejection in the problem of two bodies referred to rotating axes. It is clear that if the ejection is in the positive direction, the curve near the point of ejection lies on the negative side of the x-axis, while if the ejection is in the negative direction the curve near the point of ejection lies on the positive side of the x-axis. When the ejection is in the positive direction, the two families of periodic orbits which are near the ejectional orbit both intersect the x-axis in the negative direction from the point of ejection, and the small complete loop about the point of ejection is then described in the retrograde direction, while the partial loop is described in the

positive direction; while if the ejection is in the negative direction, the two families of periodic orbits which are near the ejectional orbit both intersect the x-axis in the positive direction from the point of ejection, but in this case the small loops are both described in the same directions as in the other case.

The x-axis consists of three parts, viz, that extending from $-\infty$ to the position of $1-\mu$, that extending from $1-\mu$ to μ, and that extending from μ to $+\infty$. If a periodic orbit intersects the x-axis perpendicularly in any one of these three parts before it goes through an ejectional form, then it will also intersect the x-axis perpendicularly in the same part after it passes through the ejectional form. Moreover, the branches of a closed orbit of ejection extend from the finite body with which there is collision in the direction opposite to that in which the neighboring periodic orbit intersects the x-axis perpendicularly.

In the case of the direct periodic orbit about $1-\mu$ which enlarges in the positive direction and approaches $1-\mu$ from the negative direction, the ejection is in the positive direction, the collision is in the negative direction, and the orbit has the form shown in Fig. 42. The computation shows that it has two loops and, therefore, that it had two cusps symmetrical with respect to the x-axis before it arrived at the ejectional form. After this orbit passes beyond the ejectional form, with decreasing values of C, it acquires a loop about $1-\mu$, which intersects the x-axis perpendicularly in the negative direction from μ and which has a double point on the x-axis between $1-\mu$ and μ.

Consider the further evolution of the periodic orbit. If the small loop about $1-\mu$ should again pass to the ejectional form, the ejectional orbit would be exactly of the type of that from which the loop developed. It is improbable that such an additional ejectional orbit exists for another value of C.

Now consider the possibility of that part of the orbit which crosses the x-axis perpendicularly in the positive direction between $1-\mu$ and μ passing through an ejectional form. It can not pass to an ejectional form with μ because, in accordance with the general conclusions respecting the motion near a point of ejection, the branches of the curve near μ would lie in the positive direction from it, and the partial loop about μ just before the ejectional form was reached would be described in the retrograde direction. But this branch of the curve could evolve to an ejectional form with $1-\mu$, when the orbit would have the form shown in Fig. 47. With decreasing values of C this orbit acquires an additional loop about $1-\mu$, which is described in the retrograde direction and which intersects the x-axis perpendicularly in the negative direction between $1-\mu$ and μ. This loop can expand and take an ejectional form with μ, then acquire a loop about μ, which can become an ejectional form with $1-\mu$, and so on, being an ejectional form first with one of the finite masses and then with the other in a never-ending sequence. The ejections from $1-\mu$ are all in the negative direction,

and from μ they are all in the positive direction. It is probable, though not certain, that this is qualitatively the course of evolution of the direct satellite orbit from which the start was made.

Now consider the direct satellite orbit about $1-\mu$ which enlarges in the negative direction and which approaches the ejectional form from the positive direction. The ejectional form was found by computation and is shown in Fig. 44. With decreasing C this orbit acquires a loop about $1-\mu$, which may pass to the ejectional form with μ, as shown in Fig. 48. The other branch which crosses the x-axis perpendicularly may pass to the ejectional form with $1-\mu$. With decreasing values of C this orbit acquires a small loop about $1-\mu$ which never again passes through the ejectional form. But the loop about μ enlarges and becomes an ejectional form with $1-\mu$ with the ejection in the negative direction. Then follows a loop which becomes an ejectional form with μ, followed by a loop about μ which becomes an ejectional form with $1-\mu$, and so on, first with one finite body and then with the other in a never-ending sequence. The ejections from $1-\mu$ are in the negative direction, and from μ they are in the positive direction.

There is a third direct satellite orbit whose evolution has not been traced. Only a conjecture can be made in regard to it, and that conjecture is that it acquires cusps and then loops, probably about the region of the equilateral triangular points.

251. On the Evolution of Retrograde Periodic Satellite Orbits.—Consider the retrograde periodic satellites about $1-\mu$. There are three such orbits, only one of which is real for large values of C. The numerical experiments which were made, §238, indicate that only one of them is real for any value of C.

For large values of C the retrograde periodic orbit about $1-\mu$ is small and nearly circular in form. As C diminishes the orbit increases in size and departs widely from a circle. Consider the question of its passing through an ejectional form with $1-\mu$. If the periodic orbit should approach the ejectional form by shrinking upon $1-\mu$ from the positive or the negative direction, just before arriving at the ejectional form it would make a *partial* loop about $1-\mu$ in the retrograde direction, and just after passage through the ejectional form it would make a *complete* loop about $1-\mu$ in the positive direction. But it was seen in §250, in connection with a consideration of an ejectional orbit in the problem of two bodies referred to rotating axes, that this is impossible. Hence the retrograde satellite orbit about $1-\mu$ can not become an ejectional orbit with $1-\mu$, at least until after it has passed through an ejectional form with μ.

Now consider the possibility of the retrograde satellite orbit about $1-\mu$ passing through a collisional form with μ. Since it intersects the x-axis between $1-\mu$ and μ such an orbit must be one in which the collision is in the negative direction, in which the ejection is in the positive direction, in which

the partial loop just before collision is described in the positive direction, and in which the complete loop just after collision is described in the retrograde direction. This is precisely the way in which such a limiting form can be passed, and the periodic orbit passes through this form. An orbit of this type, with the rôles of $1-\mu$ and μ interchanged, was computed and is shown in Fig. 39.

After the retrograde periodic orbit about $1-\mu$ passes through an ejectional form with μ, it acquires a retrograde loop about μ which crosses the x-axis in the positive direction between $1-\mu$ and μ. This loop enlarges and passes through an ejectional form with $1-\mu$, after which it acquires a retrograde loop about $1-\mu$, which, in turn, enlarges and passes through an ejectional form with μ. This process continues, the form becoming ejectional first with one finite mass and then with the other in a never-ending sequence. In all of these orbits the parts near the ejection points are on the negative side of $1-\mu$ or the positive side of μ, and never between $1-\mu$ and μ. The orbits of these series which are closed orbits of ejection with $1-\mu$ are a part of those which were shown to exist in Chapter XV for sufficiently small values of μ; and those which are closed orbits of ejection with μ are the corresponding orbits for the other finite mass.

Consider first the orbits of the type under consideration which are orbits of collision and ejection with $1-\mu$. All these orbits are orbits of ejection in the negative direction; they have double points on the x-axis in the positive direction from μ, and intersect the x-axis perpendicularly only in the negative direction from $1-\mu$. They are therefore only those orbits of §226 which are characterized by ejection in the negative direction and by even values of j; those characterized by odd values of j have a different origin.

The orbits of the type under consideration which are orbits of ejection and collision with μ also intersect the x-axis perpendicularly only on the negative side of $1-\mu$. On interchanging the rôles of $1-\mu$ and μ in §226, orbits of ejection from μ in the positive direction were proved to exist for $1-\mu$ sufficiently small. Those which are characterized by odd values of j intersect the x-axis perpendicularly on the negative side of $1-\mu$. They are of the type of the part of those under consideration which are orbits of collision and ejection with μ.

To summarize: The retrograde periodic satellite orbits about $1-\mu$, with decreasing values of C, go through an infinite series of ejectional forms with $1-\mu$, the ejections all being in the negative direction, and these orbits are those of the orbits treated in §226 which are ejected from $1-\mu$ in the negative direction and which are characterized by even values of j. The retrograde periodic satellite orbits also go through an infinite series of ejectional forms with μ, the ejections all being in the positive direction, and these orbits are those which can be shown to exist by the methods of §226, and which are characterized by ejection in the positive direction from μ and by odd values of j. There are similar retrograde periodic satellite orbits about

μ, and they go through a similar series of critical ejectional forms with both $1-\mu$ and μ. The ejections from $1-\mu$ and μ are also respectively in the negative and positive directions, but those which are ejectional forms with $1-\mu$ are characterized by odd values of j, while those which are ejectional forms with μ are characterized by even values of j. Therefore, the retrograde periodic satellites about the two finite bodies together pass through all the ejectional forms from both finite bodies, of the type treated in §226, in which the ejection from one body is in the opposite direction from the other.

252. On the Evolution of Periodic Orbits of Superior Planets.—It was shown in Chapter XII that for large values of C there are two periodic orbits in which the infinitesimal body makes simple circuits about both of the finite bodies in the retrograde direction. When the system is referred to fixed axes, one of the orbits is direct and one is retrograde. There are also four orbits in which the coördinates are complex, two of them being direct when referred to fixed axes and two being retrograde. It is not known whether or not either pair of the complex orbits becomes real with decreasing values of C. Since none of these orbits was computed, very little is positively known about their geometrical characteristics or about their evolution.

It was shown in §248 that there are two infinite sets of orbits which are orbits of ejection from one finite body and of collision with the other. One set is characterized by the fact that the ejection from each finite body is in the direction away from the other. Reasons were given in §249 for believing that they are limiting forms of the analytic continuations of the oscillating satellites about the equilibrium point b. The other set is characterized by the fact that the ejection from each finite body is toward the other finite body. These orbits are probably limiting forms of the analytic continuations of retrograde periodic planetary orbits. The probable series of changes to the first limiting form is shown qualitatively in Figs. 49 and 50. Beyond the limiting form the orbits acquire loops about each of the finite bodies.

There are two retrograde periodic orbits of the types of superior planets. Probably they both undergo evolutions to limiting forms of the types described. This conjecture is supported by the fact that two closed orbits of ejection were found by computation, §247, for a related type of orbits.

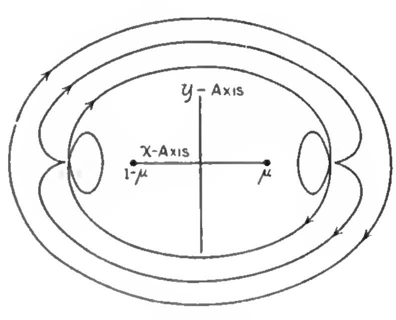


Fig. 49.

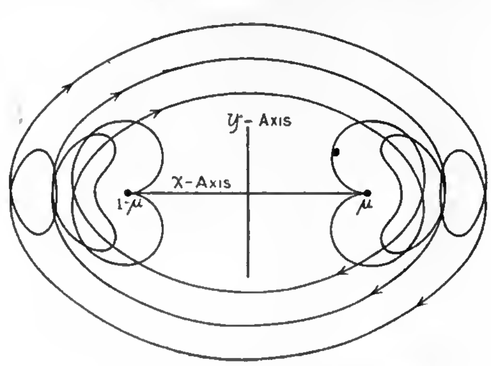


Fig. 50.